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TRANSITIVE ALGEBRA PROBLEM AND LOCAL RESOLVENT TECHNIQUES

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0. INTRODUCTION

Throughout the paper by an operator we mean a bounded linear transformation acting in a general Banach space $\mathscr{X}$. When refering to the terms "normal" or "adjoint" it is understood that the underlying space is a Hilbert space. The algebra of all operators on $\mathscr{X}$ is denoted by $\mathscr{B}(\mathscr{X})$. By a subspace of $\mathscr{X}$ we always mean a closed linear manifold in $\mathscr{X}$. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{X})$ is called *transitive* if the only invariant subspaces of $\mathscr{A}$ are $\{0\}$ and $\mathscr{X}$.

The results of this paper are contained in the next two sections. In the first section we generalize some results due to Foiaş [7], Lomonosov [13], Fong-Nordgren-Radjabalipour-Radjavi-Rosenthal [9] and Jafarian [12] as follows: If $\mathscr{A}$ is a uniformly closed algebra of operators, if K and L are two operators such that $\mathscr{A}K \subset L\mathscr{A}$, $\sigma(K)\neq\{0\}$, and $0\in\sigma(L)$, and if K (respectively L) is in the following class I (resp. class II), then $\mathscr{A}$ is not transitive.

Class I: (a) Decomposable operators.
(b) Adjoints of subdecomposable operators.
(c) Adjoints of M-hyponormal operators.

Class II: (α) Subdecomposable operators.
(β) Hyponormal operators.

(See below for the definitions.)

In [9], it is shown that if $\mathscr{A}K \subset L\mathscr{A}$ for a compact operator K and a *certain* operator L, then $\mathscr{A}$ is not transitive. Their proof is based on Lomonosov's theorem [13], [24, pages 156–159] and cannot be regarded as a local resolvent technique, while our proof is heavily based on the properties and existence of certain local resolvents. Even the techniques used in [7] and [12] are not applicable here, though their results can be obtained as corollaries of ours. We would like to mention that our results are *essentially* disjoint from those of [9] and cannot be obtained as corollaries of each other. However, we acknowledge some motivations from [9] and [12].

In the course of our study we obtained some results of possibly independent interest on local resolvents. These extend some results due to Clancey [2], Putnam [16; 17], Radjabalipour [20; 21] and Stampfli-Wadhwa [26; 27] which are gathered in the last section of the paper.

Now we define the operators mentioned in the classes I and II.

DEFINITION 1. Let M be a positive number. A Hilbert space operator T is called *M-hyponormal* if

$$(T-z)(T-z)^* \leqslant M(T-z)^*(T-z) \text{ for all } z \in \mathbf{C}.$$

Note that the class of 1-hyponormal operators is the same as the class of hyponormal operators. Recall that every subnormal operator is hyponormal. The results that we will prove for M-hyponormal operators are new even for normal operators. The special case in which K and L are equal normal operators is discussed in [12].

Now we are going to define the class of decomposable operators. First we need some notations and terminology. For an arbitrary operator $T \in \mathscr{B}(\mathscr{X})$ and an arbitrary closed subset F of $\mathbf{C}$ we let

$$X_T(F) = \{x \in \mathscr{X} : (T-z) f_x(z) \equiv x \text{ for some analytic function } f_x : \mathbf{C} \setminus F \to \mathscr{X}\}.,$$

It is easy to see that $X_T(F)$ is an invariant linear manifold for T. Note that if $z \notin \sigma(T)$ then $f_x(z) = (T-z)^{-1}x$.

Dunford's condition (C). An operator T is said to satisfy condition (C) if $X_T(F)$ is closed for all closed sets F.

Single-valued extension property (s.v.e.p.). An operator $T \in \mathscr{B}(\mathscr{X})$ is said to have the s.v.e.p. if there exists no nonzero analytic $\mathscr{X}$-valued function f such that $(T-z)f(z) \equiv 0$.

Any operator satisfying condition (C) has the s.v.e.p. [22, Theorem 2.13].

Local spectrum and local resolvent. For an operator T with the s.v.e.p. and for $x \in \mathscr{X}$ we define the *local spectrum* $\sigma_T(x)$ of x by

$$\sigma_T(x) = \cap \{F : F \text{ closed}, \ x \in X_T(F)\}.$$

The complement of $\sigma_T(x)$ is called the *local resolvent set* of x and is denoted by $\rho_T(x)$. It is easy to see that there exists a unique analytic function $R(z; x)$: $\rho_T(x) \to \mathscr{X}$ such that

$$(T-z)\, R(z; x) \equiv x.$$

The function $R(z; x)$ is called the *local resolvent* of x. It follows that if T has the s.v.e.p. then

$$X_T(F) = \{x \in \mathscr{X} : \sigma_T(x) \subset F\}$$

DEFINITION 2. An operator $T \in \mathscr{B}(\mathscr{X})$ is called decomposable if T satisfies condition (C) and

$$\sigma(T^F) \subset \mathbf{C} \setminus F^{\circ} \tag{1}$$

for all closed sets F, where T^F is the operator induced by T on $\mathscr{X}/X_T(F)$ and F° denotes the interior of F.

The equivalence of this definition with the original one given by Foiaş [6] is discussed in [22], and the present form is more suitable to our work. We refer the reader to [3] and [22] for the properties of decomposable operators; in particular, we mention that every decomposable operator T has the s.v.e.p. and

$$\sigma(T_F) \subset \sigma(T) \cap F, \tag{2}$$

for all closed sets F, where $T_F = T \mid X_T(F)$.

As usual, the restriction of a decomposable operator to an invariant subspace is called a subdecomposable operator.

1. TRANSITIVE ALGEBRAS

Now we are ready to prove the key theorem of this section. As in [9], instead of uniformly closed algebras we consider algebras which are operator ranges (A linear manifold is called an operator range if it is the range of a bounded linear transformation between two Banach spaces.)

In the rest of the paper we fix the following notation:

$$D_r = \{z \in \mathbf{C} : |z| < r\}, \quad (r > 0).$$

THEOREM 1. *Let $\mathscr{A}$ be an algebra of operators on $\mathscr{X}$ and assume that $\mathscr{A}$ is an operator range. Suppose there exist operators K and L such that $\mathscr{A}K \subset L\mathscr{A}$. Then there exists a number $a > 0$ such that*

$$\mathscr{A}X_K(\mathbf{C} \setminus D_r) \subset X_L(\mathbf{C} \setminus D_{r/a}),$$

for all $r > 0$. In particular, if $X_K(\mathbf{C} \setminus D_r) \neq \{0\}$ and $X_L(\mathbf{C} \setminus D_{r/a})$ is not dense in $\mathscr{X}$ for some $r > 0$, then $\mathscr{A}$ is not transitive. (Here $\mathscr{A}\mathfrak{M}$ denotes the set $\{Am : m \in \mathfrak{M}, A \in \mathscr{A}\}$, where $\mathfrak{M} \subset \mathscr{X}$.)

Proof. Let V be a bounded linear transformation defined on a Banach space $\mathscr{Y}$ such that $\mathscr{A} = V\mathscr{Y}$. Let $\mathscr{C} = \{y \in \mathscr{Y} : LV(y) = 0\}$ and define $S : \mathscr{Y} \to \mathscr{B}(\mathscr{X})$ and $T : \mathscr{Y}/\mathscr{C} \to \mathscr{B}(\mathscr{X})$ by $S(y) = V(y)K$ and $T(\hat{y}) = LV(y)$, where $\hat{y}$ is the equivalence class containing y. Obviously S and T are well defined bounded linear transformations and $S(\mathscr{Y}) \subset T(\mathscr{Y}/\mathscr{C})$. Moreover, T is injective. By Theorem 1 of [4] (extended to Banach spaces) there exists a bounded linear transformation $\varphi : \mathscr{Y} \to \mathscr{Y}/\mathscr{C}$ such that $S = T \circ \varphi$. Let $A = V(y)$ be an arbitrary operator in $\mathscr{A}$ and let $b > 1$ be fixed. Choose $y_0 \in \varphi(y)$ such that $\|y_0\| \leqslant b\|\varphi\| \|y\|$. Assume $y_0, y_1, \ldots, y_{n-1}$ are chosen, find $y_n \in \varphi(y_{n-1})$ such that $\|y_n\| \leqslant b\|\varphi\| \|y_{n-1}\|$. It is easy to see that $\|y_n\| \leqslant b^{n+1} \|\varphi\|^{n+1}\|y\|$.

Let $x \in X_K(\mathbf{C} \setminus D_r)$ and let $f: D_r \to \mathscr{X}$ be an analytic function such that $(K - z)f(z) \equiv x$. By induction, one can easily show that $(K - z)f^{(n)}(z) \equiv nf^{(n-1)}(z)$ for $n = 1, 2, \ldots$. (Here $f^{(n)} = \mathrm{d}^n f/\mathrm{d}z^n$.) Define

$$g(z) = \sum_{n=0}^{\infty} (z^n/n!)\, V(y_n) f^{(n)}(0)$$

for $|z| < r/(b\|\varphi\|)$. The function g is analytic and

$$(L - z)g(z) \equiv \sum_{n=0}^{\infty} (z^n/n!)\, LV(y_n) f^{(n)}(0) - \sum_{n=0}^{\infty} (z^{n+1}/n!)\, V(y_n) f^{(n)}(0) =$$

$$= LV(y_0) f(0) = V(y)\, Kf(0) = Ax.$$

This shows that $Ax \in X_L(\mathbf{C} \setminus D_{r/a})$, where $a = b\|\varphi\|$. That is $\mathscr{A}X_K\, (\mathbf{C} \setminus D_r) \subset \subset X_L(\mathbf{C} \setminus D_{r/a})$.

Now assume $X_K(\mathbf{C} \setminus D_r) \neq \{0\}$ and $X_L(\mathbf{C} \setminus D_{r/a})$ is not dense in $\mathscr{X}$. If $\mathscr{A}\, X_K (\mathbf{C} \setminus D_r) = \{0\}$, then the null space of every operator in $\mathscr{A}$ contains $X_K(\mathbf{C} \setminus D_r)$ and hence $\mathscr{A}$ cannot be transitive. If $\mathscr{A} X_K(\mathbf{C} \setminus D_r) \neq \{0\}$, then its closed span is a non-zero invariant subspace of $\mathscr{A}$ contained in the closure of $X_L(\mathbf{C} \setminus D_{r/a})$. This completes the proof of the theorem.

REMARK 1. In the proof of Theorem 1, if the operator L has the s.v.e.p., then

$$\mathscr{A}X_K(\mathbf{C} \setminus D_r) \subset \bigcap_{b>1} X_L(\mathbf{C} \setminus D_{r/b\|\varphi\|}) = X_L(\mathbf{C} \setminus D_{r/\|\varphi\|}).$$

Now we prove the main result of this section.

THEOREM 2. *Let $\mathscr{A}$ be an algebra of operators on $\mathscr{X}$ and assume that $\mathscr{A}$ is an operator range. Let K and L be operators such that $\mathscr{A}K \subset L\mathscr{A}$, $\sigma(K) \neq \{0\}$, and $0 \in \sigma(L)$. Moreover, assume that K(respectively L) is in class* I *(resp. class* II*). Then $\mathscr{A}$ is not transitive.*

Proof. In view of Theorem 1, it is enough to show that $X_K(\mathbf{C} \setminus D_r) \neq \{0\}$ for some $r > 0$ and $X_L(\mathbf{C} \setminus D_s)$ is not dense in $\mathscr{X}$ for all $s > 0$. To do this we consider different cases of K and L.

(a) If K is decomposable, then, in view of (1), $X_K(\mathbf{C} \setminus D_r) \neq \{0\}$ for every r less than the spectral radius of K.

(b) Let K be the adjoint of a subdecomposable operator (on a Hilbert space $\mathscr{X}$) and let N be the adjoint of a decomposable operator on a Hilbert space $\mathscr{H}$ containing $\mathscr{X}$ such that

$$N = \begin{bmatrix} A & B \\ 0 & K \end{bmatrix} \begin{matrix} \mathscr{X}^{\perp} \\ \mathscr{X}. \end{matrix}$$

(Note that N is also decomposable [10].) Let Q be the orthogonal projection from $\mathscr{H}$ onto $\mathscr{X}$. By [10, Lemma2] $QX_N(\mathbf{C}\setminus D_r) \subset X_K(\mathbf{C}\setminus D_r)$ for all $r > 0$. Assume, if possible, that $QX_N(\mathbf{C}\setminus D_r) = \{0\}$ for all $r > 0$. Then $\mathscr{X} \subset [X_N(\mathbf{C}\setminus D_r)]^{\perp}$ and hence it follows from (1) that $\sigma(K)$ is in the closure of D_r for all $r > 0$. This means that $\sigma(K) = \{0\}$, a contradiction. Therefore, if K is the adjoint of a subdecomposable operator, then $X_K(\mathbf{C}\setminus D_r) \neq \{0\}$ for some $r > 0$.

(c) Next let K be the adjoint of an M-hyponormal operator and assume that K is not normal (otherwise K is decomposable and we are done by the case (a)). By [21, Theorems 1 and 2] there exists a non-zero vector x and a bounded function $f\colon \mathbf{C} \to \mathscr{X}$ such that $(K - z)f(z) \equiv x$. If K has an eigenvalue $\lambda_0 \neq 0$, then $X_K(\mathbf{C}\setminus D_r) \neq \{0\}$ for all $r \leqslant |\lambda_0|$. If K has no non-zero eigenvalue, then $f(z)$ is weakly continuous for $z \neq 0$ [27, proof of Lemma 1]. Given an arbitrary Jordan curve Γ having 0 in its exterior, we write $u = \oint_\Gamma f(\lambda)\, d\lambda$ and observe that, for z in the exterior of Γ, the function

$$g(z) = \oint_\Gamma (\lambda - z)^{-1} f(\lambda)\, d\lambda$$

is analytic and $(K - z)g(z) \equiv u$. (see Stampfli [25, page 289] for details.) Assume, if possible, that $u = 0$ for all such Jordan curves Γ. It follows from Morera's Theorem that f is analytic everywhere except possibly at 0. Since f is bounded in a neighborhood of 0, it follows that f can be redefined at 0 (if necessary) to become analytic everywhere and still $(K - z)f(z) \equiv x$ for all $z \in \mathbf{C}$. Thus $x = 0$, a contradiction. Therefore, $u \neq 0$ for some curve Γ_0 and hence $X_K(\mathbf{C}\setminus D_r) \neq \{0\}$ whenever $D_r \cap \Gamma_0 = \varnothing$.

Now we turn to L.

(α) If L is subdecomposable, then $X_L(F)$ is closed for all closed sets F [19, proof of Lemma 1] and thus $\sigma(L|X_L(F)) \subset F \cap \sigma(L)$, [3, page 23] (or [22, Theorems 2.10 and 2.13]). Since $0 \in \sigma(L)$, it follows that $X_L(\mathbf{C}\setminus D_s) \neq \mathscr{X}$ for all $s > 0$.

(β) If L is hyponormal, then, again, $X_L(F)$ is closed for all closed sets F [20, Proposition 1]. Therefore by the argument given in (α), $X_L(\mathbf{C}\setminus D_s) \neq \mathscr{X}$ for all $s > 0$.

REMARK 2. An operator S is called a quasiaffine transform of an operator T if there exists an injective operator W with dense range such that $WS = TW$. It is easy to see that $WX_S(F) \subset X_T(F)$ for all closed sets F. Therefore, if K is a quasiaffine transform of K_1 and if L_1 is a quasiaffine transform of L, then Theorem 2 remains true with K and L replaced by K_1 and L_1, respectively.

REMARK 3. Two operators are quasisimilar if they are quasiaffine transforms of each other. Here we give an example of an operator which is quasisimilar to a normal operator, but is in neither of classes I or II (cf. the assertion at the end of Remark 2). Let B be an operator that is quasisimilar to a unitary operator U with $\sigma(U) = \{z : |z| = 1\}$ and $\sigma(B) = \{z : |z| \leqslant 1\}$. Such an example is given on page

262 of [28]. The operator $B \dotplus I$ is not decomposable because $\sigma(B \dotplus I) \neq \sigma(U \dotplus I$- (see [3, page 55]). In view of [22, Corollary 7.3] B $\dotplus$ I cannot be even a subdecomposable operator. Also ([21, Theorem 3]) $B \dotplus I$ is not the adjoint of an M-hyponormal operator. So $B + I$ is not in class I. Since every operator T in class II satisfies condition (C), it follows, again, from [22, Corollary 7.3] that $B + I$ is not in class II. The operator $B \dotplus I$ is quasisimilar to the normal operator $U \dotplus I$ and $U \dotplus I$ is in both classes I and II.

REMARK 4. A class of operators is introduced [11] which is called the class of quasi-decomposable operators. The class of decomposable operators is contained properly in the class of quasi-decomposable operators [1]. By [11, Section 3] every quasi-decomposable operator can be used as K or L in Theorem 2.

The following theorem is an analogue of Theorem 1 in which the algebra is replaced by a single operator.

THEOREM 3. *Let A be an operator such that* $\limsup_{n\to\infty} \|L^{-n} AK^n\|^{\frac{1}{n}} = a < \infty$ *for some injective operator L and some operator K. Then* $AX_K(\mathbf{C} \setminus D_r) \subset X_L(\mathbf{C} \setminus D_{r/a})$ *for every* $r > 0$. *In particular, if* $a \leqslant 1$ *and* $K = L$ *then* $X_K(\mathbf{C} \setminus D_r)$ *is an invariant linear manifold of* A.

Proof. Let $x \in X_K(\mathbf{C} \setminus D_r)$ and let $f : D_r \to \mathscr{X}$ be an analytic function such that $(K - z)f(z) \equiv x$. Let

$$g(z) = \sum_{n=0}^{\infty} (z^n/n!)\, L^{-n-1}\, A\, K^{n+1}\, f^{(n)}(0)$$

for $|z| < r/a$. By an argument similar to the one given in the proof of Theorem 1, one can observe that g is analytic and $(L - z)g(z) \equiv Ax$. The rest of the proof is trivial.

EXAMPLE 1. Let $\mathscr{H}$ be a Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots\}$. Define $K e_m = b^m e_m$ and $S e_m = e_{m+1}$, where $0 < b \leqslant 1$ is an arbitrary positive number. It is easy to see that $K^{-n}SK^n = b^{-n}S$ and $K^{-n}S^*K^n = b^nS^*$. Therefore, $\lim \|K^{-n}SK^n\|^{\frac{1}{n}} = b^{-1} \geqslant 1$, and $\lim \|K^{-n} S^* K^n\|^{\frac{1}{n}} = b \leqslant 1$. So, in Theorem 3, all positive numbers a are possible. However, if J is an invertible operator and A is a nonquasinilpotent operator, then $\|J^{-n}AJ^n\|$ is not less than the spectral radius of A and hence $\limsup \|J^{-n}AJ^n\|^{\frac{1}{n}} \geqslant 1$.

Note. In case that K and L are two equal positive operators sharper versions of Theorem 3 are obtained in [23].

The following two corollaries can be proved in the same way that Corollaries 1 and 2 of [9] are proved except at the end, where instead of Theorem 4 of [9] we apply our Theorems 1 and 2.

COROLLARY 1. *Let A be an operator for which there exist a bounded open set $\mathscr{D}$ containing $\sigma(A)$, an analytic function φ taking $\mathscr{D}$ into $\mathscr{D}$ and a decomposable operator*

K such that $AK = K\varphi(A)$ *and* $0 \in \sigma(K) \neq \{0\}$. *Then A has a nontrivial invariant subspace.*

COROLLARY 2. *If A is a power bounded operator and if there exists an integer k and a decomposable operator K such that* $AK = KA^k$ *and* $0 \in \sigma(K) \neq \{0\}$, *then A has a nontrivial invariant subspace.*

REMARK 5. In view of Remark 4, Corollaries 1 and 2 remain true if the word "decomposable" is replaced by the word "quasidecomposable".

2. LOCAL INVERSE AND LOCAL RESOLVENT

Local inverses which are so named for the first time here have been studied implicitly in the works of Putnam [16; 17; 18], Stampfli [25], Radjabalipour [20, 21], Stampfli-Wadhwa [26; 27] and Clancey [2]. Here we will obtain some results concerning the boundedness, continuity, and analyticity of these functions which are generalizations of some results due to the above mentioned authors.

DEFINITION 3. Let $T \in \mathscr{B}(\mathscr{X})$ and $x \in \mathscr{X}$. Assume there exists an $\mathscr{X}$-valued function f defined on a subset Δ of $\mathbf{C}$ such that $(T - z)f(z) \equiv x$. The function f is called a *local inverse* of x (with respect to T).

It is obvious that if T has no eigenvalue in Δ then the function f in Definition 3 is unique.

Our first result is about the lower semi-continuity of the norm of a local inverse. This extends a result of Clancey [2] to reflexive Banach spaces by a different proof. (The proof given in [2] does not apply to Banach spaces.)

THEOREM 4. *Let* $\mathscr{X}$ *be a reflexive Banach space, let* $T \in \mathscr{B}(\mathscr{X})$, *and let* $x \in \mathscr{X}$. *Assume that f is a local inverse of x. Then f can be redefined so that* $\|f\|$ *becomes lower semi-continuous.*

Proof. First observe that if M is a (strongly) closed subspace of $\mathscr{X}$ and if $y \in \mathscr{X}$, then there exists at least one element $u \in y + M$ such that

$$\|u\| = \inf\{\|v\| : v \in y + M\}.$$

Now, for each z in the domain of f, let N_z be the null space of $(T - z)$, and let $g(z)$ be a vector in $f(z) + N_z$ whose norm is minimal. Obviously $(T - z)\,g(z) \equiv x$ for all z in the domain of f. We show that $\|g\|$ is lower semi-continuous on the common domain of f and g. Let α be an arbitrary nonnegative number. Let $\{z_n\}$ be a sequence such that $\|g(z_n)\| \leqslant \alpha$ and $\lim z_n = z_0$ for some z_0 in the domain of f. We have to show that $\|g(z_0)\| \leqslant \alpha$. Since $\{g(z_n)\}$ is bounded, it has a subsequence $\{g(z_{n_k})\}$ converging weakly to a vector v such that $\|v\| \leqslant \alpha$. Therefore $(T - z_{n_k})\,g(z_{n_k})$ converges weakly to $(T - z_0)v$ and hence $(T - z_0)v = x$. So $v \in f(z_0) + N_{z_0}$ and thus $\|g(z_0)\| \leqslant \|v\| \leqslant \alpha$. This completes the proof of the theorem.

REMARK 6. Let T be a nonzero quasinilpotent operator and let $0 \neq x = Ty$ for some vector y. Let

$$f(z) = \begin{cases} (T-z)^{-1}x & \text{if } z \neq 0 \\ y & \text{if } z = 0. \end{cases}$$

By the above theorem $\|f\|$ is lower semi-continuous for some choice of y. Note that f is an analytic function with a singularity at 0. However, $\|f\|$ is not upper semi-continuous for any choice of y, because otherwise $\|f\|$ must be bounded in a neighborhood of the origin which implies that f has a removable singularity at 0 and $x = 0$, a contradiction.

The next property that we study is the weak continuity of local inverses. The local inverse in Remark 6 is not weakly continuous at 0 because it is not bounded around the origin. It is easy to see that in reflexive Banach spaces a *bounded* local inverse is weakly continuous if the operator has no eigenvalue in the domain of the local inverse [27, proof of Lemma 1]. However, if the operator has some eigenvalues in the domain of a bounded local inverse, then the local inverse may be weakly discontinuous.

We raise the following question.

QUESTION 1. Can every bounded local inverse be replaced by a weakly continuous one?

As a partial answer we prove the following theorem.

THEOREM 5. *Let $\mathscr{X}$ be a reflexive Banach space, $T \in \mathscr{B}(\mathscr{X})$, $x \in \mathscr{X}$, and let N_z denote the null space of $T - z$. Suppose that $f: \Delta \to \mathscr{X}$ is a bounded local inverse of x with respect to T. Assume for each $z \in \Delta$ there exists an invariant subspace M_z of T such that $\mathscr{X} = N_z \oplus M_z$, and the projections P_z onto M_z parallel to N_z are uniformly bounded. Then the local inverse f can be replaced by a weakly continuous one.*

Proof. First observe that the range of $T - z$ is a subset of M_z and thus $x \in M_z$ for all z in Δ. Let $g(z) \equiv P_z f(z)$, $(z \in \Delta)$. Obviously $g(z)$ is bounded on Δ and $(T - z)$ $g(z) \equiv x$. Let $\lambda \neq \mu$, $\lambda \in \Delta$, and $\mu \in \Delta$. We have

$$x = (T-\mu)\, g(\mu) = (T-\mu)\,[(I-P_\lambda)\, g(\mu) + P_\lambda\, g(\mu)] =$$

$$= [(\lambda-\mu)(I-P_\lambda)\, g(\mu)] \oplus (T-\mu)P_\lambda\, g(\mu).$$

Since $x \in M_z$ for all $z \in \Delta$, it follows that $(I - P_\lambda)\, g(\mu) = 0$ and thus $g(\mu) \in M_\lambda$. Note that $g(\lambda) \in M_\lambda$ by definition. Hence $g(\lambda) \in M_z$ for all λ and z in Δ. Let $\{z_n\}$ be any sequence in Δ converging to the point $z_0 \in \Delta$. We show that $\{g(z_n)\}$ converges weakly to $g(z_0)$. If not, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$, a linear functional φ, and a positive number ε_0 such that

$$|\varphi(g(\lambda_k)) - \varphi(g(z_0))| \geqslant \varepsilon_0,$$

where $\lambda_k = z_{n_k}$ for all k. Since $g(\lambda_k)$ is bounded, there exists a subsequence $\{g(\lambda_{k_i})\}$ of $\{g(\lambda_k)\}$ which converges weakly to a vector y. Moreover, $(T - z_0)y = x$. Since $g(\lambda_{k_i}) \in M_{z_0}$ for all i, $y \in M_{z_0}$ and thus $y = g(z_0)$. Hence

$$\lim |\varphi(g(\lambda_{k_i})) - \varphi(g(z_0))| = 0$$

a contradiction. The proof of the theorem is complete.

REMARK 7. If all eigenvalues of an operator T on a Hilbert space $\mathscr{H}$ are reducing, then $\mathscr{H} = N_z \oplus \bar{R}_z$ for all $z \in \mathbf{C}$, where R_z denotes the range of $T - z$. Therefore, Theorem 5 applies to hyponormal operators. Also, if T is a scalar type spectral operator, that is, if $T = \int_{\mathbf{C}} \lambda dE(\lambda)$ for some spectral measure E, then $N_z = E(\{z\})$ and hence $\mathscr{X} = N_z \oplus M_z$, where $M_z = E(\mathbf{C} \setminus \{z\})\mathscr{X}$. Here $\|P_z\|$ does not exceed the bound of the spectral measure E.

In the next proposition we will use the boundedness or weak continuity of the local inverse to obtain some nontrivial spectral manifolds of the form $X_T(F)$, whose existence played an important role in Section 1. The proposition is essentially proved in [25].

PROPOSITION 1. *Let $\mathscr{X}$ be a reflexive Banach space, Δ be a Cauchy domain, and let $f: \rho(T) \cup \partial\Delta \to \mathscr{X}$ be a function such that $(T - z)f(z) \equiv x$ for some $T \in \mathscr{B}(\mathscr{X})$ and some $x \in \mathscr{X}$. Assume that f is bounded on $\partial\Delta$. If f is not weakly continuous, then $X_T(\partial\Delta)$ is nontrivial; if f is weakly continuous, then $x = u + v$, where*

$$u = \frac{1}{2\pi i}\int_{+\partial\Delta} f(\lambda)\, d\lambda \in X_T(\bar{\Delta}) \qquad v = \frac{1}{2\pi i}\int_{+\partial(D \setminus \Delta)} f(\lambda)\, d\lambda \in X_T(\mathbf{C} \setminus \Delta),$$

where D is an open disc containing $\sigma(T) \cup \bar{\Delta}$. In particular, at least one of $X_T(\bar{\Delta})$ or $X_T(\mathbf{C} \setminus \Delta)$ is nonzero.

Proof. If f is not weakly continuous, then N_z (the null space of $T - z$) is not the zero subspace for some $z \in \partial\Delta$. Hence $X_T(\partial\Delta) \supset N_z \neq \{0\}$. If f is weakly continuous, then u and v are well-defined [25, Scholium] and $x = u + v$. Let

$$g(z) = \frac{1}{2\pi i}\int_{+\partial\Delta} (\lambda - z)^{-1} f(\lambda)\, d\lambda, \quad (z \notin \bar{\Delta}),$$

$$h(z) = \frac{1}{2\pi i}\int_{+\partial(D \setminus \Delta)} (\lambda - z)^{-1} f(\lambda)\, d\lambda, \quad (z \in \Delta).$$

By [25, proof of Scholium], $(T - z)g(z) \equiv u$, $(T - z)\,h(z) \equiv v$, and hence $u \in X_T(\bar{\Delta})$, $v \in X_T(\mathbf{C} \setminus \Delta)$. Since at least one of u or v is nonzero, the proof is complete.

The next proposition gives some sufficient conditions implying Dunford's condition (C) whose usefulness was observed in Section 1. First we need the following definition.

DEFINITION 4. Let m be a positive integer. An operator $T \in \mathscr{B}(\mathscr{X})$ satisfies a *local growth condition of order m* if for every closed set $\delta \subset \mathbf{C}$ and every $x \in X_T(\delta)$ there exists an analytic function $f: \mathbf{C} \setminus \delta \to \mathscr{X}$ such that $(T - z)f(z) \equiv x$ and

$$\|f(z)\| \leqslant K[\operatorname{dist}(z, \delta)]^{-m}\|x\|$$

where K is a positive number independent of δ and x.

All hyponormal operators satisfy a local growth condition of order 1 [20], [25], and all spectral operators of type $m - 1$ satisfy a local growth condition of order m [5, proof of Theorem XV. 6.7].

The proof of Proposition 2 is similar to the proof of [25, Theorem 2]. We present the proof for minor differences.

PROPOSITION 2. *Let T be an operator on a reflexive Banach space $\mathscr{X}$. Assume T satisfies a local growth condition of order m. Then $X_T(F)$ is closed for all closed sets F, that is, T satisfies condition (C).*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $X_T(F)$ and let $x = \lim x_n$. For each n, let $f_n : \mathbf{C}\setminus F \to \mathscr{X}$ be an analytic function such that $(T - z)f_n(z) = x_n$ and $\|f_n(z)\| \leqslant K[\operatorname{dist}(z, \delta)]^{-m}\|x_n\|$. Now, following the proof of [25, Theorem 2], we see that a subsequence of $\{f_n\}$ converges to an analytic function $f : \mathbf{C}\setminus F \to \mathscr{X}$ such that $(T - z) f(z) \equiv x$. This completes the proof.

COROLLARY 3. *If the adjoint of an M-hyponormal operator satisfies a local growth condition of order $m \geqslant 1$, then it has a nontrivial invariant subspace. In particular, if a cosubnormal operator satisfies a local growth condition, then it is decomposable.*

Proof. Let T be the adjoint of an M-hyponormal operator satisfying a local growth condition, and assume without loss of generality that T is not normal. By [21, Theorems 1 and 2] there exists a nonzero vector x and a bounded function $f: \mathbf{C} \to \mathscr{X}$ such that $(T - z)f(z) \equiv x$. Let Δ be an open disc such that $\bar{\Delta} \cap \sigma(T) \neq \emptyset$ and $\sigma(T) \setminus \bar{\Delta} \neq \emptyset$. (Note that T cannot be quasinilpotent [21, Corollary 5].) In view of Propositions 1 and 2 one of $X_T(\bar{\Delta})$ or $X_T(\mathbf{C} \setminus \Delta)$ is a (closed) invariant subspace of T which is not equal to $\{0\}$. Therefore, there exists a proper closed subset F of $\sigma(T)$ such that $X_T(F)$ is closed and different from $\{0\}$. Since T satisfies condition (C), it has the s.v.e.p. and hence $\sigma(T|X_T(F)) \subset F \cap \sigma(T)$ [3, page 23], [22, Theorem 2.13]. Thus $X_T(F)$ is a nontrivial invariant subspace of T. In particular, if T is cosubnormal, then it follows from [19, Lemma 2] that T is decomposable.

Note. Every hyponormal operator satisfies a local growth condition of order 1 [20], [25]; however, the invariant subspace problem for hyponormal operators is still unsolved.

In the case of the local growth conditions of order 1, we have the following interesting result.

PROPOSITION 3. *Let $\mathscr{X}$ be a reflexive Banach space, $T \in \mathscr{B}(\mathscr{X})$, and let $0 \neq x \in \mathscr{X}$. Assume that T satisfies a local growth condition of order 1 and x has a local inverse* *defined on an open disc Δ such that $\Delta \cap \sigma_T(x) \neq \emptyset$. Then f can be replaced by a function g such that g is bounded on a closed disc contained in Δ with center in $\sigma_T(x)$. In particular, T has a nontrivial invariant subspace.*

Proof. Using Theorem 4, replace f by a function g such that $\|g(z)\|$ is lower semi-continuous. Now imitate the proof of [2, Theorem 1] to obtain the first part of the proposition. For the last part of the proposition, apply Propositions 1 and 2 as in the proof of Corollary 3.

REMARK 8. Let μ be a Borel measure supported on a compact subset of **C**. If $f \in L^2(\mu)$ is such that the functions $g_z(\lambda) = (\lambda - z)^{-1}f(\lambda)$ belong to $L^2(\mu)$ for all z in some open set G, then in view of [16] G does not intersect the support of f. (Note that the multiplication by z in $L^2(\mu)$ defines a normal operator and the local spectrum of f is its support.) Consequently, a similar statement is true in $L^p(\mu)$ for $2 < p \leqslant \infty$. We are grateful to Professor Gill Martin who pointed out a direct proof of these facts as well as brought to our attention that if $1 \leqslant p < 2$ and if $f \in L^p(\mu)$ is essentially bounded, then the functions $g_z(\lambda)=(\lambda-z)^{-1}f(\lambda)$ belong to $L^p(\mu)$ for all $z \in$ **C**. Our Proposition 3 shows that if $f \in L^p(\mu)$ for some $p \in (1, \infty)$ and if the functions $g_z(\lambda) = (\lambda - z)^{-1}f(\lambda)$ belong to $L^p(\mu)$ for all z in some open set G intersecting the support of f, then $\int |g_z|^p d\mu$ is uniformly bounded for z in an open disc contained in G with center in the support of f.

In the remainder of this section we apply the above results to the class of dominant operators which are defined below.

DEFINITION 5. A Hilbert space operator T is called dominant if for each $z \in$ **C** there exists a positive number C_z such that

$$(T - z)(T - z)^* \leqslant C_z(T - z)^*(T - z).$$

Obviously every M-hyponormal operator is dominant. An important result about dominant operators, related to our work in this paper, is that given a non-normal dominant operator T there exists a non-zero vector x which has a local inverse with respect to T^* defined on **C**.

COROLLARY 4. *Let A be the adjoint of a dominant operator and suppose A satisfies a local growth condition of order 1. Then A has a non-trivial invariant subspace.*

Proof. If A is not normal, then there exists a nonzero vector x and an $\mathscr{X}$-valued function f defined on **C** such that $(A - z)f(z) \equiv x$, [21, Remark 3]. Now apply Proposition 3.

In [21, Theorem 3], it is shown that if T and T^* are both M-hyponormal, then they are normal. Counterexamples are given in the case T and T^* are dominant [21]. The following corollary may be of interest; it shows that if T and T^* are dominant and T satisfies a local growth condition of order 1, then T is normal.

COROLLARY 5. *Let A be the adjoint of a dominant operator, and let T be a dominant (respectively, subspectral) operator satisfying a local growth condition of* *order* 1. *Assume* $WA = TW$ *for some injective operator* W. *Then* A *is normal. If, moreover,* W *has a dense range, then* T *is a normal (resp., similar to a normal) operator.*

Proof. Assume, if possible, that A is not normal. Then there exists a nonzero vector x and an $\mathcal{X}$-valued function f such that $(A - z)f(z) \equiv x$ [21, Remark 3]. Then $(T - z)Wf(z) \equiv Wx$. In view of Proposition 3, there exists a bounded function g defined on an open disc D intersecting $\sigma_T(Wx)$ such that $(T - z)g(z) = Wx$. Now, by [27, Lemma 1] (resp., [8]) g can be chosen to be analytic on D. Thus $\sigma_T(Wx) \cap \cap D = \emptyset$ a contradiction. Therefore, A is normal. The rest of the proof follows from [21, Theorem 3(a, b)].

It is well known that the spectrum of a non-normal hyponormal operator has positive measure [15]. However, a dominant operator can be even quasinilpotent. The following corollary shows that if A is the adjoint of a dominant operator having spectrum on a smooth curve and if A satisfies a local growth condition of order 1, then it is normal.

COROLLARY 6. *Let* A *be the adjoint of a dominant operator, and let* T *be an operator satisfying a local growth condition of order* 1 *and having spectrum on a smooth curve* J. *Assume* $WA = TW$ *for some injective operator* W. *Then* A *is normal.*

Proof. Assume, if possible, that A is not normal. By following the proof of Corollary 5, we obtain a nonzero vector x, an open disc D, and a bounded function $g : D \to \mathcal{X}$ such that $(T - z)Wg(z) = Wx$ and $D \cap \sigma_T(Wx) \neq \emptyset$. Now for each z on J we have $\mathcal{X} = N_z \oplus M_z$, where N_z and M_z are the null space and the closure of the range of $T - z$, respectively [14, page 62]. Hence, by Theorem 5, $Wg(z)$ can be replaced by a function which is weakly continuous in D and analytic in $D \setminus J$. Therefore, $Wg(z)$ is analytic throughout D and hence $\sigma_T(Wx) \cap D = \emptyset$, a contradiction.

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