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"LOCALIZED" SELF-ADJOINTNESS OF SCHRÖDINGER OPERATORS

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Let $q \in L^2_{\text{loc}}(\mathbf{R}^m)$ be a real valued function.
Assume

$$A = -\Delta + q \quad \text{with} \quad D(A) = \mathscr{D}(\mathbf{R}^m)\,[=C_0^\infty(\mathbf{R}^m)]$$

satisfies

$$(1) \qquad A \geqslant 0$$

i.e.

$$\int |\nabla \varphi|^2 \mathrm{d}x + \int q|\varphi|^2 \mathrm{d}x \geqslant 0 \qquad \forall\ \varphi \in \mathscr{D}(\mathbf{R}^m).$$

Suppose in addition that for every $x_0 \in \mathbf{R}^m$ there exist $\eta \in \mathscr{D}_+(\mathbf{R}^m)$ with $\eta = 1$ near x_0, $\delta > 0$ and C such that

$$(2) \qquad -\Delta + q\eta \geqslant \delta[-\Delta + q^+\eta] - C,$$

(3) $\qquad -\Delta + q\eta$ is essentially self-adjoint (on $\mathscr{D}(\mathbf{R}^n)$ in $L^2(\mathbf{R}^n)$).

Our main result is the following

THEOREM 1. *A is essentially self-adjoint in $L^2(\mathbf{R}^m)$.*

REMARK. Assumptions (2) and (3) hold for example if $q^- \in L^p_{\text{loc}}(\mathbf{R}^m)$ where $p = 2$ when $m \leqslant 3$, $p > 2$ when $m = 4$, $p = \dfrac{m}{2}$ when $m \geqslant 5$ (see T. Kato [5] and also M. Reed and B. Simon [6] Theorem X.29).
In this case, the conclusion of Theorem 1 was obtained by Chernoff [2], (Theorem 4.6) (under slightly more general assumptions, using a completely different approach). Related results can be found in [3], [4], [7], [8], [9].

In the proof of Theorem 1 we shall use the following

LEMMA 1. *Assume (1). Let $v \in H^1(\mathbf{R}^m)$ be such that $q^+|v|^2 \in L^1(\mathbf{R}^m)$. Then $q^-|v|^2 \in L^1(\mathbf{R}^m)$, and*

$$\int |\nabla v|^2 \mathrm{d}x + \int q|v|^2 \mathrm{d}x \geqslant 0.$$

Proof. The conclusion holds obviously if $v \in H^1(\mathbf{R}^m) \cap L^\infty(\mathbf{R}^m)$ has compact support (use smoothing by convolution). In the general case, truncate v and multiply by cut-off functions.

LEMMA 2. *Let $Q \in L^2_{\mathrm{loc}}(\mathbf{R}^m)$ be a real valued function.*
Assume

$$(4) \qquad -\Delta + Q \text{ is essentially self-adjoint (on } \mathscr{D}(\mathbf{R}^m) \text{ in } L^2(\mathbf{R}^m)),$$

$$(5) \qquad -\Delta + Q \geqslant \delta[-\Delta + Q^+] - C$$

for some $\delta > 0$ and some C.
Let $v \in L^2(\mathbf{R}^m)$ be such that $-\Delta v + Qv \in H^{-1}(\mathbf{R}^m)$.
Then $v \in H^1(\mathbf{R}^m)$ and $Q^+|v|^2 \in L^1(\mathbf{R}^m)$.

Proof. Let $B = -\Delta + Q + C$ with $D(B) = \mathscr{D}(\mathbf{R}^m)$. By (4) and (5), B is essentially self-adjoint and $B \geqslant 0$. Consequently $N(I + B^*) = \{0\}$.
Let

$$W = \{w \in H^1(\mathbf{R}^m),\ Q^+|w|^2 \in L^1(\mathbf{R}^m)\}$$

with the Hilbert norm

$$\|w\|^2 = \int |\nabla w|^2 dx + \int Q^+|w|^2 dx + \int |w|^2 dx.$$

We deduce from Lemma 1 and (5) that if $w \in W$, then $Q^-|w|^2 \in L^1(\mathbf{R}^m)$ and

$$\int |\nabla w|^2\, dx + \int Q|w|^2 dx \geqslant \delta \int |\nabla w|^2 + \delta \int Q^+|w|^2 - C \int |w|^2.$$

Consequently, if T is given in $H^{-1}(\mathbf{R}^m)$ there exists a unique $w \in W$ such that

$$(6) \qquad \int \nabla w \nabla \bar{\psi}\, dx + \int Q w \bar{\psi}\, dx + \int (C+1)\, w \bar{\psi}\, dx = \langle T, \psi \rangle \quad \forall\, \psi \in W$$

(by Lax-Milgram).
If $v \in L^2(\mathbf{R}^m)$ is such that $-\Delta v + Qv \in H^{-1}(\mathbf{R}^m)$ we may choose in (6) $T = -\Delta v + Qv + (C+1)v$. It follows that $v - w \in N(I + B^*)$ and thus $v = w$; in particular $v \in W$.

Proof of Theorem 1. It suffices to show that $N(I + A^*) = \{0\}$. So let $u \in L^2(\mathbf{R}^m)$ be such that

$$(7) \qquad u - \Delta u + qu = 0 \text{ in } \mathscr{D}'(\mathbf{R}^m).$$

We have to prove that $u = 0$. The proof is divided into two steps.

Step 1. If $u \in L^2(\mathbf{R}^m)$ satisfies (7), then $u \in H^1_{\mathrm{loc}}(\mathbf{R}^m)$ and $q^+|u|^2 \in L^1_{\mathrm{loc}}(\mathbf{R}^m)$.

Step 2. If $u \in H^1_{\mathrm{loc}}(\mathbf{R}^m) \cap L^2(\mathbf{R}^m)$ and $q^+|u|^2 \in L^1_{\mathrm{loc}}(\mathbf{R}^m)$; then (7) implies $u = 0$.

Step 1. Let η be as in (2) — (3). Let $\tilde{\eta} \in \mathscr{D}(\mathbf{R}^m)$ be such that $\tilde{\eta}(x_0) = 1$ and

$$\operatorname{Supp} \tilde{\eta} \subset \{x; \eta(x) = 1\}.$$

Thus $\tilde{\eta}(1 - \eta) = 0$.
It follows from (7) that in $\mathscr{D}'(\mathbf{R}^m)$ we have

$$-\Delta(\tilde{\eta}u) = -\tilde{\eta}\Delta u - 2\frac{\partial}{\partial x_i}\left(\frac{\partial\tilde{\eta}}{\partial x_i}u\right) + (\Delta\tilde{\eta})u =$$

$$= \tilde{\eta}(-u - qu) - 2\frac{\partial}{\partial x_i}\left(\frac{\partial\tilde{\eta}}{\partial x_i}u\right) + (\Delta\tilde{\eta})u$$

and therefore

$$-\Delta(\tilde{\eta}u) + q\eta(\tilde{\eta}u) = -\tilde{\eta}u - 2\frac{\partial}{\partial x_i}\left(\frac{\partial\tilde{\eta}}{\partial x_i}u\right) + (\Delta\tilde{\eta})u,$$

which belongs to $H^{-1}(\mathbf{R}^m)$.
We deduce from Lemma 2 applied with $Q = q\eta$ and $v = \tilde{\eta}u$ that $\tilde{\eta}u \in H^1(\mathbf{R}^m)$ and $q^+\eta|\tilde{\eta}u|^2 \in L^1(\mathbf{R}^m)$.

Step 2. Let $\zeta \in \mathscr{D}_+(\mathbf{R}^m)$ and set $T = \zeta qu$. We have $T \in L^1(\mathbf{R}^m)$ and also $T \in H^{-1}(\mathbf{R}^m)$ since

$$T = \zeta(\Delta u - u) = \frac{\partial}{\partial x_i}\left(\zeta\frac{\partial u}{\partial x_i}\right) - \frac{\partial\zeta}{\partial x_i}\frac{\partial u}{\partial x_i} - \zeta u.$$

On the other hand

$$\operatorname{Re} T\zeta\bar{u} = \zeta^2 q|u|^2 \geqslant -\zeta^2 q^-|u|^2.$$

The last expression belongs to $L^1(\mathbf{R}^m)$ (by Lemma 1 used with ζu in place of u). We may therefore apply a result of [1] to conclude that

$$\operatorname{Re}\left[-\int\zeta\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}(\zeta\bar{u})\,dx - \int\frac{\partial\zeta}{\partial x_i}\frac{\partial u}{\partial x_i}\zeta\bar{u}\,dx - \int\zeta^2|u|^2\,dx\right] = \int\zeta^2 q|u|^2\,dx,$$

i.e.

$$\int|\nabla(\zeta u)|^2dx + \int|\nabla\zeta|^2|u|^2dx - \int\zeta^2|u|^2dx = \int\zeta^2 q|u|^2dx.$$

From Lemma 1 (used with ζu) we derive that

$$\int\zeta^2|u|^2dx \leqslant \int|\nabla\zeta|^2|u|^2dx.$$

Choosing $\zeta(x) = \zeta_0(n^{-1}x)$ where $\zeta_0 \in \mathscr{D}_+(\mathbf{R}^m)$ is fixed with $\zeta_0(0) = 1$ we see as $n \to \infty$ that $u = 0$.

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