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BI-INVARIANT SUBSPACES OF WEAK CONTRACTIONS

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1. INTRODUCTION

For a bounded linear operator T acting on a complex, separable Hilbert space H, let Alg T, $\{T\}''$ and $\{T\}'$ denote the weakly closed algebra generated by T and I, the double commutant and the commutant of T, respectively. A subspace K of H is said to be *bi-invariant* (resp. *hyperinvariant*) for T if K is invariant for every operator in $\{T\}''$ (resp. $\{T\}'$). Let Lat T, Lat$''T$ and Hyperlat T denote the lattices of invariant subspaces, bi-invariant subspaces and hyperinvariant subspaces of T, respectively. The following trivial relations hold: Alg $T \subseteq \{T\}'' \subseteq \{T\}'$ and Lat $T \supseteq$ Lat$''T \supseteq$ Hyperlat T.

For various classes of operators, among which are normal operators and operators acting on a finite dimensional space, the elements of Lat$''T$ have been completely determined (cf. [4]). In particular, if T satisfies the double commutant property, that is, if Alg $T = \{T\}''$, then Lat$''T$ coincides with Lat T.

The purpose of the present paper is to study Lat$''T$ for completely non-unitary (c.n.u.) weak contractions with finite defect indices. Before in a series of papers [13], [14] and [15] we investigated the elements of Hyperlat T for such operators. (These were preceded by the work of Sickler [5].) We gave specific descriptions of the elements of Hyperlat T and showed that Hyperlat T is preserved, as a lattice, under quasi-similarities of this type of operator. In this paper we extend some of these results to Lat$''T$. As before we shall develop the theory in two stages, first for C_{11} contractions and then for weak contractions. In Section 2 we fix the notation and terminology and briefly review some basic results needed in the later work. In Section 3 we consider c.n.u. C_{11} contractions with finite defect indices. Specific descriptions of the elements in Lat$''T$ are given in Theorem 3.5. As a corollary, we show that Lat$''T$ is also preserved under quasi-similarities (Corollary 3.7). The former result is then extended to c.n.u. weak contractions with finite defect indices in Section 4 (Theorem 4.1). We also give necessary and sufficient conditions, in terms of the characteristic function of T, that two of Lat T, Lat$''T$ and Hyperlat T be equal to each other (Theorems 4.4, 4.5 and Corollary 4.6). In particular, for the operators considered any two of these lattices are equal if and only if the corresponding algebras of operators Alg T, $\{T\}''$ and $\{T\}'$ are equal. Note that whether these hold for general operators is unknown (cf. [4]).

2. PRELIMINARIES

A contraction T ($\|T\| \leq 1$) is *completely non-unitary (c.n.u.)* if there exists no nontrivial reducing subspace on which T is unitary. The *defect indices* of T are, by definition, $d_T = \text{rank } (I - T^*T)^{\frac{1}{2}}$ and $d_{T^*} = \text{rank } (I - TT^*)^{\frac{1}{2}}$. $T \in C_{\cdot 1}$ (resp. $C_{1\cdot}$) if $T^{*n}x \not\to 0$ (resp. $T^n x \not\to 0$) for all $x \neq 0$; $C_{11} = C_{\cdot 1} \cap C_{1\cdot}$. T is a *weak contraction* if (i) its spectrum $\sigma(T)$ does not fill the open unit disc, and (ii) $I - T^*T$ is of finite trace. Weak contractions have equal defect indices. Note that C_{11} contractions with finite defect indices are weak contractions.

Let $\mathbf{C}$ be the complex plane. For positive integer n, let L^2_n and H^2_n denote the standard Lebesgue and Hardy spaces of $\mathbf{C}^n$-valued functions defined on the unit circle Λ. We use t to denote the argument of a function defined on Λ. For the sake of brevity, a statement involving t is said to be true if it holds for almost all t with respect to the Lebesgue measure. For an arbitrary contraction T, let Θ_T denote its *characteristic function*. If T is c.n.u. and has defect indices $d_T = d_{T^*} \equiv n < \infty$, then Θ_T is an $n \times n$ matrix-valued function defined on Λ. In the discussion of the following we shall consider the *functional model* of such a contraction, that is, we consider T being defined on $H \equiv [H^2_n \oplus \overline{\Delta L^2_n}] \ominus \{\Theta_T w \oplus \Delta w : w \in H^2_n\}$ by $T(f \oplus g) = P(e^{it}f \oplus e^{it}g)$ for $f \oplus g \in H$, where $\Delta(t) = (I - \Theta_T(t)^* \Theta_T(t))^{\frac{1}{2}}$ and P denotes the (orthogonal) projection onto H. There is a one-to-one correspondence between the invariant subspaces of T and the regular factorizations of Θ_T. Let $K \subseteq H$ be an invariant subspace for T with the corresponding regular factorization $\Theta_T = \Theta_2 \Theta_1$. If $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ is the triangulation on $H = K \oplus K^\perp$, then the characteristic functions of T_1, T_2 coincide with the purely contractive parts of Θ_1, Θ_2, respectively. For the details, the readers are referred to [6]. Operators in $\{T\}'$ have the form $P\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$, where A is a bounded analytic function while B and C are bounded measurable functions satisfying the conditions $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + C\Delta = \Delta A_0$, where A_0 is another bounded analytic function (cf. [8]).

For arbitrary operators T_1, T_2 on H_1, H_2, respectively , $T_1 \prec T_2$ denotes that T_1 is a *quasi-affine transform* of T_2, that is, there exists a one-to-one operator X from H_1 onto a dense linear manifold of H_2 (called *quasi-affinity)* such that $XT_1 = T_2 X$. T_1, T_2 are *quasi-similar* ($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$.

A c.n.u. C_{11} contraction T with finite defect indices is quasi-similar to a uniquely determined operator, called the *Jordan model* of T, of the form $M_{E_1} \oplus \ldots \oplus M_{E_k}$, where $E_1, \ldots, E_k$ are measurable subsets of Λ satisfying $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_k$ and M_{E_j} denotes the operator of multiplication by e^{it} on the space $L^2(E_j)$ of square-integrable functions on E_j, $j = 1, \ldots, k$ (cf. [12], Theorem 2). Indeed,

$E_j = \{t: \text{rank}\, \Delta(t) \geqslant j\}, j = 1, \ldots, k,$ and, in particular, $E_1 = \{t : \Theta_T(t)$ not isometric $\}$. Let U and V denote the operators of multiplication by e^{it} on $\overline{\Delta L^2_n}$ and $\overline{\Delta_* L^2_n}$, respectively, where $\Delta_* = (I - \Theta_T\, \Theta_T^*)^{\frac{1}{2}}$. It is known that T is quasi-similar to U as well as to V (cf. [6], Prop. II. 3.5). In this case, both U and V are unitarily equivalent to the Jordan model of T (cf. [2], Lemma 4.1).

If T is a weak contraction, then Θ_T admits a *scalar multiple*, that is, there exist a scalar valued analytic function $\delta \not\equiv 0$ and a contractive analytic f unction Ω such that $\Omega\Theta_T = \Theta_T\Omega = \delta I$. For a c.n.u. weak contraction T on H we may consider its $C_0 - C_{11}$ *decomposition.* Let $H_0, H_1 \subseteq H$ be the invariant subspaces for T such that $T_0 \equiv T|H_0$ and $T_1 \equiv T|H_1$ are the C_0 and C_{11} *parts* of T, respectively. Then H_0 and H_1 correspond to the $*$-canonical factorization $\Theta_T = \Theta_{*e}\Theta_{*i}$ and the canonical factorization $\Theta_T = \Theta_i\Theta_e$ of Θ_T, respectively. They are even hyperinvariant for T and satisfy $H_0 \vee H_1 = H$ and $H_0 \cap H_1 = \{0\}$. For the details consult [6], Chapter VIII. A weak contraction is *multiplicity-free* if it admits a cyclic vector (cf. [12] for equivalent conditions).

3. C_{11} CONTRACTIONS

Throughout this section T denotes a c.n.u. C_{11} contraction with equal defect indices $n < \infty$ on $H \equiv [H^2_n \oplus \overline{\Delta L^2_n}] \ominus \{\Theta_T w \oplus \Delta w : w \in H^2_n\}$. We start the proof of our main result with the following

3.1 LEMMA. *Let T be as above and let U be the operator of multiplication by e^{it} on $\overline{\Delta L^2_n}$. If $S = P\begin{bmatrix} A & 0 \\ B & C\end{bmatrix}$ is an operator in $\{T\}''$, then C is in $\{U\}''$.*

Proof. Let W be an operator in $\{U\}'$. Let δ be a scalar multiple of Θ_T and let Ω be a contractive analytic function such that $\Omega\Theta_T = \Theta_T\Omega = \delta I_{C^n}$. For each $m \geqslant 1$, let $F_m = \left\{t : |\delta(t)| \geqslant \frac{1}{m}\right\}$. Then $F_m \uparrow \Lambda$. Define $S_m = P\begin{bmatrix} 0 & 0 \\ -\chi_{F_m} W\Delta \frac{1}{\delta}\Omega & \chi_{F_m} W\end{bmatrix}$.
It is easily seen that $S_m \in \{T\}'$. Since $S \in \{T\}''$, we have $S_m S = S S_m$ and it follows that $\chi_{F_m} WC = C\chi_{F_m} W$ for all $m \geqslant 1$. As $m \to \infty$, we obtain $WC = CW$. This shows that $C \in \{U\}''$.

3.2 LEMMA. *For $j = 1, 2,$ let T_j be a c.n.u. C_{11} contraction with finite defect ndices and let $\Delta_j = (I - \Theta_{T_j}^*\Theta_{T_j})^{\frac{1}{2}}$. Then the following are equivalent to each other:*

(1) T_1 is quasi-similar to T_2;

(2) rank $\Delta_1(t) =$ rank $\Delta_2(t)$ a.e.

Proof. Since (1) is equivalent to the fact that T_1 and T_2 have the same Jordan model, the equivalence of (1) and (2) follows immediately.

3.3. LEMMA. *Let T be as before. If $K_1, K_2 \in Lat''T$, $K_1 \subseteq K_2$ and $T|K_1$ is quasi-similar to $T|K_2$, then $K_1 = K_2$.*

Proof. Since $\sigma(T|K_j) \subseteq \sigma(T)$ (cf. [3], Theorem 3), $T|K_j$ is also a C_{11} contraction for $j = 1, 2$ (cf. [6], Theorem VII. 6.3). Let Θ_j be the characteristic function of $T|K_j$ and $\Delta_j = (I - \Theta_j^* \Theta_j)^{\frac{1}{2}}$, $j = 1, 2$. The quasi-similarity of $T|K_1$ and $T|K_2$ implies that rank $\Delta_1(t) =$ rank $\Delta_2(t)$ a.e., by Lemma 3.2. On the other hand, since $K_1 \subseteq K_2$, we have $\Theta_2 = \Omega\Theta_1$ for some contractive analytic function Ω (cf. [6], Prop. VII. 2.4). Hence rank$\Delta_2(t) =$ rank $\Delta(t) +$ rank $\Delta_1(t)$ a.e., where $\Delta = (I - \Omega^*\Omega)^{\frac{1}{2}}$ (cf. [6], Prop. VII.3.3). It follows that rank$\Delta(t) = 0$ a.e., whence Ω is an inner function. It is easily seen that Ω is also outer. Hence Ω is a constant unitary function (cf. [6], Prop. V. 2.3) and we obtain $K_1 = K_2$.

It was shown in [13] that for K_1, K_2 in Hyperlat T, the preceding lemma holds even without the assumption $K_1 \subseteq K_2$. (However this assumption is essential here as may be seen from the discussion below.) It is instructive to contrast these results with the corresponding ones for normal operators with finite multiplicities.

The quasi-affinity X in the next lemma has been considered by Sickler [5] and also implicitly in [14].

3.4 LEMMA. *Let T be as before and let V be the operator of multiplication by e^{it} on the space $\overline{\Delta_* L_n^2}$, where $\Delta_* = (I - \Theta_T\Theta_T^*)^{\frac{1}{2}}$. Let $X: H \to \overline{\Delta_* L_n^2}$ be defined by $X(f \oplus g) = -\Delta_* f + \Theta_T g$ for $f \oplus g \in H$. Then X is a quasi-affinity intertwining T and V.*

Proof. For any $f \oplus g \in H$,

$$XT(f \oplus g) = XP(e^{it}f \oplus e^{it}g) = X((e^{it}f \oplus e^{it}g) - (\Theta_T w \oplus \Delta w)) =$$
$$= -\Delta_*(e^{it}f - \Theta_T w) + \Theta_T(e^{it}g - \Delta w) =$$
$$= e^{it}(-\Delta_* f + \Theta_T g) + (\Delta_*\Theta_T - \Theta_T\Delta)w =$$
$$= e^{it}X(f \oplus g) = VX(f \oplus g),$$

where $w \in H_n^2$ and we make use of the fact $\Delta_*\Theta_T = \Theta_T\Delta$. This shows that X intertwines T and V.

To show that X is a quasi-affinity, let $K = \{f \oplus g \in H: -\Delta_* f + \Theta_T g = 0\}$ and $L = \overline{XH}$. $K = \{0\}$ follows from Theorem 3.5 of [14]. On the other hand, since $T \prec V|L$ and $T \sim V$, we infer from Lemma 4.1 of [2] that V is unitarily equivalent to $V|L$. Note that L is a reducing subspace for V and hence $L \in \mathrm{Lat}''V$. As remarked above, for normal operators with finite multiplicities these imply that $L = \overline{\Delta_* L_n^2}$. Hence T is a quasi-affinity, as asserted.

As indicated above, for normal operators bi-invariant subspaces are exactly reducing subspaces, the structure of which is well-known. The following main theorem says that the inverse images under X of bi-invariant subspaces of V give all the elements in Lat$''T$.

3.5 THEOREM. *Let T be a c.n.u. C_{11} contraction with equal defect indices $n < \infty$ on $H \equiv [H_n^2 \oplus \overline{\Delta L_n^2}] \ominus \{\Theta_T w \oplus \Delta w : w \in H_n^2\}$. Let V be the operator of multiplication by e^{it} on $\overline{\Delta_* L_n^2}$ and X be the quasi-affinity from H to $\overline{\Delta_* L_n^2}$ defined by $X(f \oplus g) = -\Delta_* f + \Theta_T g$. Let $K \subseteq H$ be an invariant subspace for T with the corresponding regular factorization $\Theta_T = \Theta_2 \Theta_1$. Then the following are equivalent:*

(1) *K is bi-invariant for T;*

(2) *$T|K$ is of class C_{11};*

(3) *the intermediate space of the factorization $\Theta_T = \Theta_2\Theta_1$ is of dimension n;*

(4) *$K = X^{-1}(L)$ for some bi-invariant subspace $L \subseteq \overline{\Delta_* L_n^2}$ for V.*

Proof. (1) $\Rightarrow$ (2). This follows from the fact that $\sigma(T|K) \subseteq \sigma(TK)$ (cf. the first paragraph in the proof of Lemma 3.3).

(2) $\Leftrightarrow$ (3). This is an immediate consequence of Theorem VII.6.3 of [6].

(2) $\Rightarrow$ (4). Note that $\overline{XK}$ is invariant for V and $T|K \prec V|\overline{XK}$. Since $T|K$ is of class C_{11}, we infer that $T K$ is quasi-similar to the unitary operator $V|\overline{XK}$. Hence $\overline{XK}$ reduces V and $\overline{XK} \in \mathrm{Lat}''V$. Let $K_1 = X^{-1}(\overline{XK})$. Then $K \subseteq K_1$ and both are invariant for T.

We first show that $K_1 \in \mathrm{Lat}''T$. Let $S = P\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ be an operator in $\{T\}''$, where A, B and C satisfy $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + C\Delta = \Delta A_0$ for some bounded analytic function A_0. For $f \oplus g \in K_1$, consider $XS(f \oplus g) = -\Delta_* Af + \Theta_T(Bf + Cg)$. Since

$$-\Delta_* Af = -\Delta_* A\Theta_T\Theta_T^{-1}f = -\Delta_*\Theta_T A_0\Theta_T^{-1}f =$$

$$= -\Theta_T\Delta A_0\Theta_T^{-1}f = -\Theta_T(B\Theta_T + C\Delta)\Theta_T^{-1}f =$$

$$= -\Theta_T Bf - \Theta_T C\Delta\Theta_T^{-1}f,$$

we have

$$XS(f \oplus g) = (-\Theta_T Bf - \Theta_T C\Delta\Theta_T^{-1}f) + \Theta_T(Bf + Cg) =$$

$$= -\Theta_T C\Delta\Theta_T^{-1}f + \Theta_T Cg = \Theta_T C\Theta_T^{-1}(-\Delta_* f + \Theta_T g),$$

where we make use of the facts that $\Theta_T(t)^{-1}$ exists for almost all t and $\Delta_*\Theta_T = \Theta_T\Delta$. Note that V is unitarily equivalent to U, the operator of multiplication by e^{it} on $\overline{\Delta L_n^2}$. Let $Y : \overline{\Delta_* L_n^2} \to \overline{\Delta L_n^2}$ be the unitary transformation such that $YV = UY$. Consider Θ_T as a multiplication operator mapping $\overline{\Delta L_n^2}$ to $\overline{\Delta_* L_n^2}$. Since $Y\Theta_T \in \{U\}'$, we infer from Lemma 3.1 that $CY\Theta_T = Y\Theta_T C$. Hence

$$YXS(f \oplus g) = Y\Theta_T C\Theta_T^{-1}(-\Delta_* f + \Theta_T g) = CY(-\Delta_* f + \Theta_T g),$$

whence $XS(f \oplus g) = Y^{-1}CY(-\Delta_* f + \Theta_T g)$. Since $C \in \{U\}''$, we have $Y^{-1}CY \in \{V\}''$. Thus $XS(f \oplus g) \in Y^{-1}CY(\overline{XK}) \subseteq \overline{XK}$. This shows that $K_1 \in \mathrm{Lat}''T$.

Note that $X|K_1 : K_1 \to \overline{XK}$ is a quasi-affinity intertwining $T|K_1$ and $V|\overline{XK}$. Since $T|K_1$ is of class C_{11}, we infer as before that $T|K_1$ is quasi-similar to $V|\overline{XK}$ Hence $T|K_1$ is quasi-similar to $T|K$. It follows from Lemma 3.3 that $K = K_1 = X^{-1}(\overline{XK})$, completing the proof.

(4) $\Rightarrow$ (1). This is actually contained in the proof of (2) $\Rightarrow$ (4).

3.6 Corollary. *Let T, V and X be as in Theorem 3.5. Then $Lat''T \cong Lat''V$. Moreover the isomorphisms are implemented by the mappings $K \to \overline{XK}$ and $L \to X^{-1}(L)$ where $K \in Lat''T$ and $L \in Lat''V$. In this case, the C_{11} contraction $T|K$ is quasi-similar to the unitary operator $V|\overline{XK}$.*

Proof. From the proof of Theorem 3.5 we have $X^{-1}(\overline{XK}) = K$ for any $K \in \text{Lat}''T$. Hence to complete the proof it suffices to show that $\overline{XX^{-1}(L)} = L$ for $L \in \text{Lat}''V$. Obviously, $\overline{XX^{-1}(L)} \subseteq L$. Let $L_1 = L \ominus \overline{XX^{-1}(L)}$. We have $\overline{XX^{-1}(L_1)} \subseteq \overline{XX^{-1}(L)} \cap \cap L_1 = \{0\}$. It follows that $\overline{XX^{-1}(L_1)} = \{0\}$ and hence $X^{-1}(L_1) = \{0\}$. Note that by Theorem 3.5 $\overline{XX^{-1}(L)}$ is in Lat$''V$ whence L_1 is also in Lat$''V$.

Since for almost all t, $\Delta_*(t)$ is a self-adjoint operator on $\mathbf{C}^n$ bounded by 0 and 1, there exists an orthonormal base $\{\Psi_k(t)\}_1^n$ of $\mathbf{C}^n$ such that $\Delta_*(t)\,\Psi_k(t) = \delta_k(t)\Psi_k(t)$, $k = 1, 2, \ldots, n$, where the eigenvalues $\delta_k(t)$ are arranged in non-increasing order: $1 \geqslant \delta_1(t) \geqslant \delta_2(t) \geqslant \ldots \geqslant \delta_n(t) \geqslant 0$ (cf. [6], p. 272). Assume that $x = x_1\Psi_1 + \ldots \ldots + x_n\Psi_n$ is a non-zero element in L_1. For each positive integer m, let $F_m = \left\{t : x_j(t) \neq 0 \text{ and } \delta_j(t) \geqslant \frac{1}{m}\right\}$, where j is the largest integer such that $x_j \not\equiv 0$. Note that except for t in a null set $x_j(t) \neq 0$ implies that $\delta_j(t) \neq 0$. (This follows from the fact that the mapping $x \to (x_1, \ldots, x_n)$ furnishes a unitary transformation from $\overline{\Delta_* L^2_n}$ onto $L^2(E_1) \oplus \ldots \oplus L^2(E_n)$ which extends the densely defined mapping $\Delta_* v \to ((v, \Psi_1)\delta_1, \ldots, (v, \Psi_n)\,\delta_n)$; cf. [6], p. 272.) We conclude that $\chi_{F_m} x \not\equiv 0$ for some m. Let δ be a scalar multiple of Θ_T and let Ω be a contractive analytic function such that $\Omega\Theta_T = \Omega\Theta_T = \delta_I$. Consider the element $f \oplus g = P\left(0 \oplus \Delta\Omega\chi_{F_m}\left(\frac{x_1}{\delta_1}\Psi_1 + \ldots + \frac{x_j}{\delta_j}\Psi_j\right)\right)$ in H. Note that

$$\begin{aligned} X(f \oplus g) &= \Theta_T \Delta\Omega\chi_{F_m}\left(\frac{x_1}{\delta_1}\Psi_1 + \ldots + \frac{x_j}{j}\Psi_j\right) = \\ &= \Delta_*\delta\chi_{F_m}\left(\frac{x_1}{\delta_1}\Psi_1 + \ldots + \frac{x_j}{\delta_j}\Psi_j\right) = \\ &= \delta\chi_{F_m}(x_1\Psi_1 + \ldots + x_j\Psi_j) = \delta\chi_{F_m} x. \end{aligned}$$

Since $\delta\chi_{F_m} x \in L_1$, we have $f \oplus g \in X^{-1}(L_1) = \{0\}$. Thus $f \oplus g = 0$ and from the computations above $\delta\chi_{F_m} x = 0$, which is a contraction. Therefore $L_1 = \{0\}$ and hence $\overline{XX^{-1}(L)} = L$, completing the proof.

3.7 Corollary. *Let T_1, T_2 be c.n.u. C_{11} contractions with finite defect indices. If T_1 is quasi-similar to T_2, then $Lat''T_1 \cong Lat''T_2$.*

Proof. Since the quasi-similarity of T_1 and T_2 implies that they are quasi-similar to the same unitary operator, the conclusion now follows from Corollary 3.6.

The next theorem gives necessary and sufficient conditions that Lat $T =$ Lat$''T$ for the operators we considered.

3.8 THEOREM. *Let T be a c.n.u. C_{11} contraction with equal defect indices $n < \infty$. Then the following are equivalent:*

(1) *$Alg\ T = \{T\}''$;*

(2) *$Lat\ T = Lat''T$;*

(3) *for any $K \in Lat\ T$, $T|K$ is of class C_{11};*

(4) *the intermediate space of any regular factorization of Θ_T is of dimension n;*

(5) *$\Theta_T(t)$ is isometric for t in a set of positive Lebesgue measure.*

Proof. Note that (1) $\Rightarrow$ (2) is trivial and the equivalence of (2), (3) and (4) follows immediately from Theorem 3.5.

(3) $\Rightarrow$ (5). Assume the contrary, that is, $\Theta_T(t)$ is not isometric for almost all t. Then the Jordan model of T is of the form $M \equiv M_\Lambda \oplus M_{E_2} \oplus \ldots \oplus M_{E_k}$ acting on $L^2 \oplus L^2(E_2) \oplus \ldots \oplus L^2(E_k)$. Let V be the operator defined before. Since $L^2 \oplus \oplus 0 \oplus \ldots \oplus 0$ reduces M and M is unitarily equivalent to V, we infer from Corollary 3.6 that there exists a subspace $K \in$ Lat$''T$ such that $T|K$ is quasi-similar to M_Λ. Thus $T|K$ is a c.n.u. multiplicity-free C_{11} contraction with $\Lambda \setminus \{t : \Theta_{T|K}(t)$ not isometric$\}$ of Lebesgue measure zero. By [14], Corollary 4.2, we conclude that there exists a subspace $K_1 \in$ Lat $(T|K)$ such that $(T|K)|K_1 = T|K_1$ is not of class C_{11}. This contradicts (3).

(5) $\Rightarrow$ (4). Let $\Theta_T = \Theta_2\Theta_1$ be a regular factorization. Then we have rank $\Delta(t) =$ rank $\Delta_1(t) +$ rank $\Delta_2(t)$ a.e., where $\Delta = (I - \Theta_T^*\Theta_T)^{\frac{1}{2}}$ and $\Delta_j = (I - \Theta_j^*\Theta_j)^{\frac{1}{2}}$, $j = 1, 2$. Since rank $\Delta(t) = 0$ for t in a set of positive Lebesgue measure, say α, the same is true for rank $\Delta_1(t)$ and rank $\Delta_2(t)$. Thus $\Theta_1(t)$ and $\Theta_2(t)$ are isometric on α. It follows that the intermediate space of $\Theta_T = \Theta_2\Theta_1$ must be of dimension n.

(5) $\Rightarrow$ (1). Let S be an operator in $\{T\}''$. It suffices to show that Lat $T^{(n)} \subseteq \subseteq$ Lat $S^{(n)}$ for all $n \geqslant 1$, where for any operator L, $L^{(n)}$ denotes the direct sum of n copies of L. Note that $\Theta_{T^{(n)}} = \Theta_T(t)^{(n)}$ is isometric if and only if $\Theta_T(t)$ is. Since $T^{(n)}$ is also a c.n.u. C_{11} contraction with finite defect indices, (5) implies that Lat $T^{(n)} = =$ Lat$''T^{(n)}$. Thus Lat $T^{(n)} \subseteq$ Lat $S^{(n)}$ follows from the observation that $S^{(n)} \in \{T^{(n)}\}''$.

In the remainder of this section we consider the splitting property of $\{T_1 \oplus T_2\}'$, Alg $(T_1 \oplus T_2)$ and Lat $(T_1 \oplus T_2)$ when T_1, T_2 are the type of operators we considered above (cf. [1] for general T_1, T_2).

3.9 LEMMA. *For $j = 1, 2$, let T_j be a c.n.u. C_{11} contraction with finite defect indices and let $E_j = \{t : \Theta_{T_j}(t)$ not isometric$\}$. Then the following are equivalent to each other:*

(1) $\{T_1 \oplus T_2\}' = \{T_1\}' \oplus \{T_2\}'$;

(2) *$E_1 \cap E_2$ has Lebesgue measure zero.*

Proof. Let M_1, M_2 denote the Jordan models of T_1, T_2, respectively. Then (1) holds if and only if $\{M_1 \oplus M_2\}' = \{M_1\}' \oplus \{M_2\}'$ (cf. [1], Lemma 4.1). However the latter condition is equivalent to (2), by Theorem 3 of [2].

Note that the preceding lemma also follows from Prop. 4.2 of [1].

3.10 THEOREM. *For $j = 1, 2$, let T_j and E_j be as in Lemma 3.9. Then the following are equivalent:*

(1) *$Alg\ (T_1 \oplus T_2) = Alg\ T_1 \oplus Alg\ T_2$;*

(2) *$Lat\ (T_1 \oplus T_2) = Lat\ T_1 \oplus Lat\ T_2$;*

(3) *$E_1 \cap E_2$ has Lebesgue measure zero and $\Lambda \setminus (E_1 \cup E_2)$ has positive Lebesgue measure.*

Proof. (1) $\Rightarrow$ (2). This is proved in Prop. 1.3 of [1].

(2) $\Rightarrow$ (3). The first assertion of (3) follows from the fact that (2) implies $\{T_1 \oplus T_2\}' = \{T_1\}' \oplus \{T_2\}'$ (cf. [1], Prop. 1.3) and Lemma 3.9. Assume that $\Lambda \setminus (E_1 \cup E_2)$ has Lebesgue measure zero. Since $\Theta_{T_1 \oplus T_2}(t)$ is isometric if and only if both $\Theta_{T_1}(t)$ and $\Theta_{T_2}(t)$ are, we infer from Theorem 3.8 that $\text{Lat}\,(T_1 \oplus T_2) \neq \text{Lat}''(T_1 \oplus T_2)$. On the other hand, from the assumption we deduce that both $\Theta_{T_1}(t)$ and $\Theta_{T_2}(t)$ are isometric for t in sets of positive Lebesgue measure. Hence $\text{Lat}\ T_1 \oplus \text{Lat}\ T_2 = \text{Lat}''T_1 \oplus \text{Lat}''T_2 = \text{Lat}''(T_1 \oplus T_2)$, by Theorem 3.8 and [1], Prop. 1.3. This shows that $\text{Lat}\,(T_1 \oplus T_2) \neq \text{Lat}\ T_1 \oplus \text{Lat}\ T_2$, contradicting (2).

(3) $\Rightarrow$ (1). (3) implies that $\Theta_{T_1}(t)$, $\Theta_{T_2}(t)$ and $\Theta_{T_1 \oplus T_2}(t)$ are isometric for t in sets of positive Lebesgue measure. Thus $\text{Alg}\,(T_1 \oplus T_2) = \{T_1 \oplus T_2\}'' = \{T_1\}'' \oplus \{T_2\}'' = \text{Alg}\ T_1 \oplus \text{Alg}\ T_2$, by Theorem 3.8 and Lemma 3.9.

4. WEAK CONTRACTIONS

In this section we extend some of the results in Section 3 for C_{11} contractions to weak contractions.

The next theorem describes the elements of $\text{Lat}''T$ for weak contractions

4.1 THEOREM. *Let T be a c.n.u. weak contraction with equal defect indices $n < \infty$ on H, and let H_0, H_1 be invariant subspaces for T such that $T_0 = T|H_0$ and $T_1 = T|H_1$ are the C_0 and C_{11} parts of T, respectively. Let $K \subseteq H$ be an invariant subspace for T with the corresponding regular factorization $\Theta_T = \Theta_2\Theta_1$. Then the following are equivalent:*

(1) *K is bi-invariant for T;*

(2) *$T|K$ is a weak contraction;*

(3) *the intermediate space of the factorization $\Theta_T = \Theta_2\Theta_1$ is of dimension* n;

(4) *$K = K_0 \vee K_1$, where $K_0 \subseteq H_0$, $K_1 \subseteq H_1$ are bi-invariant subspaces for T_0, T_1. respectively.*

Proof. (1) $\Rightarrow$ (2). For bi-invariant K, we have $\sigma(T|K) \subseteq \sigma(T)$. (2) follows immediately.

(2) $\Rightarrow$ (3). If $T|K$ is a weak contraction then it has equal defect indices, which implies (3).

(3) $\Rightarrow$ (2). $I-(T|K)^*(T|K)$ certainly has finite rank. Hence to complete the proof we have only to show that $\sigma(T|K) \neq \bar{D}$, where D denotes the open unit disc. Indeed, T is a weak contraction implies that $\sigma(T) \neq \bar{D}$. Therefore det $\Theta_T(\lambda_0) \neq 0$ for some $\lambda_0 \in D$ (cf. [6], Theorem VI. 4.1). Hence det $\Theta_1(\lambda_0) \neq 0$. Using Theorem VI. 4. 1 of [6] again, we conclude that $\lambda_0 \notin \sigma(T|K)$, whence $\sigma(T|K) \neq \bar{D}$.

(2) $\Rightarrow$ (4). Let K_0, K_1 be subspaces of K such that $T|K_0$ and $T|K_1$ are the C_0 and C_{11} parts of $T|K$, respectively. Then we have $K = K_0 \vee K_1$ and $K_0 \subseteq H_0$, $K_1 \subseteq H_1$ (cf. [6], Theorem VIII. 2.1 and Prop. VIII. 2.2). K_0, being invariant for the $C_0(N)$ contraction T_0, is bi-invariant (cf. [9], Theorem 3.1). On the other hand, since $T|K_1$ is of class C_{11}, we conclude from Theorem 3.5 that K_1 is bi-invariant for T_1. This proves (4).

(4) $\Rightarrow$ (1). Let S be an operator in $\{T\}''$. Since H_0 and H_1 are hyperinvariant for T, they are invariant under S. Let $S_0 = S|H_0$ and $S_1 = S|H_1$. We claim that $S_0 \in \{T_0\}''$. Indeed, it was proved in Theorem 3.1 of [15] that $H_0 = \overline{WH}$ for some $W \in \{T\}''$. For any V in $\{T_0\}'$, consider VW as an operator on H. It is easily seen that $VW \in \{T\}'$. Hence $SVW = VWS = VSW$. This shows that $S_0V = VS_0$ on H_0. Hence $S_0 \in \{T_0\}''$ as asserted and we have $S_0K_0 \subseteq K_0$. In a similar fashion we can show that $S_1K_1 \subseteq K_1$. Thus $SK \subseteq K$ for any $S \in \{T\}''$ and K is bi-invariant for T.

4.2 Corollary. *Let T, T_0 and T_1 be as in Theorem 4.1. Then the following lattices are isomorphic: Lat''T, Lat''$T_0 \oplus$ Lat''T_1 and Lat''$(T_0 \oplus T_1)$.*

Proof. Since T_0 and T_1 are of class C_{00} and of class C_{11}, respectively, Lat''$T_0 \oplus$ $\oplus$ Lat''$T_1 \cong$ Lat''$(T_0 \oplus T_1)$ follows easily from Prop. 1.3 and Lemma 4.4 of [1]. Lat''$T \cong$ Lat''$T_0 \oplus$ Lat''T_1 follows from Theorem 4.1 and [15], Lemma 3.2.

At this connection we should point out that whether two quasi-similar c.n.u. weak contractions with finite defect indices have isomorphic bi-invariant subspace lattices is still unknown. The difficulty lies in that we don't know whether this holds for $C_0(N)$ contractions. (However the corresponding result for hyperinvariant subspace lattice is true; cf. [15], Corollary 3.4.)

The next result generalizes Lemma 3.3.

4.3 Corollary. *Let T be as in Theorem 4.1. If $K_1, K_2 \in$ Lat''T, $K_1 \subseteq K_2$ and $T|K_1$ is quasi-similar to $T|K_2$, then $K_1 = K_2$.*

Proof. A straightforward argument, using the $C_0 - C_{11}$ decompositions of $T|K_1$ and $T|K_2$ and Corollary 1 of [11], reduces the assertion to those of their C_0 and C_{11} parts. The latter follow from Corollary 2 of [7] (for $C_0(N)$ contractions) and Lemma 3.3 (for C_{11} contractions).

The next theorem generalizes Theorem 3.8.

4.4 THEOREM. *Let T, T_0 and T_1 be as in Theorem* 4.1. *Then the following conditions are equivalent:*

(1) *$Alg\ T = \{T\}''$;*

(2) *$Lat\ T = Lat''T$;*

(3) *$Alg\ T_1 = \{T_1\}''$;*

(4) *$Lat\ T_1 = Lat''T_1$;*

(5) *for any $K \in Lat\ T$, $T|K$ is a weak contraction;*

(6) *the intermediate space of any regular factorization of Θ_T is of dimension n;*

(7) *$\Theta_T(t)$ is isometric for t in a set of positive Lebesgue measure.*

Proof. The equivalence of (2), (5) and (6) follows immediately from Theorem 4.1. That (3) and (4) are equivalent to (7) follows from Theorem 3.8 and the fact that $\Theta_T(t)$ is isometric if and only if $\Theta_{T_1}(t)$ is (since Θ_{T_1} coincides with the purely contractive part of the outer factor of Θ_T). Also note that (1) $\Rightarrow$ (2) is trivial and (7) $\Rightarrow$ (1) can be proved along the same line of arguments as in the corresponding implication in Theorem 3.8. Thus to complete the proof we have only to show that (5) $\Rightarrow$ (7).

Assume that (7) does not hold, that is, $\Theta_T(t)$ is not isometric for almost all t. By the preceding remark, the same is true for $\Theta_{T_1}(t)$. We infer from Theorem 3.8 that there exists a subspace $K_1 \in$ Lat T_1 such that $T|K_1$ is not of class C_{11}. Certainly $T|K_1$ cannot be a weak contraction (cf. [6], Theorem VII. 6. 3). This contradicts (5) and completes the proof.

The implication (7) $\Rightarrow$ (5) of the preceding theorem is proved in Prop. VIII. 2. 3 of [6] for weak contractions with defect indices not necessarily finite.

4.5 THEOREM. *Let T, T_0 and T_1 be as in Theorem* 4.1. *Then the following conditions are equivalent:*

(1) *T is multiplicity-free;*

(2) *$\{T\}'' = \{T\}'$;*

(3) *$Lat''\ T = Hyperlat\ T$;*

(4) *$\{T_0\}'' = \{T_0\}'$ and $\{T_1\}'' = \{T_1\}'$;*

(5) *$Lat''T_0 = Hyperlat\ T_0$ and $Lat''T_1 = Hyperlat\ T_1$.*

For other equivalent conditions for a weak contraction being multiplicity-free, compare Theorem 5 of [12].

Proof. The equivalence of (1), (2) and (4) is established in Theorem 5 of [12 ; (2) $\Rightarrow$ (3) is trivial.

(5) $\Rightarrow$ (4). Since Lat $T_0 =$ Lat$''T_0$ for $C_0(N)$ contractions (cf. [9], Theorem 3.1), Lat$''T_0 =$ Hyperlat T_0 implies that Lat $T_0 =$ Hyperlat T_0. By Corollary 4.4 of [9] we have $\{T_0\}'' = \{T_0\}'$. As for T_1, let V and X be defined as in Theorem 3.5. Note that the mapping $K \to \overline{XK}$ implements both the isomorphism from Lat$''T_1$ to Lat$''V$ and the one from Hyperlat T_1 to Hyperlat V (cf. [13], Corollary 1). Hence Lat$''T_1 =$ = Hyperlat T_1 implies that Lat$''V =$ Hyperlat V. For normal operators this is

equivalent to V being cyclic. Therefore, T_1 is also cyclic and satisfies $\{T_1\}'' = \{T_1\}'$ (cf. [12], Theorem 5).

(3) $\Rightarrow$ (5). By Theorem 4.1 we have $\mathrm{Lat}'' T_0 \subseteq \mathrm{Lat}'' T = \mathrm{Hyperlat}\ T$. Now using the structure of hyperinvariant subspaces of T (cf. [15], Theorem 3.3), we deduce that any subspace $K_0 \subseteq H_0$ which is hyperinvariant for T must be hyperinvariant for T_0. It follows that $\mathrm{Lat}'' T_0 = \mathrm{Hyperlat}\ T_0$. The same argument applies to T_1.

Combining Theorems 4.4 and 4.5 we have

4.6. COROLLARY. *Let T, T_0 and T_1 be as in Theorem* 4.1. *Then the following conditions are equivalent:*

(1) *$\mathrm{Alg}\ T = \{T\}'$;*

(2) *$\mathrm{Lat}\ T = \mathrm{Hyperlat}\ T$;*

(3) *$\mathrm{Alg}\ T_0 = \{T_0\}'$ and $\mathrm{Alg}\ T_1 = \{T_1\}'$;*

(4) *$\mathrm{Lat}\ T_0 = \mathrm{Hyperlat}\ T_0$ and $\mathrm{Lat}\ T_1 = \mathrm{Hyperlat}\ T_1$;*

(5) *T is multiplicity-free and $\Theta_T(t)$ is isometric for t in a set of positive Lebesgue measure.*

The preceding corollary generalizes Corollary 3.6 of [15] and the main result in [10].

For the splitting property for weak contractions, we have the following generalization of Lemma 3.9. Note that this also generalizes Theorem 4.6 of [1].

4.7 THEOREM. *For $j = 1, 2$, let T_j be a c.n.u. weak contraction with finite defect indices and let T_{j0} and T_{j1} denote its C_0 and C_{11} parts. Let $E_j = \{t : \Theta_{T_j}(t)$ not isometric$\}$ and let φ_j be the minimal function of T_{j0}. Then the following are equivalent:*

(1) *$\{T_1 \oplus T_2\}' = \{T_1\}' \oplus \{T_2\}'$;*

(2) *$\{T_{10} \oplus T_{20}\}' = \{T_{10}\}' \oplus \{T_{20}\}'$ and $\{T_{11} \oplus T_{21}\}' = \{T_{11}\}' \oplus \{T_{21}\}'$;*

(3) *$E_1 \cap E_2$ has Lebesgue measure zero and $\varphi_1 \wedge \varphi_2 = 1$, that is, φ_1, φ_2 have no common nontrivial inner divisor.*

Proof. The equivalence of (2) and (3) follows from Lemma 3.9 and [1], Theorem 3.1. The equivalence of (1) and (2) is an easy consequence of Lemmas 4.3, 4.4 and Prop. 4.5 of [1].

Conditions guaranteeing the splitting of $\mathrm{Alg}(T_1 \oplus T_2)$ and $\mathrm{Lat}(T_1 \oplus T_2)$ for weak contractions will be given in [17], Theorem 4.

Added in proof. The question posed after Corollary 4.2 has been solved positively, that is, quasi-similar $C_0(N)$ contractions have isomorphic lattices of (bi-) invariant subspaces. Hence the same holds for c.n.u. weak contractions with finite defect indices (cf. [16], Theorem 3).

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