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# COMPACT PERTURBATIONS OF DEFINITIZABLE OPERATORS

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Let $\mathcal{H}$ be a Kreĭn space [1] with the indefinite scalar product $[x, y]$, $x, y \in \mathcal{H}$. The $J$-selfadjoint operator $A$ in $\mathcal{H}$ (with dense domain $\mathcal{D}(A)$) is called *definitizable*, if there exists a polynomial $p$ such that $[p(A)x, x] \geqq 0$ for all $x \in \mathcal{D}(A^n)$ where $n$ denotes the degree of $p$[1].

The non-real part $\sigma_0(A)$ of the spectrum of a definitizable operator $A$ with $\rho(A) \neq \emptyset$ consists of no more than a finite number of points which lie symmetrically with respect to the real axis (see [8], [5]). We denote the Riesz-Dunford projector corresponding to $\sigma_0(A)$ by $E_0$.

A definitizable operator $A$ with $\rho(A) \neq \emptyset$ has a spectral function (see [8], [1]; here we use the notation and results of [5]). That is, we have a finite (possibly empty) set $c(A) \subset \overline{\mathbf{R}}$ $(= \mathbf{R} \cup \{\infty\})$ with the following property: If $\mathfrak{B}(A)$ denotes the Boolean algebra of subsets of $\overline{\mathbf{R}}$ generated by the closed and open intervals whose endpoints do not belong to $c(A)$, there exists a homomorphism $E$ from $\mathfrak{B}(A)$ into a Boolean algebra of $J$-selfadjoint projectors in $\mathcal{H}$ such that for $\Delta \in \mathfrak{B}(A)$ we have

$$E(\overline{\mathbf{R}}) = I - E_0; \tag{1}$$

$$AE(\Delta) \supset E(\Delta)A; \tag{2}$$

$$\sigma(A \mid E(\Delta)\mathcal{H}) \subset \overline{\Delta}; \tag{3}$$

(4) $t \in c(A)$ if and only if the subspace $E(\Delta)\mathcal{H}$ is indefinite[2] for all $\Delta$ containing $t$.

The elements of $c(A)$ are called *critical points* of $A$. If $t \in c(A)$, $t \neq \infty$, then (see e.g. [1]) we have $p(t) = 0$ for each definitizing polynomial $p$ of $A$.

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[1] In [1] $J$-selfadjoint operators are called selfadjoint and instead of "definitizable" the term "positizable" is used.

[2] We recall that an element $x \in \mathcal{H}$ is said to be *positive, non-negative, neutral*, etc., if $[x, x] > 0$, $[x, x] \geqq 0$, $[x, x] = 0$, etc. A subspace of $\mathcal{H}$ is called *positive, non-negative, neutral*, etc., if all its non-zero elements have this property; it is called *indefinite* if it contains positive and negative elements. Furthermore, we set $\mathfrak{P}_\pm := \{x \in \mathcal{H} : \pm [x, x] \geqq 0\}$, $\mathfrak{P}_0 := \mathfrak{P}_+ \cap \mathfrak{P}_-$.

For $t \in \overline{\mathbf{R}}$ we denote by $\varkappa_+(t;A)$, $(\varkappa_-(t;A))$ the minimum of the numbers[3] $\varkappa_+(E(\Delta)\mathscr{H})$, $(\varkappa_-(E(\Delta)\mathscr{H}))$, where $\Delta$ runs over all open $\Delta \in \mathfrak{B}(A)$ with $t \in \Delta$, and put

$$\varkappa(t;A) := \min(\varkappa_+(t;A), \varkappa_-(t;A)).$$

This quantity is positive if and only if $t \in c(A)$; it is called the *rank of indefiniteness* of $t$ (with respect to $A$). Instead of $\varkappa(t;A)$, $\varkappa_\pm(t;A)$ we shall often write $\varkappa(t)$, $\varkappa_\pm(t)$. By $c_\infty(A)$ we denote the set of those $t \in c(A)$ for which $\varkappa(t;A)=\infty$; $\sigma_{0,\infty}(A)$ is the set of non-real eigenvalues of $A$ with infinite-dimensional root subspaces.

It is the aim of this note to study (relatively) finite-dimensional and compact perturbations of definitizable operators, esp. the behaviour of the sets of critical points under such perturbations. We mention that some, partly more special, results of this kind were proved and used in [10], [12], [2].

## 1. FINITE-DIMENSIONAL PERTURBATIONS OF DEFINITIZABLE OPERATORS

We start with the following simple

PROPOSITION 1. *If $A$ is definitizable, $\rho(A) \neq \emptyset$, then*

$$c(A) \setminus \sigma_p(A) \subset c_\infty(A).$$

*In particular, if $\infty$ is a critical point of $A$ then it belongs to $c_\infty(A)$.*

*Proof.* Consider $t_0 \in c(A) \setminus \sigma_p(A)$ and assume e.g. $\varkappa_-(t_0) < \infty$. Then there exists an open set $\Delta \in \mathfrak{B}(A)$, $\Delta \cap c(A) = \{t_0\}$, such that $\varkappa_-(t_0) = \varkappa_-(E(\Delta)\mathscr{H})$, that i $E(\Delta)\mathscr{H}$ is a Pontrjagin space [1]. Pontrjagin's theorem [1], Theorem IX.7.2, now implies that $A$ has a non-positive eigenvector corresponding to some eigenvalue $t_1 \in \Delta$, $\varkappa_-(t_1) > 0$. Since $t_1 \neq t_0$, we have

$$\varkappa_-(t_1) + \varkappa_-(t_0) \leqq \varkappa_-(E(\Delta)\mathscr{H}) = \varkappa_-(t_0),$$

which is impossible.

The next proposition shows that for a definitizable operator $A$ satisfying $\rho(A) \neq \emptyset$ in a Kreĭn space $\mathscr{H}$ with $\varkappa_+(\mathscr{H}) = \varkappa_-(\mathscr{H}) = \infty$ there exists a finite-dimensional perturbation such that the critical points of finite rank of indefiniteness and the non-real eigenvalues of finite algebraic multiplicity disappear.

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[3] If $\mathscr{L}$ is a linear space with scalar product $[.,.]$, possibly degenerated, we denote by $\varkappa_+(\mathscr{L};[.,.])$, $(\varkappa_-(\mathscr{L};[.,.]))$ the least upper bound ($\leqq \infty$) of the dimensions of positive (resp. negative) subspaces of $\mathscr{L}$. Instead of $\varkappa_\pm(\mathscr{L};[.,.])$ we often write $\varkappa_\pm(\mathscr{L})$.

PROPOSITION 2. *Let $A$, $\rho(A) \neq \emptyset$, be a definitizable operator in the Kreĭn space $\mathscr{H}$, $\varkappa_+(\mathscr{H}) = \varkappa_-(\mathscr{H}) = \infty$. Then there exists a definitizable operator $A_1$, $\rho(A_1) \neq \emptyset$, of the form*

$$(1) \qquad A_1 = A + F, \ F \text{ finite-dimensional},$$

*such that*

$$(2) \qquad c(A_1) = c_\infty(A_1) = c_\infty(A), \ \sigma_0(A_1) = \sigma_{0,\infty}(A_1) = \sigma_{0,\infty}(A).$$

*Proof.* Put $\sigma_0' := \sigma_0(A) \setminus \sigma_{0,\infty}(A)$. Then $A_0 := A - AE(\sigma_0'; A)$[4] differs from $A$ by a finite-dimensional operator and we have

$$\sigma_0(A_0) = \sigma_{0,\infty}(A_0) = \sigma_{0,\infty}(A).$$

Evidently, $c_\infty(A_0) = c_\infty(A)$ and $A_0$ is definitizable by $p_0$: $p_0(t) = t^2 p(t)$, if $p$ is a definitizing polynomial of $A$.

Consider $t_0 \in c(A_0) \setminus c_\infty(A_0)$. Then $t_0 \neq \infty$ by Proposition 1. Assume, e.g., $\varkappa(t_0; A_0) = \varkappa_-(t_0; A_0)$ and choose an open interval $\Delta \in \mathfrak{B}(A_0)$, $t_0 \in \Delta$, such that $\varkappa_-(t_0; A_0) = \varkappa_-(E_0(\Delta)\mathscr{H})$, where $E_0(\cdot)$ is the spectral function of $A_0$. We consider the restriction $A_\Delta$ of $A_0$ to $E_0(\Delta)\mathscr{H}$ and a fundamental decomposition [1] of $E_0(\Delta)\mathscr{H}$:

$$(3) \qquad E_0(\Delta)\mathscr{H} = \mathscr{H}_+^{(\Delta)} \dotplus \mathscr{H}_-^{(\Delta)}, \ \dim \mathscr{H}_-^{(\Delta)} = \varkappa_-(t_0; A_0) < \infty.$$

The condition $\varkappa_-(\mathscr{H}) = \infty$ implies that either $\sigma_{0,\infty}(A_0) \neq \emptyset$ or $\{t: \varkappa_-(t; A_0) = = \infty\} \neq \emptyset$. In the first case we choose $\lambda_0 \in \sigma_{0,\infty}(A_0)$ and a $\varkappa_-(t_0; A_0)$-dimensional subspace $\mathscr{L}'_+$ of $\mathscr{H}_+^{(\Delta)}$. Denote by $\mathscr{L}_+$ the orthogonal complement of $\mathscr{L}'_+$ in the Hilbert space $\mathscr{H}_+^{(\Delta)}$ (with respect to $[.,.]$). Then we have the decomposition $E_0(\Delta)\mathscr{H} = \mathscr{L}_+ \dotplus \dotplus \mathscr{L}'_+ \dotplus \mathscr{H}_-^{(\Delta)}$. If the corresponding matrix representation of $A_\Delta$ is

$$A_\Delta = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix},$$

we define

$$(4) \qquad A'_\Delta := \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & \operatorname{Re} \lambda_0 I & \operatorname{Im} \lambda_0 I \\ 0 & -\operatorname{Im} \lambda_0 I & \operatorname{Re} \lambda_0 I \end{pmatrix}.$$

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[4] $E(\sigma; A)$ denotes the Riesz-Dunford-Taylor projector corresponding to the spectral set $\sigma$ of $A$. If no confusion arises we write $E(\sigma)$ instead of $E(\sigma; A)$.

Next consider the case $\sigma_{0,\infty}(A_0) = \emptyset$. If $\{t: \varkappa_-(t;A_0)=\infty\} = \{\infty\}$ we easily find (cf. [5]) a finite real point $s_0$ near $\infty$ satisfying $\varkappa_+(s_0;A_0) = 0$. If $\{t: \varkappa_-(t;A_0) = \infty\} \neq \{\infty\}$ we choose $s_0 \in \{t: \varkappa_-(t;A_0) = \infty\} \setminus \{\infty\}$. If

$$A_\Delta = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

is the matrix representation of $A_\Delta$ with respect to the decomposition in (3) we consider the operator

$$A'_\Delta := \begin{pmatrix} C_{11} & 0 \\ 0 & s_0 I \end{pmatrix}. \tag{5}$$

In both cases $A'_1 := A_0(I - E_0(\Delta)) + A'_\Delta E_0(\Delta)$ (with $A'_\Delta$ defined by (4) or (5)) is $J$-selfadjoint, definitizable, $\rho(A'_1) \neq \emptyset$, it differs from $A_0$ by a finite-dimensional operator and has the properties

$$c_\infty(A'_1) = c_\infty(A_0) = c_\infty(A), \quad c(A'_1) = c(A_0) \setminus \{t_0\},$$

$$\sigma_0(A'_1) = \sigma_{0,\infty}(A'_1) = \sigma_{0,\infty}(A_0) = \sigma_{0,\infty}(A).$$

Repeating this argument we find an operator $A_1$ satisfying (1) and (2).

In the proof of Theorem 1 we use the following simple observation. Let $B_0$, $B_1$ be bounded linear operators in a Banach space $\mathscr{X}$ and suppose $\dim (B_1 - B_0) = m < \infty$. Then for an arbitrary polynomial $p$ of degree $n$: $p(z) = \sum_{j=0}^{n} \gamma_j z^j$, $\gamma_n \neq 0$, we have $\dim (p(B_1) - p(B_0)) \leqq nm$. This follows from the identity

$$p(B_1) - p(B_0) = \sum_{k=1}^{n} B_0^{k-1}(B_1 - B_0) \sum_{j=k}^{n} \gamma_j B_1^{j-k}.$$

THEOREM 1. *Let $A_0$, $A_1$ be J-selfadjoint operators in the Kreĭn space $\mathscr{H}$ such that $\rho(A_0) \cap \rho(A_1) \neq \emptyset$. Suppose that $A_0$ is definitizable and*

$$\dim ((A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1}) = m < \infty \tag{6}$$

*for some (and hence for all) $\lambda \in \rho(A_0) \cap \rho(A_1)$. Then $A_1$ is definitizable and we have*

$$c_\infty(A_0) = c_\infty(A_1), \ \sigma_{0,\infty}(A_0) = \sigma_{0,\infty}(A_1). \tag{7}$$

*If a definitizing polynomial $p_0$ of $A_0$ is of degree $n_0$, then there exists a definitizing polynomial $p_1$ of $A_1$ such that the degree of $p_1$ is not greater than $4m\left[\frac{n_0+1}{2}\right] + n_0$,* *and we have*

(8) $$\sum_{t\in c(A_1)\setminus c(A_0)} \varkappa(t;A_1)+\sum_{\lambda\in\sigma_0(A_1)\setminus\sigma_0(A_0),\ \mathrm{Im}\,\lambda>0} \dim E(\{\lambda\};A_1)\leqq 2m\left[\frac{n_0+1}{2}\right].$$

*In particular, the set* $(c(A_1)\setminus c(A_0))\cup(\sigma_0(A_1)\setminus\sigma_0(A_0))$ *has at most* $2m\left[\frac{n_0+1}{2}\right]$ *points in the closed upper half plane.*

*Proof.* 1. The relation

$$((A_0-\lambda I)^{-1}-(A_1-\lambda I)^{-1})(A_0-\lambda I)(A_0-\lambda_0 I)^{-1}=$$

$$=(A_1-\lambda_0 I)(A_1-\lambda I)^{-1}((A_0-\lambda_0 I)^{-1}-(A_1-\lambda_0 I)^{-1}),$$

$$(\lambda,\lambda_0\in\rho(A_0)\cap\rho(A_1))$$

implies that condition (6) is independent of the point $\lambda\in\rho(A_0)\cap\rho(A_1)$. We fix some non-real $z_0\in\rho(A_0)\cap\rho(A_1)$. Then also $\bar z_0\in\rho(A_0)\cap\rho(A_1)$, and the function $r$:

$$r(z):=p_0(z)\,(z-z_0)^{-n}(z-\bar z_0)^{-n},\quad n:=\left[\frac{n_0+1}{2}\right],$$

is definitizing for $A_0$:

$$[r(A_0)x,x]=[p_0(A_0)(A_0-z_0I)^{-n}(A_0-\bar z_0I)^{-n}x,x]\geqq 0\qquad (x\in\mathscr{H}).$$

Here we may suppose that the definitizing polynomial $p_0$ of $A_0$ and hence also $r$ are real on $\mathbf{R}$ (see [8], [5]). Consider the representation

$$r(z)=\gamma_0+\sum_{j=1}^{n}[\gamma_j(z-z_0)^{-j}+\bar\gamma_j(z-\bar z_0)^{-j}]$$

of the function $r$. Then

$$r(A_1)-r(A_0)=\sum_{j=1}^{n}\{\gamma_j[(A_1-z_0I)^{-j}-(A_0-z_0I)^{-j}]+\bar\gamma_j[(A_1-\bar z_0I)^{-j}-(A_0-\bar z_0I)^{-j}]\}$$

and the above remark and (6) imply that

(9) $$\dim(r(A_1)-r(A_0))\leqq 2nm.$$

The operator $r(A_1)$ is again $J$-selfadjoint. We put

$$\{x,y\}:=[r(A_1)x,y]\qquad (x,y\in\mathscr{H})$$

and $\hat{\mathscr{H}} := \mathscr{H}/\mathscr{N}_0$, $\mathscr{N}_0 := \{x \in \mathscr{H} : \{x, y\} = 0 \text{ for all } y \in \mathscr{H}\}$. By (9) we have

(10) $$\varkappa := \varkappa_-(\mathscr{H}; \{.,.\}) = \varkappa_-(\hat{\mathscr{H}}; \{.,.\}) \leqq 2nm.$$

The space $\hat{\mathscr{H}}$ can be completed to a Pontrjagin space $\tilde{\mathscr{H}}$.

Setting

$$R_z := (A_1 - zI)^{-1} \ (z \in \rho(A_1))$$

we define the operator $\hat{R}_z$ in $\hat{\mathscr{H}}$ by the relation

$$\hat{R}_z \hat{x} := \widehat{R_z x} \ (x \in \mathscr{H},\ \hat{x} = x + \mathscr{N}_0).$$

This definition is correct because of

$$\{R_z x, y\} = \{x, R_{\bar{z}}\, y\} \quad (x, y \in \mathscr{H}, z \in \rho(A_1)).$$

Evidently, we have

$$(\hat{I} + (z_0 - \bar{z}_0)\hat{R}_{z_0})(\hat{I} + (\bar{z}_0 - z_0)\,\hat{R}_{\bar{z}_0}) =$$

$$= (\hat{I} + (\bar{z}_0 - z_0)\,\hat{R}_{\bar{z}_0})(\hat{I} + (z_0 - \bar{z}_0)\,\hat{R}_{z_0}) = \hat{I}.$$

Hence $\hat{I} + (z_0 - \bar{z}_0)\,\hat{R}_{z_0}$ is an isometric operator in $\tilde{\mathscr{H}}$ whose domain and range are the dense subspace $\hat{\mathscr{H}}$. Then $\hat{I} + (z_0 - \bar{z}_0)\,\hat{R}_{z_0}$ and also $\hat{R}_{z_0}$ are continuous by [1], Theorem IX.3.1 or [4], Theorem 2.3. The closure of $\hat{R}_{z_0}$ in $\tilde{\mathscr{H}}$ is denoted by $\tilde{R}_{z_0}$.

For the $J$-unitary operator $\tilde{I} + (z_0 - \bar{z}_0)\,\tilde{R}_{z_0}$ in the Pontrjagin space $\tilde{\mathscr{H}}$ there exists a $\varkappa$-dimensional (cf. (10)) non-positive invariant subspace $\mathscr{M}$ of $\tilde{\mathscr{H}}$ (see [1], Theorem IX.7.1 or [4], Theorem 3.7). We find a polynomial $q$ of degree $\leqq \varkappa$ such that $q(\tilde{R}_{z_0} | \mathscr{M}) = 0$. Then

$$\{q(\tilde{R}_{\bar{z}_0})\, q(\tilde{R}_{z_0})\tilde{x},\ \tilde{x}\} \geqq 0, \quad (\tilde{x} \in \tilde{\mathscr{H}}).$$

For the polynomial

$$p_1(z) := p_0(z)\,(z - \bar{z}_0)^{\varkappa}(z - z_0)^{\varkappa}\, \bar{q}\left(\frac{1}{z - \bar{z}_0}\right) q\left(\frac{1}{z - z_0}\right)$$

the degree of which is not greater than

$$2\varkappa + n_0 \leqq 4m\left[\frac{n_0 + 1}{2}\right] + n_0$$

we have

$$[p_1(A_1)x, x] = [p_0(A_1)(A_1 - \bar{z}_0 I)^{\varkappa} (A_1 - z_0 I)^{\varkappa} \bar{q}(R_{\bar{z}_0})\, q(R_{z_0})\, x, x] =$$

$$= [r(A_1)(A_1 - z_0 I)^{\varkappa+n}\, \bar{q}(R_{\bar{z}_0})\, q(R_{z_0})\, x, (A_1 - z_0 I)^{\varkappa+n} x] =$$

$$= \{\bar{q}(R_{\bar{z}_0})\, q(R_{z_0})(A_1 - z_0 I)^{\varkappa+n} x, (A_1 - z_0 I)^{\varkappa+n} x\} \geqq 0 \quad (x \in \mathscr{D}(A_1^{2\varkappa+n_0})).$$

2. Let $t_j$ , $j=1, \ldots, k$, be the elements of $c(A_1)\setminus c(A_0)$ and let $S := \sigma_0(A_1)\setminus\setminus\sigma_0(A_0)$. In order to prove (8) we first observe that the rational function $r$ with $[r(A_0)x, x] \geqq 0$, $(x \in \mathscr{H})$, can be chosen so that $r(t_j) \neq 0$, $j = 1, 2, \ldots, k$, and $\operatorname{Im} r(z) \neq 0$ for $z \in S$ (see [5]). For each $t_j$ , $j = 1, \ldots, k$, we choose an open set $\Delta_j \in \mathfrak{B}(A_1)$ such that $t_j \in \Delta_j$, $\varkappa_\pm(E(\Delta_j; A_1)\mathscr{H}) = \varkappa_\pm(t_j; A_1)$ and $\delta_j r(t) > 0$ if $t \in \bar{\Delta}_j$ and $\delta_j = +1$ or $-1$. Denote the restriction of $A_1$ to $E(\Delta_j; A_1)\mathscr{H}$ by $A_1^{\Delta_j}$. The spectrum of $\delta_j r(A_1^{\Delta_j})$ in $E(\Delta_j; A_1)\mathscr{H}$ is positive, hence $\delta_j r(A_1^{\Delta_j})$ has a bounded inverse and a $J$-selfadjoint square root with positive spectrum. If $x \in E(\Delta_j; A_1)\mathscr{H}$ it follows that

$$\{x, x\} = [r(A_1^{\Delta_j})\, x, x] = \delta_j[(\delta_j r(A_1^{\Delta_j}))^{\frac{1}{2}}\, x, (\delta_j r(A_1^{\Delta_j}))^{\frac{1}{2}}\, x],$$

hence

$$\varkappa(t_j; A_1) \leqq \varkappa_-\,(E(\Delta_j; A_1)\mathscr{H};\ \{.,.\}), \qquad j = 1, \ldots, k. \tag{11}$$

It is easily seen that for the restriction $A_1^S$ of $A_1$ to $E(S;A_1)\mathscr{H}$, the operator $r(A_1^S)$ has a $J$-selfadjoint square root with bounded inverse. Hence for $x \in E(S;A_1)\mathscr{H}$ we have

$$\{x, x\} = [r(A_1^S)\, x, x] = [(r(A_1^S))^{\frac{1}{2}}\, x, (r(A_1^S))^{\frac{1}{2}}\, x]$$

and it follows

$$\sum_{\lambda \in S \cap \{\operatorname{Im}\lambda > 0\}} \dim E(\{\lambda\};A_1)\mathscr{H} \leqq \varkappa_-(E(S;A_1)\mathscr{H};\ \{.,.\}). \tag{12}$$

Adding equations (11) and (12) and making use of (10) we obtain (8).

3. The second relation in (7) follows from well-known results in perturbation theory. To prove the first relation we may suppose $\varkappa_-(\mathscr{H}) = \varkappa_+(\mathscr{H}) = \infty$. Otherwise $c_\infty(A_0) = c_\infty(A_1) = \emptyset$.

Assume $t \notin c_\infty(A_0)$. Then, by Proposition 2, there exists a finite-dimensional $J$-selfadjoint operator $F$ such that $t \notin c(A_0 + F)$. Now relation (8) applied to the definitizable operators $A_1$ and $A_0 + F$ instead of $A_1$ and $A_0$ gives $\varkappa(t; A_1) < \infty$, that is $t \notin c_\infty(A_1)$, or $c_\infty(A_1) \subset c_\infty(A_0)$. As $A_0$ and $A_1$ can be replaced by each other the required equality follows.

REMARK. If, under the conditions of Theorem 1, the operators $A_0$ and $A_1$ are bounded, the right-hand side in (8) can be replaced by $mn_0$. This follows immediately from the fact that in this case in the proof of Theorem 1 we can use $p_0$ instead of $r$.

As a corollary of Theorem 1 we have: *The class of bounded definitizable J-selfadjoint operators is closed with respect to finite-dimensional perturbations.*

By relation (7) of Theorem 1, the sets of critical points with infinite rank of indefiniteness and of non-real eigenvalues with infinite multiplicity of a definitizable operator do not change under a relatively finite-dimensional perturbation, that is a perturbation satisfying (6). On the other hand, if $\varkappa_+(\mathscr{H}) = \varkappa_-(\mathscr{H}) = \infty$, then by Proposition 2 we can always choose a relatively finite-dimensional perturbation for which the critical points of finite rank of indefiniteness and the non-real eigenvalues of finite multiplicity disappear.

## 2. COMPACT PERTURBATIONS OF STRONGLY STABLE OPERATORS

For the following it is convenient to fix a fundamental decomposition $\mathscr{H}=\mathscr{H}_+ \dotplus \mathscr{H}_-$ of the Kreĭn space $\mathscr{H}$. Let $P_+$ and $P_-$ be the corresponding $J$-selfadjoint projectors. We shall consider the definite scalar product $(.,.)$ defined by

$$(x, y) := [Jx, y], \quad J := P_+ - P_- \quad (x, y \in \mathscr{H})$$

and we set

$$\|x\|^2 := (x, x) \qquad (x \in \mathscr{H}).$$

A $J$-selfadjoint operator $A$ in the Kreĭn space $\mathscr{H}$ is called *strongly stable*, if $\sigma(A)$ is the union of two disjoint closed subsets $\sigma_+(A)$, $\sigma_-(A)$ of $\mathbf{R}$ at least one of which is bounded such that

$$E(\sigma_\pm(A))\,\mathscr{H} \subset \mathfrak{P}_\pm.$$

Obviously, the $J$-selfadjoint operator $A$ is strongly stable if and only if it is definitizable and $c(A) = \sigma_0(A) = \emptyset$. In this case each polynomial $p$ of even degree whose zeros are in $\rho(A)$ and which has the property $p(t) \gtrless 0$ if $t \in \sigma_\pm(A)$ is definitizing for $A$.

If $A$ is strongly stable, it is similar to a selfadjoint operator. Indeed, it is easy to see that $A$ is selfadjoint with respect to the positive scalar product $(.,.)_0$ defined by

$$(x, y)_0 := [(E(\sigma_+(A)) - E(\sigma_-(A)))x, y] \qquad (x, y \in \mathscr{H}).$$

Then for $\|x\|_0^2 := (x, x)_0$ and

$$\gamma(A) := \inf_{x \neq 0} \frac{\|x\|_0}{\|x\|} > 0, \qquad \Gamma(A) := \sup_{x \neq 0} \frac{\|x\|_0}{\|x\|} < \infty$$

we have

$$\gamma(A)\|x\| \leqq \|x\|_0 \leqq \Gamma(A)\|x\| \qquad (x \in \mathscr{H}). \tag{13}$$

In the study of strongly stable operators it is often sufficient to consider bounded strongly stable operators. Indeed, if $A$ is a $J$-selfadjoint operator with $\rho(A) \cap \mathbf{R} \neq \emptyset$, $t_0 \in \rho(A) \cap \mathbf{R}$ and $\varphi_{t_0}(t) := (t - t_0)^{-1}$, it is easy to see that $A$ is definitizable if and only if $\varphi_{t_0}(A) = (A - t_0 I)^{-1}$ has this property. Moreover, $\Delta \in \mathfrak{B}(A)$ if and only if $\varphi_{t_0}(\Delta) \in \mathfrak{B}(\varphi_{t_0}(A))$ and we have

$$E(\Delta; A) = E(\varphi_{t_0}(\Delta); \varphi_{t_0}(A)). \tag{14}$$

Hence $A$ is strongly stable if and only if $\varphi_{t_0}(A)$ is so. By definition a strongly stable operator has a real regular point if $\varkappa_+(\mathscr{H}) \neq 0$ and $\varkappa_-(\mathscr{H}) \neq 0$.

Bounded strongly stable operators can be characterized in the following way: The bounded $J$-selfadjoint operator $A$ in the Kreĭn space $\mathscr{H}$ is strongly stable if and only if it is similar to a selfadjoint operator and there exists a $\delta > 0$ such that all the $J$-selfadjoint operators $B$ with the property $\|B - A\| \leqq \delta$ are also similar to selfadjoint operators. This is just the "selfadjoint version" of [7], Theorem 8.

We denote by $\Pi$ the class of definitizable $J$-selfadjoint operators $A$ in the Kreĭn space $\mathscr{H}$ with the properties $\rho(A) \neq \emptyset$, $\sigma_{0,\infty}(A) = \emptyset$ and $c_\infty(A) = \emptyset$. Evidently, if $\mathscr{H}$ is a Pontrjagin space then $\Pi$ is the set of all $J$-selfadjoint operators in $\mathscr{H}$. In an arbitrary Kreĭn space, if $A$ is $J$-selfadjoint and $t_0 \in \rho(A) \cap \mathbf{R}$, then (14) implies that $A \in \Pi$ if and only if $(A - t_0 I)^{-1} \in \Pi$.

THEOREM 2. *Let $A_0$ and $A_1$ be $J$-selfadjoint operators in the Kreĭn space $\mathscr{H}$ with $\rho(A_0) \cap \rho(A_1) \neq \emptyset$ and*

$$(A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1} \in \mathfrak{S}_\infty{}^{5)} \tag{15}$$

*for some (and hence for all) $\lambda \in \rho(A_0) \cap \rho(A_1)$. If $A_0$ is strongly stable, then $A_1$ belongs to the class $\Pi$.*

*Proof.* Condition (15) implies that $\sigma(A_1)$ is discrete in $\rho(A_0)$ (see [3]). Therefore $\lambda$ in (15) can be chosen real. By the above remark, it is sufficient to prove the following special case of Theorem 2: Let $A_0$ and $A_1$ be bounded $J$-selfadjoint operators such that $D := A_1 - A_0 \in \mathfrak{S}_\infty$. If $A_0$ is strongly stable, then $A_1 \in \Pi$.

To see this we choose $\delta > 0$ such that $\|B - A_0\| \leqslant \delta$ implies that the $J$-selfadjoint operator $B$ is also strongly stable. Consider a decomposition $D = D' + D''$ where $D'$ and $D''$ are $J$-selfadjoint operators such that $\|D'\| \leqq \delta$ and $D''$ is finite-dimensional. Then $A_0 + D'$ is strongly stable, that is $c(A_0 + D') = \sigma_0(A_0 + D') = \emptyset$, therefore, by Theorem 1, $A_0 + D' + D'' \in \Pi$ and the theorem is proved.

COROLLARY. *The class $\Pi$ is closed with respect to $J$-selfadjoint compact perturbations, that is, if $A \in \Pi$, and $B \in \mathfrak{S}_\infty$ is $J$-selfadjoint, then $A + B \in \Pi$.*

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[5] $\mathfrak{S}_\infty$ is the class of compact operators in $\mathscr{H}$. If $A \in \mathfrak{S}_\infty$ we denote by $s_j(A)$, $j = 1, 2, \ldots$, $s_1(A) \geqq s_2(A) \geqq \ldots$, the $s$-numbers of $A$, that is the eigenvalues of $(A^*A)^{1/2}$ counted according to their multiplicity (see [3]).

Indeed, if $A \in \Pi$ then by Proposition 2 we can choose a finite-dimensional operator $B'$ such that $A + B'$ is strongly stable. Hence, by Theorem 2, $A + B = A + B' + (B - B')$ is of class $\Pi$.

As in Theorem 1, an upper bound for the number of non-real eigenvalues and critical points of $A_1$ in Theorem 2 can be given. If, e.g., $A_0$ and $A_1$ in Theorem 2 are bounded operators, we have the following result:

Let $m$ be the smallest integer such that

$$s_{\left[\frac{m}{2}\right]+1}(A_1 - A_0) < \frac{1}{2}\delta(A_0)\,\gamma(A_0)\,\Gamma(A_0)^{-1},\quad \delta(A_0) := \operatorname{dist}(\sigma_+(A_0), \sigma_-(A_0)), \tag{16}$$

and denote by $n_0$ the degree of a definitizing polynomial of $A_0$. Then

$$\sum_{t \in c(A_1)} \varkappa(t; A_1) + \sum_{\lambda \in \sigma_0(A_1),\ \operatorname{Im}\lambda > 0} \dim E(\{\lambda\}; A_1)\,\mathscr{H} \leqq mn_0. \tag{17}$$

In particular, the set $c(A_1) \cup (\sigma_0(A_1) \cap \{\lambda\colon \operatorname{Im}\lambda > 0\})$ consists of no more than $mn_0$ points.

Indeed, we find real numbers $\alpha_0, \alpha_1, \dots, \alpha_{n_1}, \alpha_{n_1+1}$, $n_1 \leqq n_0$, satisfying

$$\operatorname{dist}(\alpha_j, \sigma(A_0)) \geqq \frac{1}{2}\,\delta(A_0) \tag{18}$$

such that we have

$$\sigma(A_0) \subset \bigcup_{j=1}^{n_1+1} (\alpha_{j-1},\ \alpha_j)$$

and for each $j = 1, \dots, n_1 + 1$

$$\text{either}\quad \sigma(A_0) \cap (\alpha_{j-1}, \alpha_j) \subset \sigma_+(A_0) \quad\text{or}\quad \sigma(A_0) \cap (\alpha_{j-1}, \alpha_j) \subset \sigma_-(A_0)$$

holds. It is easy to see that there exists a definitizing polynomial $p$ for $A_0$ such that the zeros of $p$ are among the $\alpha_1, \dots, \alpha_{n_1}$ and simple.

Let $D'$ be a $J$-selfadjoint operator with

$$\|D'\|_0 < \frac{1}{2}\,\delta(A_0). \tag{19}$$

We consider the operators $A_0 + \varepsilon D'$, $\varepsilon \in [0, 1]$. Since $A_0$ is selfadjoint with respect to $(.,.)_0$ from (19) and (18) we derive that

$$\|\varepsilon(A_0 - \alpha_j I)^{-1} D'\|_0 < \varepsilon[\operatorname{dist}(\alpha_j, \sigma(A_0))]^{-1}\frac{1}{2}\,\delta(A_0) \leqq \varepsilon \leqq 1, \qquad j = 1, \dots, n_1.$$

Thus, by the relation

$$(A_0 + \varepsilon D' - \alpha_j I)^{-1} = (I + \varepsilon(A_0 - \alpha_j I)^{-1} D')^{-1} (A_0 - \alpha_j I)^{-1},\quad j = 1, \dots, n_1,$$

we obtain

$$\alpha_j \in \rho(A_0 + \varepsilon D'), \quad j = 1, \ldots, n_1,$$

for every $\varepsilon \in [0, 1]$. Now using [11], Lemma 1 we see that $A_0 + D'$ is strongly stable with a definitizing polynomial $p$ of degree not greater than $n_0$. Further, by (13), the relation

$$\|D'\| < \frac{1}{2} \delta(A_0)\, \gamma(A_0)\, \Gamma(A_0)^{-1}$$

implies (19).

If $\mathscr{F}_k$ denotes the set of $k$-dimensional operators and $\mathscr{F}'_k$ the set of $J$-selfadjoint $k$-dimensional operators, then (see [3]) we have for every compact operator $T$:

$$s_{\left[\frac{m}{2}\right]+1}(T) = \inf_{F \in \mathscr{F}_{\left[\frac{m}{2}\right]}} \|T - F\| = \inf_{F \in \mathscr{F}_{\left[\frac{m}{2}\right]}} \frac{1}{2} (\|T - F\| + \|T^+ - F^+\|) \geqq$$

$$\geqq \inf_{F \in \mathscr{F}_{\left[\frac{m}{2}\right]}} \|T - \frac{1}{2}(F + F^+)\| \geqq \inf_{F \in \mathscr{F}'_m} \|T - F\|,$$

where $T^+$ is the $J$-adjoint of $T$. Together with condition (16) this inequality implies that

$$\frac{1}{2} \delta(A_0)\, \gamma(A_0)\, \Gamma(A_0)^{-1} > \inf_{F \in \mathscr{F}'_m} \|A_1 - A_0 - F\|.$$

Hence there exists an $m$-dimensional $J$-selfadjoint operator $F'$ such that $A_1 - F'$ is strongly stable with a definitizing polynomial of degree $\leqq n_0$. Now the conclusion (17) follows from the Remark after Theorem 1.

REMARK. In Theorem 2 as well as in (17) condition (15) can be weakened: It is enough to suppose that the difference $(A_1 - \lambda I)^{-1} - (A_0 - \lambda I)^{-1}$ is the sum of a sufficiently small and a finite-dimensional operator.

Finally, we prove a "negative" result about compact perturbations of definitizable operators.

PROPOSITION 3. *Let $A$ be a bounded definitizable operator in the Kreĭn space $\mathscr{H}$, which is not of class $\Pi$, that is $\sigma_{0,\infty}(A) \cup c_\infty(A) \neq \emptyset$. Then there exists a compact $J$-selfadjoint operator $K$ such that $A + K$ is not definitizable.*

*Proof.* (1) If $\sigma_{0,\infty}(A) \neq \emptyset$, we may suppose that $\sigma(A) = \{\alpha, \bar{\alpha}\}$, $\alpha \neq \bar{\alpha}$. Consider the decomposition

$$\mathscr{H} = \mathscr{H}_\alpha \dot{+} \mathscr{H}_{\bar{\alpha}} \quad , \quad \mathscr{H}_\alpha := E(\{\alpha\})\mathscr{H}, \ \mathscr{H}_{\bar{\alpha}} := E(\{\bar{\alpha}\})\mathscr{H}.$$

Then $A$ has the matrix form

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_0^+ \end{pmatrix},$$

where $A_0^+$ is the adjoint of $A_0$ with respect to the duality $[.,.]$ between $\mathcal{H}_\alpha$ and $\mathcal{H}_{\bar{\alpha}}$ (see [1]). The lengths of the Jordan chains of $A_0$ corresponding to $\alpha$ are bounded by the order of the zero $\alpha$ of the definitizing polynomial (see [8] or [5]). Thus $A_0 - \alpha I$ has an infinite-dimensional null space $\mathcal{N}_\alpha$. Let $(e_i)$ be an infinite orthonormal system in the Hilbert space $\mathcal{H}_\alpha$ (with respect to $(.,.)$), $e_i \in \mathcal{N}_\alpha$, $i = 1, 2, \ldots$, and

$$K_0 := \sum_{i=1}^{\infty} \varepsilon_i (.,e_i)\, e_i.$$

Here $\varepsilon_i$ are non-zero complex numbers with $\varepsilon_i \neq \varepsilon_j$ for $i \neq j$ such that the sum converges in the operator norm (or in one of the $\mathfrak{S}_p$-norms [6], $p \geqq 1$). For

$$A + K := \begin{pmatrix} A_0 + K_0 & 0 \\ 0 & A_0^+ + K_0^+ \end{pmatrix}$$

we have

$$(A + K)\, e_i = (\alpha + \varepsilon_i)\, e_i.$$

Thus $\alpha \neq \bar{\alpha}$ is an accumulation point of $\sigma(A + K)$ and $A + K$ is not definitizable.

(2) Assume $c_\infty(A) \neq \emptyset$ for the rest of the proof. Evidently, we may suppose $0 \in c_\infty(A)$. Moreover, it is sufficient to prove the statement for the restriction of $A$ to $E(\Delta)\mathcal{H}$ where $E$ is the spectral function of $A$, $\Delta \in \mathfrak{B}(A)$ with $0 \in \Delta$. Therefore we may suppose in the following

(20) $$[A^n x, x] \geqslant 0 \ (x \in \mathcal{H}) \text{ for some integer } n,\ c_\infty(A) = \{0\}.$$

(a) Suppose first that the root subspace $\mathcal{L}_0$ of $A$ corresponding to zero contains an infinite-dimensional subspace $\mathcal{Y} \subset \mathfrak{P}_0$ invariant under $A$. As the Jordan chains of $A$ have length $\leqq n + 1$, the null space of $A$ also contains an infinite-dimensional subspace $\mathcal{Y}' \subset \mathfrak{P}_0$. We choose an infinite orthonormal system $(e_j)$ in $\mathcal{Y}'$ and define $P_j := (., e_j)e_j$, $K_j = i\,\varepsilon_j P_j - i\,\varepsilon_j JP_jJ$ with $\varepsilon_j$ real, $\varepsilon_j \neq \varepsilon_k$ if $j \neq k$, $j, k = 1, 2, \ldots$. The operators $K_j$ are $J$-selfadjoint and two-dimensional and we have $K_j e_j = i\varepsilon_j e_j$, $K_j Je_j = -i\varepsilon_j Je_j$, $K_k e_j = K_k Je_j = 0$ for $k \neq j$. If the $\varepsilon_j$ are chosen small enough then $K := \sum_{j=1}^{\infty} K_j$ is compact, even belongs to an arbitrary class $\mathfrak{S}_p$, $p \geqq 1$. Then $A + K$ is $J$-selfadjoint and has infinitely many non-real eigenvalues, hence it is not definitizable.

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[6] For the definition of the classes $\mathfrak{S}_p$, $p \geqq 1$, see [3].

(b) Assume now (20) with $n \geqq 3$. Then there exists (see [9]) a subspace $\mathcal{N}' \subset \mathfrak{P}_0$ such that

$$(i)\quad A\mathcal{N}' \subset \mathcal{N}',\ (ii)\ \overline{(A\mathcal{N}'^{[\perp]} + \mathcal{N}')} \cap (A\mathcal{N}'^{[\perp]})^{[\perp]} \subset \mathcal{N}'^{7)}.$$

The subspace $\mathcal{N}'$ is contained in the root space $\mathcal{L}_0$. Indeed, $x \in \mathcal{N}'$ implies $[A^n x, x]=0$, hence by (20) and the Schwarz' inequality we find $A^n x=0$. If $\dim \mathcal{N}' = \infty$ the statement follows from (a). If $\dim \mathcal{N}' < \infty$, consider the subspace $\mathcal{N}' + J\mathcal{N}'$ of $\mathcal{H}$. Let $P_0$ be the selfadjoint and $J$-selfadjoint projector onto $\mathcal{N}' + J\mathcal{N}'$. Then with respect to the decomposition $\mathcal{H} = (I - P_0)\mathcal{H} \oplus \mathcal{N}' \oplus J\mathcal{N}'$ the operator $A$ has the representation

$$A = \begin{pmatrix} A_{11} & 0 & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + A'.$$

with a finite-dimensional operator $A'$. Evidently, $[A_{11}^n x, x] \geqq 0$ $(x \in (I - P_0)\mathcal{H})$, and Theorem 1 implies $0 \in c_\infty(A_{11})$. Moreover, from $(ii)$ it follows (see [9]) that

$$\overline{\mathcal{R}(A_{11}) + \mathcal{N}(A_{11})} = (I - P_0)\,\mathcal{H}.$$

Therefore, if $n \geqq 3$ we have also $[A_{11}^{n-2}x, x] \geqq 0$ $(x \in (I - P_0)\mathcal{H})$. If we show that there exists a compact operator $K_{11}$ in $(I - P_0)\mathcal{H}$ such that $A_{11} + K_{11}$ is not definitizable, then also

$$\begin{pmatrix} A_{11} + K_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and hence, by Theorem 1, also

$$A + \begin{pmatrix} K_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not definitizable. Therefore the case of a general $n$ in (20) reduces to one of the following cases:

(21) $$[A^2 x, x] \geqq 0 \quad (x \in \mathcal{H})$$

or

(22) $$[Ax, x] \geqq 0 \quad (x \in \mathcal{H}).$$

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[7] For a subspace $\mathcal{N}$ of $\mathcal{H}$ we write $\mathcal{N}^{[\perp]} := \{x : [x, \mathcal{N}] = \{0\}\}$.

Moreover, in both cases we may suppose that $\overline{\mathscr{R}(A) + \mathscr{N}(A)} = \mathscr{H}$, $c_\infty(A) = \{0\}$ and, by the remark preceding (20), that if for some $\varepsilon > 0$ we have $(-\varepsilon, 0) \cap \sigma(A) = \emptyset$ or $(0, \varepsilon) \cap \sigma(A) = \emptyset$ then $\sigma(A)$ is even non-negative or non-positive, respectively.

(c) Suppose now (21). Then $\mathscr{R}(A) \subset \mathfrak{P}_+$ and $\mathscr{N}(A) = \mathscr{R}(A)^{[\perp]}$ contains an infinite-dimensional non-positive subspace. According to (a) we may assume that $\mathscr{N}(A)$ contains even an infinite-dimensional negative subspace $\mathscr{N}$.

If $\sigma(A) \neq \{0\}$, then by the remark at the end of (b) there exists a sequence $(\lambda_j) \subset \sigma(A) \subset \mathbf{R}$ such that $\lambda_j \neq 0$, $\lambda_j \neq \lambda_k$ if $j \neq k$, $j, k = 1, 2, \ldots, \lambda_j \to 0$ $(j \to \infty)$.

Consider an infinite system $(e_j)$ in $\mathscr{N}$ with $[e_j, e_k] = 0$ for $j \neq k$. Define

$$P_j = [e_j, e_j]^{-1} [., e_j] e_j$$

and consider a sequence $(k_\nu)$ of natural numbers such that $K := \sum_\nu \lambda_{k_\nu} P_\nu$ converges in the norm of the class $\mathfrak{S}_p, p \geqq 1$. Then the operator $A + K$ has infinitely many critical points $\lambda_{k_\nu}$, $\nu = 1, 2, \ldots$, hence it is not definitizable.

If $\sigma(A) = \{0\}$ then $\mathscr{H} = \mathscr{L}_0$. Then there exist (see [9]) a maximal non-negative and a maximal non-positive subspace of $\mathscr{H}$ both invariant under $A$. We may suppose (according to (a)) that both subspaces have only finite-dimensional isotropic parts, and it is easy to see that there exist a positive and a negative infinite-dimensional subspace of $\mathscr{H}$ both invariant under $A$. Now by a similar construction as in the case $\sigma(A) \neq \{0\}$ one can find a perturbation of $A$ which is not definitizable, moreover, the perturbation can be chosen from an arbitrary class $\mathfrak{S}_p, p \leqq 1$.

(d) It remains to consider the case (22). If $(-\infty, 0) \subset \rho(A)$ or $(0, \infty) \subset \rho(A)$ then there exists a definitizing polynomial of even degree and (c) applies. Therefore we may suppose that for each $\varepsilon > 0$ we have

$$(23) \qquad (-\varepsilon, 0) \cap \sigma(A) \neq \emptyset, \quad (0, \varepsilon) \cap \sigma(A) \neq \emptyset.$$

Consider the selfadjoint non-negative operator $JA$. Assume $\dim \mathscr{N}(JA) < \infty$. Then there is no interval $(0, \delta)$, $\delta > 0$, which belongs to $\rho(JA)$, as otherwise we could find a finite-dimensional selfadjoint operator $F$ such that $0 \in \rho(JA + F)$. This would imply $0 \in \rho(A + JF)$, a contradiction to our assumption $0 \in c_\infty(A)$ by Theorem 1.

With the aid of a theorem of Weyl-von Neumann (see [6], Theorem X.2.1) it is easy to see that there exists a selfadjoint operator $K_1 \in \mathfrak{S}_2$ (or even $\in \mathfrak{S}_p, p > 1$) such that $JA + K_1$ is non-negative and there exists a sequence $(\lambda_j)$ of positive eigenvalues of $JA + K_1$ with $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$. Let $e_j$ be an eigenvector of $JA + K_1$ corresponding to $\lambda_j$, $\|e_j\| = 1, j = 1, 2, \ldots$, and define

$$K_2 := -\sum_{j=1}^{\infty} \lambda_j (., e_j) e_j.$$

Then $K_2 \in \mathfrak{S}_2$, and $JA + K_1 + K_2$ is selfadjoint, non-negative and we have $\dim \mathscr{N}(JA + K_1 + K_2) = \infty$.

Then also

$$\dim \mathcal{N}(A + JK_1 + JK_2) = \infty.$$

As the perturbation $JK_1 + JK_2$ is compact, in the case (23) for the proof of the proposition we may additionally require $\dim \mathcal{N}(A) = \infty$.

Now, under conditions (22) and (23), if $\mathcal{N}(A)$ contains an infinite-dimensional subspace of $\mathfrak{P}_0$ the statement follows from (a). If $\mathcal{N}(A)$ contains an infinite-dimensional positive or negative subspace the statement follows as in (c).

Proposition 3 is proved.

REMARK. As the proof shows, the operator $K$ in Proposition 3 can be chosen in an arbitrary class $\mathfrak{S}_p$, $p > 1$. Very likely, the restriction $p \neq 1$ is just a technical one.

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