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A NEW PROOF OF THE SZEGÖ LIMIT THEOREM AND NEW RESULTS FOR TOEPLITZ OPERATORS WITH DISCONTINUOUS SYMBOL

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§ 1. INTRODUCTION

The Strong Szegö Limit Theorem computes the limit as $N \to \infty$ of

$$D_N[\varphi] = \det P_N T_\varphi P_N$$

after appropriate normalization. Here T_φ is the Toeplitz operator with (nice) generating function φ and P_N is the projection onto the first $N+1$ coordinates. To an operator theorist the result seems strange since Toeplitz operators are not generally determinant class operators or even close to them. Thus even existence of the limit seems bizarre. In this paper we show that $P_N T_\varphi P_N$ has a direct connection with a determinant class operator and, in fact, splits into two canonical and extremely stable parts. From this we can swiftly obtain the Strong Szegö Limit Theorem.

The basic argument which illustrates the above claim is as follows. Suppose $\varphi = \varphi_+ \varphi_-$ where $\varphi_+, \overline{\varphi}_-, \dfrac{1}{\varphi_+}, \dfrac{1}{\overline{\varphi}_-} \in H^\infty$. Then $T_\varphi = T_{\varphi_-} T_{\varphi_+}$. Now it is very simple to verify that

$$P_N T_{\varphi_+} = P_N T_{\varphi_+} P_N \text{ and } T_{\varphi_-} P_N = P_N T_{\varphi_-} P_N.$$

Thus, if the order of $T_{\varphi_-} T_{\varphi_+}$ in $D_N[\varphi]$ were reversed we would have

$$\det P_N T_{\varphi_+} T_{\varphi_-} P_N = \det P_N T_{\varphi_+} P_N \det P_N T_{\varphi_-} P_N .$$

Let f_k denote the k^{th} Fourier coefficient of the function f. Then the above term is easily computed to be $[\varphi_{+0} \varphi_{-0}]^{N+1}$. The order, however, can be reversed if we are willing to compensate with a multiplicative commutator. To wit,

$$P_N T_\varphi P_N = P_N T_{\varphi_+} P_N T_{\varphi_+}^{-1} T_{\varphi_-} T_{\varphi_+} T_{\varphi_-}^{-1} P_N T_{\varphi_-} P_N.$$

Note that the middle term is a multiplicative commutator, namely, it has the form $\{A, B\} = ABA^{-1}B^{-1} = I + [A, B]A^{-1}B^{-1}$. Furthermore, multiplicative commutators of Toeplitz operators with smooth generating function are determinant class since $[T_\varphi, T_\psi]$ is trace class. Therefore,

$$\lim_{N\to\infty} D_N[\varphi]/[\varphi_{+0}\varphi_{-0}]^{N+1} =$$

$$(1.1) \qquad = \lim_{N\to\infty} \det P_N\{T_{\varphi_+}^{-1}, T_{\varphi_-}\}P_N =$$

$$= \det\{T_{\varphi_+^{-1}}, T_{\varphi_-}\}.$$

This argument gives us existence of the limit provided a nice outer factorization $\varphi = \varphi_+\varphi_-$ exists.

Furthermore a general theory of traces of additive commutators and determinants of multiplicative commutators was developed in [H-H] which gives broadly applicable formulas for computation. In particular, the simple computation for additive commutators done in § 1 [H-H] and the fact that invariants for multiplicative commutators actually reduce to ones for additive commutators give immediately Szegö's classical explicit formula for the limit. The fact that multiplicative invariants reduce to additive ones was conjectured by Helton and Howe and proved by Pincus ([P], § 10 [H-H], [Bn]). Also Berger and Shaw [B-S] proved a special case of the additive formula for Toeplitz operators and used it very effectively for other purposes. The first application of [H-H] to the Szegö Limit Theorem was made by Widom [W1].He showed the limit in (1.1) is det $T_\varphi T_{\varphi^{-1}}$ and expressed this directly as an exponential of a trace of a sum of commutators of Toeplitz operators. This establishes the connection and computes the limit explicitly. Earlier Gohberg and Feldman [G-F] used a consideration involving additive commutators to obtain the original version of Lemma 3.4 herein; then this was used to obtain the $\varphi_{+0}\varphi_{-0}$ term in (1.1). The main idea of our paper is that working directly with multiplicative commutators gives a saving which becomes dramatic when applied to Toeplitz operators with matrix or with discontinuous symbol.

In this paper we apply the argument above to two different situations. In Section 2 we use it to give a very fast proof of the Widom-Szegö Limit theorem for Toeplitz operators with matrix symbol. In Section 3 we apply it to Toeplitz operators with discontinuous generating function. This is of interest in studying the Ising model at a phase transition and most treatments of the problem are in the physics literature, cf. [M-W]. Our results simplify existing approaches and add considerably to what is known.

Now we set notation and terminology. The Hardy space H^p is defined by

$$H^p = \{f \in L^p(\mathbf{T}) \,|\, f_k = 0, k < 0\}.$$

We will be most concerned with H^2 in what follows. The projection from L^p to H^p is continuous for every $\infty > p > 1$ and it will be denoted by P. The dependence on p will be clear from the content.

The operators that will appear in the next sections will generally be either Toeplitz or multiplication operators. If $\varphi \in L^\infty$, they are defined classically as follows:

$$M_\varphi\colon L^2 \to L^2, \quad M_\varphi(f) = \varphi f \text{ (multiplication)}$$

$$T_\varphi\colon H^2 \to H^2, \quad T_\varphi(f) = P(\varphi f) \text{ (Toeplitz)}.$$

We will, however, be concerned with two different generalizations of these operators. The first concerns the matrix-valued symbol case. For a matrix valued function φ, T_φ is defined on

$$H^2(\mathbf{C}^n) = H^2 \oplus \ldots \oplus H^2$$

in the obvious way. The matrix $P_N T_\varphi P_N$ is then block Toeplitz and the $(i-j)^{\text{th}}$ block is φ_{i-j}.

The other generalization will be used in the third section and it will allow us to consider T_φ for $\varphi \in L^p$. In fact, if we define

$$T_\varphi\colon H^s \to H^t, \quad T_\varphi(f) = P(\varphi f),$$

it is easily verified that this definition yields a bounded operator as long as $\dfrac{1}{t} \geqslant \dfrac{1}{s} + \dfrac{1}{p}$ and $\infty > t > 1$. If $\varphi \in L^p$ and $\psi \in L^q$, then $T_\varphi T_\psi$ is a bounded operator from $H^s \to H^t$ provided $\dfrac{1}{t} \geqslant \dfrac{1}{s} + \dfrac{1}{p} + \dfrac{1}{q}\cdot$ The above remarks can also be made to apply to multiplication operators.

The notation $\tilde{\varphi}$ will indicate $\varphi\left(\dfrac{1}{z}\right) = \varphi(e^{-i\theta})$ for any $\varphi \in L^p$ and the unitary operator $\mathscr{U}\colon L^p \to L^p$ (or $H^p \to H^p$) is defined by $\mathscr{U}(\varphi) = \tilde{\varphi}$.

Finally, we end the Introduction with some remarks on the geometric mean. For a function $\varphi \in L^p$ such that $\log \varphi \in L^1$ the geometric mean $G[\varphi]$ is classically defined as $\exp\left[\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \log \varphi(\theta)\, d\theta\right]\cdot$ If a function is matrix valued then $G[\varphi]$ is defined as $\exp\left[\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \log \det \varphi(\theta) d\theta\right]\cdot$ We will define the geometric mean of a function φ with a factorization $\varphi_-\varphi_+$ by $\det(\varphi_{+0})\det(\varphi_{-0})$. This definition agrees with the classical definition when φ has a L^1 logarithm and it is more convenient for our purposes.

§2. MATRIX-VALUED SYMBOLS

A variant of the basic idea described in §1 gives Widom's theorem for block Toeplitz matrices immediately.

THEOREM 2.1 (Widom [W1]) *Suppose φ and $\varphi^{-1} \in L^\infty_{C^n}(\mathbf{T})$ satisfy either*

(a) *φ has a factorization $\varphi = \varphi_+\varphi_-$ such that*

$$\varphi_+, \varphi_-^*, \varphi_+^{-1}, \varphi_-^{*-1} \in H^\infty_{C^n}(\mathbf{T}),$$

or

(b) *the Toeplitz operators T_φ and $T_{\varphi^{-1}}$ are invertible and the matrices $A_N = P_N T_{\varphi^{-1}}^{-1} P_N$ satisfy $\|A_N^{-1}\| < M$ where $\| \quad \|$ denotes the usual operator norm on $H^2_{C^n}(\mathbf{T})$. If $[T_{\varphi^{-1}}, T_\varphi]$ is trace class (which is true if φ and $\varphi^{-1} \in C^3$ or if φ is an invertible element of K_n, the Banach algebra of all $n \times n$ matrix valued functions f satisfying $\operatorname{esssup}\|f(e^{i\theta})\| + \left\{\sum_{k=-\infty}^{\infty} |k|\,\|f_k\|^2\right\}^{\frac{1}{2}} < \infty$), then*

$$\lim_{n\to\infty} D_N(\varphi) G_{[\varphi]}^{-(N+1)} = \det(T_\varphi T_{\varphi^{-1}}),$$

where $G[\varphi] = \det(\varphi_{+0})\det(\varphi_{-0})$.

New proof under hypothesis (a). The difficulty with block Toeplitz matrices is that T_φ does not factor as $T_{\varphi_-}T_{\varphi_+}$ as it does for scalar φ. However, $T_{\varphi^{-1}}$ factors as $T_{\varphi_-^{-1}}T_{\varphi_+^{-1}}$. Thus

$$P_N T_\varphi P_N = P_N T_{\varphi_+} T_{\varphi_+^{-1}} T_\varphi T_{\varphi_-^{-1}} T_{\varphi_+^{-1}} T_{\varphi_+} T_{\varphi_-} P_N =$$

$$= P_N T_{\varphi_+} P_N P_N T_{\varphi_+^{-1}} T_\varphi T_{\varphi^{-1}} T_{\varphi_+} P_N P_N T_{\varphi_-} P_N ,$$

and so

$$\det P_N T_\varphi P_N = G[\varphi_+]^{N+1} \det P_N T_{\varphi_+^{-1}} T_\varphi T_{\varphi^{-1}} T_{\varphi_+} P_N G[\varphi_-]^{N+1}.$$

The conclusion of the theorem is now immediate.

New proof under hypothesis (b). Write

$$P_N T_\varphi P_N = P_N T_{\varphi^{-1}}^{-1} T_{\varphi^{-1}} T_\varphi P_N =$$

$$= P_N T_{\varphi^{-1}}^{-1} (I + \mathscr{J}) P_N ,$$

where $\mathscr{J}$ is trace class. This has the form

$$P_N T_\varphi P_N = P_N T^{-1}_{\varphi^{-1}} P_N (P_N + G_N \mathscr{J} P_N),$$

where $G_N = (P_N T^{-1}_{\varphi^{-1}} P_N)^{-1} P_N T^{-1}_{\varphi^{-1}}$. Soon we shall show that $G_N \to I$ strongly from which one can easily check (cf. [W1], Prop. 2.1) that $G_N \mathscr{J} P_N \to \mathscr{J}$ is trace norm. This implies

$$\lim_{n\to\infty} \frac{\det P_N T_\varphi P_N}{\det P_N T^{-1}_{\varphi^{-1}} P_N} = \det (I + \mathscr{J}) = \det (T_{\varphi^{-1}} T_\varphi) =$$

$$= \det (T_\varphi T_{\varphi^{-1}}).$$

To check that G_N really makes sense, to check that $G_N \to I$, and to evaluate $\det P_N T^{-1}_{\varphi^{-1}} P_N$ we resort to factoring φ as $\varphi_+ \varphi_-$. It is well known (cf. [W2], Theorem 1) that T_φ and $T_{\varphi^{-1}}$ invertible imply that such a factorization exists with

$$\varphi_+,\ \varphi_-^*,\ \varphi_+^{-1},\ \varphi_-^{-1*} \in H^2_{\mathbf{C}^n}(\mathbf{T}).$$

Since these functions are in L^2,

$$(P_N T^{-1}_{\varphi^{-1}} P_N x, y) = (P_N T_{\varphi_-} P_N x, P_N T^*_{\varphi_+} P_N y),$$

for all x, y despite the fact that T_{φ_-} and T_{φ_+} may be unbounded. Since P_N is finite rank we get

$$P_N T^{-1}_{\varphi^{-1}} P_N = P_N T_{\varphi_+} P_N P_N T_{\varphi_-} P_N,$$

and consequently the argument in part (a) applies to give $\det P_N T^{-1}_{\varphi^{-1}} P_N = G[\varphi]$. Moreover, the identity implies readily that $(P_N T^{-1}_{\varphi^{-1}} P_N)^{-1} = A_N^{-1}$ exists. Also since $A_N \to A = T^{-1}_{\varphi^{-1}}$ strongly and $\|A_N^{-1}\| < M$ we also have $A_N^{-1} \to (T^{-1}_{\varphi^{-1}})^{-1}$ strongly and thus $G_N \to I$ strongly. To see this write

$$A_N^{-1}x = A_N^{-1}AA^{-1}x - A_N^{-1}(A_N A^{-1}x) + A_N^{-1}(A_N A^{-1}x) =$$

$$= A^{-1}x - A_N^{-1}(A - A_N)(A^{-1}x).$$

The strong convergence follows from this identity.

The preceding discussion simplifies much of the literature for a function φ that is sufficiently nice, at least in the case where $[T_{\varphi^{-1}}, T_\varphi]$ is trace class. The results of Widom in [W1] for φ in $H^\infty_{\mathbf{C}^n} + C_{\mathbf{C}^n}$ do not require the invertibility conditions of Theorem 2.1 (b). One could extend Theorem 2.1 to that case by using the density arguments in [W1].

§3. SINGULAR GENERATING FUNCTION

The main result is a simple formula which computes how φ and ψ interact in the asymptotic expansion of $D_N[\varphi\psi]$, even when φ and ψ have (disjoint) rough places. What we see is that φ and ψ don't interact in the highest order terms of the expansion, so these expansions have a "local" character. Let us begin by describing precisely the classes of functions we study, then we give the main theorem.

DEFINITION. $\mathscr{E}_p$ is the class of measurable functions which factor as $\varphi = \varphi_+\varphi_-$ with $\varphi_+^{\pm 1} \in H^p$ and $\varphi_-^{\pm 1} \in \bar{H}^p$. The *singular points* of φ are

closure $\{e^{i\theta} \in \mathbf{T}$: one of $\varphi_\pm^{\pm 1}$ do not belong to C near $e^{i\theta}\}$.

Various descriptions of this class will be given later. Note if $\varphi \in \mathscr{E}_p$, then there is a determination of the r^{th} root φ^{+} which is in $\mathscr{E}_{p/r}$. So if φ is in any $\mathscr{E}_p$ class a root of it is in a high one.

THEOREM 1.1. *If φ and ψ in $\mathscr{E}_p$ for p sufficiently large ($p > 50$ will do) have disjoint singular points, then*

$$\lim_{N\to\infty} \frac{D_N[\varphi\psi]}{D_N[\varphi]D_N[\psi]} = \det\{T_{\varphi_-}, T_{\psi_+}\} \det\{T_{\psi_-}, T_{\varphi_+}\}.$$

The multiplicative commutators each have the form $I + \tau$ where τ is a trace class operator on H^2. Thus the determinants make good sense; also they can be computed, to wit

$$\det\{T_{\psi_-}, T_{\varphi_+}\} =$$

$$= \exp \frac{1}{2\pi i} \sum_{j=1}^{n} \left(\int_{l_j}^{\zeta_j} \frac{\varphi'_+}{\varphi_+} \ln \psi_- d\theta - \int_{\zeta_j}^{l_{j+1}} \frac{\psi'_-}{\psi_-} \ln \varphi_+ \, d\theta - \ln \varphi_+ \ln \psi_- \Big|_{l_j}^{\zeta_j} \right),$$

where the $[l_j, \zeta_j]$ are any intervals so that each one contains no singularity of φ_+^{-1}, all singularities of ψ_- are contained in $\bigcup_{j=1}^{n} [l_j, \zeta_j]$ and $l_1 = 0$, $l_{n+1} = 2\pi$. Here $\arg(\psi_-(\theta))$ *and* $\arg \varphi_+^{-1}(\theta)$ *are taken to be in $(-\pi, \pi)$.*

The theorem reduces the problem of computing asymptotic expansions for any function φ in $\mathscr{E}_{50}$ to computing the contribution of each singularity alone. Moreover, since Theorem 3.1 gives C^2 "locally" at an isolated singularity z_0 (if φ does not vanish) only behavior of the first two derivatives of φ and their jumps at z_0 can contribute to the high order terms of the expansion of $D_N[\varphi]$. Thus Theorem 3.1 reduces the problem of computing expansions for non-vanishing φ in $\mathscr{E}_p$, with isolated jumps, at which φ and all of its derivatives have finite right and left limits to computing the asymptotics of $D_N[\varphi]$ for φ of the form

$$\varphi(z) = (z-a)^\delta (z-b)^\mu (z-c)^\tau,$$

where the branch cuts all pass through one point z_0 on the circle.

Our result was motivated originally by a conjecture of Hartwig and Fischer [H-F]. They were concerned with functions of the form

$$\varphi(\theta) = \tau(\theta) \prod_{r=1}^{R} \left(1 - \frac{z}{z_r}\right)^{\alpha_r+\beta_r} \left(1 - \frac{z_r}{z}\right)^{\alpha_r-\beta_r}, \tag{3.0}$$

where $\tau(\theta)$ is a sufficiently nice function and z_r is on the unit circle. If $\alpha_r = 0$, then $\varphi(\theta)$ is piecewise continuous and if $\beta_r = 0$, then $\varphi(\theta)$ has possibly zeros and/or singularities. Hartwig and Fischer conjectured the form of the answer and in particular gave a complete characterization of how the factors $\left(1 - \frac{z}{z_r}\right)$, $\left(1 - \frac{z_r}{z}\right)$ should interact in the asymptotic formula. The conjecture was proved for some values of α_r and β_r, see [L], [H-F], [W3], [Ba]. The authors noticed how these formulas could be written in terms of determinants of multiplicative commutators and this led to Theorem 3.1. Notice that Theorem 3.1 gives information about the conjecture of Hartwig and Fischer for complex values of α_r and β_r. In particular Theorem 3.1 implies that the high order asymptotics of $D_N[\varphi]$ for φ of the form (3.0) with $|\alpha_r|$ and $|\beta_r| < \frac{1}{400}$ can be computed provided one can compute expansions for each $D_N\left[\left(1 - \frac{z}{z_r}\right)^{\alpha_r+\beta_r} \left(1 - \frac{z_r}{z}\right)^{\alpha_r-\beta_r}\right]$ individually.

In many cases an analytic continuation argument will probably allow us to extend the decomposition above to much larger classes of α_r and β_r. The point is that Theorem 3.1 establishes such a decomposition for an open set in $\mathbf{C}^{2n}$ of α's and β's and that forces uniqueness of any analytic continuation which might exist. This observation applies much more generally as we now describe. Suppose $\varphi, \psi \in \mathscr{E}_p$ for some p. Then roots φ^t, ψ^s exist and Theorem 3.1 applies and says that $C_N(t, s) = \frac{D_N(\varphi^t \psi^s)}{D_N(\varphi^t)\, D_N(\psi^s)}$ has a limit $C(t, s)$ for small enough t, s. Since each C_N is analytic a normal families argument implies that a subsequence converges to a function C^j *on any region where the* C_N *are uniformly* bounded. Our theorem forces each C^j to agree with C on a small open set. Thus we may conclude $\lim_{N\to\infty} C_N$ exists on any region where it is uniformly bounded (and computed thereby analytic continuation).

The analytic continuation argument just described relies on Theorem 3.1 holding for φ, ψ in $\mathscr{E}_p$ with some $p < \infty$. For $p = \infty$ the theorem is much easier to prove because $T_{\varphi_\pm}$ are bounded on H^2. Our main technical innovation here is to develope H^p arguments for dealing with T_ψ for ψ in L^r. Thus the basic Hilbert space tools involving determinant class and trace class operators must be extended to Banach spaces and to nuclear operators.

Suppose E and F are Banach spaces. Recall that $A: E \to F$ is nuclear if it has the representation $A = \sum_{i=0}^{\infty} z_i^* \otimes y_i$, where $z_i^* \in E^*$ and $y_i \in F$ and

$$s(A) \stackrel{\Delta}{=} \sum_{i=1}^{\infty} \|z_i^*\|_{E^*} \|y_i\|_F < \infty.$$

Here $\otimes$ is defined by

$$(z_i^* \otimes y_i)(x) = z_i^*(x) y_i ,$$

for all x in E. Call any $s(A)$ gotten as above a bound for A. Let $\|A\|_{\mathscr{L}(E,F)}$ denote the usual operator norm for A. For details about nuclear operators see [B]. The next proposition is a straightforward generalization of Proposition 1.1 in [W1].

LEMMA 3.2. *Let $A: E \to F$ be a compact operator (the norm limit of finite rank operators) and suppose*

$$C_N: D \to E \quad \text{and} \quad B_N: F \to G.$$

If B_N converges strongly to B and $C_N^*: E^* \to D^*$ converges strongly to C^*, then

(a) *$B_N A C_N \to BAC$ in the $\mathscr{L}(D, G)$ norm,*

(b) *if A is nuclear, if both B_N, C_N^* are uniformly bounded and if either of them converges strongly to zero, then there is a sequence of bounds $s(B_N A C_N)$ on $B_N A C_N$ which converges to 0.*

Proof. Choose a finite rank operator

$$J = \sum_{i=1}^{K} z_i^* \otimes y_i \tag{3.1}$$

so that $\|A - J\|_{\mathscr{L}(E,F)} < \varepsilon$. Then

$$\|[B_N J C_N - BJC]\, x\|_G = \|(B_N - B)\, J C_N x + BJ(C_N - C)x\|_G ,$$

and which is less than

$$\|x\|_D \sum_i^K \|z_i\|_E \|C_N\|_{\mathscr{L}(D,E)} \|(B_N - B)\, y_i\|_G + \|x\|_D \sum_i^K \|(C_N^* - C^*)(z_i^*)\|_{D^*} \|B y_i\|_G .$$

The strong convergence hypothesis says that for fixed K each of these two terms converge to zero. This proves (a).

To prove (b) represent A as in (3.1) with $K = \infty$. Then

$$B_N A C_N x = \sum_i^{\infty} z_i^*(C_N(x))\, B_N y_i ;$$

set

$$s(B_N A C_N) = \sum_i^{\infty} \|C_N^* z_i^*\|_{E^*} \|B_N y_i\|_F.$$

Split $\sum^{\infty}$ into $\sum^{K} + \sum_{K+1}$ where K is chosen large enough to make $\sum_{K+1}$ less than $\frac{\varepsilon}{2}$ independent of N. Strong convergence of C_N^* or B_N to zero implies that $\sum^{K}$ converges to 0 as $N \to \infty$ and so it is eventually less than $\frac{\varepsilon}{2}$. So for large enough N

$$s(B_N A C_N) < \varepsilon.$$

In addition to Toeplitz operators T_φ and multiplication operators M_φ we shall need Hankel operators H_φ defined by $M_{z^{-1}} \cdot u \cdot (M_{\tilde{\varphi}} - T_{\tilde{\varphi}})$ where u and $\tilde{\varphi}$ are as defined in the Introduction. Note φ and $\tilde{\varphi}$ have singular points which are complex conjugates of each other. Basic facts about Toeplitz and Hankel operators are summarized in

LEMMA 3.3. *Let $\varphi \in L^p$ and define t by*

$$\frac{1}{t} = \frac{1}{p} + \frac{1}{w}.$$

Then

(a) *$T_\varphi: H^w \to H^t$ is bounded when $t \neq 1, \infty$; also $\|T_\varphi\|_{\mathscr{L}(w,t)} \leqslant K_t \|\varphi\|_p$.*
(b) *$H_\varphi: H^w \to H^t$ is compact if $w > t > 1$.*
(c) *$(T_\varphi)^*: H^{t'} \to H^{w'}$ equals $T_{\bar{\varphi}}$ on $H^{t'} \cap H^t$.*
(d) *If $\varphi, \psi \in L^{2p}$, then*

$$T_{\varphi\psi} - T_\varphi T_\psi = H_\varphi H_{\tilde{\psi}}$$

on H^w provided $w > p'$. Consequently if $\varphi \in \overline{H}^{2p}$ and $\psi \in H^{2p}$, then $T_{\varphi\psi} = T_\varphi T_\psi$.

(e) *If $\varphi \in C^3$, then $H_\varphi: H^w \to H^\infty$ is nuclear for any $w \geqslant 1$.*
(f) *If φ in L^p and ψ in L^q have disjoint singular points, then $H_\varphi H_{\tilde{\psi}} : H^w \to H^s$ is nuclear provided $w > 1$ and $\frac{1}{s} \geqslant \frac{1}{p} + \frac{1}{q}$.*

Proof. a) If $f \in H^w$ then Holder's inequality says

$$\int |M_\varphi f|^t \leqslant \left(\int |f|^w\right)^{t/w} \left(\int |\varphi|^p\right)^{t/p}.$$

Thus M_φ is in $\mathscr{L}(w, t)$. The projection P from L^t to H^t has norm K_t provided $t \neq 1, \infty$. Thus T_φ and H_φ have norm bounded by K_t.

(b) If ψ is a trigonometric polynomial then the range of H_ψ is finite dimensional. Any φ in L^p can be approximated in $\| \quad \|_p$ by trigonometric polynomials, and since $\|H_\varphi - H_\psi\|_t \leqslant K_t\|\varphi - \psi\|_p$ the operator H_φ is the norm limit of finite rank operators.

(c) is straightforward.

(d) First note that part (a) implies $T_\varphi T_\psi$, $T_{\varphi\psi}$, and $H_\varphi H_{\tilde{\psi}}$ are in $\mathscr{L}(w, s)$ provided $\dfrac{1}{w} + \dfrac{1}{2p} + \dfrac{1}{2p} = \dfrac{1}{s}$; thus $w > p'$ insures $s > 1$. That the identity we wish to prove holds on $P_N H^w$ is standard. So it holds on all H^w by continuity. For $\overline{\varphi}$ in H^p the operator $H_\varphi = 0$.

(e) If $\varphi \in C^3$, then H_φ can be written as an integral operator with two times differentiable kernel. In fact the operator H_φ can be written as

$$H_\varphi(f)(\xi) = M_{z^{-1}}\left(\frac{1}{2\pi i}\int_\Gamma \left(\frac{\tilde{\varphi}\left(\frac{1}{\xi}\right) - \tilde{\varphi}(z)}{\frac{1}{\xi} - z}\right) f(z)\,dz\right)$$

where Γ is the unit circle. To see this recall $H_\varphi(f)(\xi) = M_{\xi^{-1}}\cdot\mathscr{U}\cdot(M_{\tilde{\varphi}} - T_{\tilde{\varphi}})(f)$. The operator

$$\mathscr{U}\cdot(M_{\tilde{\varphi}} - T_{\tilde{\varphi}})(f) =$$

$$= \mathscr{U}\cdot(\tilde{\varphi}(f)(\xi) - P(\tilde{\varphi}(f)(\xi))) =$$

$$= \mathscr{U}\cdot\left(\frac{1}{2\pi i}\int_\Gamma \frac{\tilde{\varphi}(\xi)f(\xi) - \tilde{\varphi}(z)f(z)}{\xi - z}\,dz\right) = \qquad \text{for } |\xi| < 1$$

$$= \mathscr{U}\cdot\left(\frac{1}{2\pi i}\int_\Gamma \frac{\tilde{\varphi}(\xi)f(\xi) - \tilde{\varphi}(\xi)f(z) + \tilde{\varphi}(\xi)f(z) - \tilde{\varphi}(z)f(z)}{\xi - z}\,dz\right) =$$

$$= \mathscr{U}\cdot\left(\frac{1}{2\pi i}\int_\Gamma \frac{(\tilde{\varphi}(\xi) - \tilde{\varphi}(z))f(z)}{\xi - z}\,dz\right) = \qquad (\text{since } f \text{ is in } H^2)$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{\varphi}\left(\frac{1}{\xi}\right) - \tilde{\varphi}(z)}{\frac{1}{\xi} - z}\,dz.$$

This kernel then will be C^2 if $\varphi(z)$ is C^3 and thus H_φ maps H^1 into C^2. The imbedding of C^2 in L^r for any $r \geqslant 1$ is well known to be nuclear. To check this one must show that the operator $\sum_{-\infty}^{\infty} x_j \otimes z^j : C^2 \to L^\infty$ is nuclear; there $z^j = e^{ij\theta}$ and x_j is the linear functional on C^2 given by $x_j(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f e^{-ij\theta}\, d\theta$. Clearly $\|z_j\|_{L^\infty} = 1$ and integration by parts gives

$$|x_j(f)| \leqslant \frac{1}{|j|^2}\left\|\frac{d^2 f}{d\theta^2}\right\|_\infty \leqslant \frac{1}{|j|^2}\|f\|_{C^2}.$$

Thus

$$\sum_j \|x_j\|_{(C^2)^*}\|z^j\|_\infty \leqslant \sum \frac{1}{|j|^2} < \infty.$$

Thus $H_\varphi : H^1 \to H^\infty$ is nuclear. Note, weaker conditions on φ would do.

(f) Choose smooth functions f, g such that $f + g = 1$ and both $f\varphi$ and $g\psi$ are in C^3. Use part (d) to obtain

$$H_\varphi H_{\tilde\psi} = T_{\varphi(f+g)\psi} - T_\varphi T_{f+g} T_\psi =$$

$$= T_{\varphi f\psi} - T_{\varphi f}T_\psi + T_{\varphi g\psi} - T_\varphi T_{g\psi} - H_\varphi H_{\tilde f} T_\psi - T_\varphi H_g H_{\tilde\psi} =$$

$$= H_{\varphi f}H_{\tilde\psi} + H_\varphi H_{g\tilde\psi} - H_\varphi H_{\tilde f}T_\psi - T_\varphi H_g H_{\tilde\psi}.$$

The product of a nuclear operator with a bounded operator is nuclear so parts (e), (a) imply that each term of this last expression is nuclear from H^w to H^s. The argument just given is a variant of one first given in [Wf].

The following more specialized fact about Toeplitz operators extends Theorem 1.1, p. 71 [G-F] (see also: Proposition 1.2 [W1]) to L^p symbols.

LEMMA 3.4. *Suppose* $\varphi_+^{\pm 1}, \overline{\varphi}_-^{\pm 1} \in H^{2p}$ *and set* $\varphi = \varphi_+\varphi_-$. *Then*

(a) $T_N[\varphi] := P_N T_\varphi P_N$ *is invertible for sufficiently large* N *and* $T_N[\varphi]^{-1}: H^w \to H^t$ *converges strongly to* $T_{\varphi_+^{-1}}T_{\varphi_-^{-1}}$. *Here* $\frac{1}{t} = \frac{5}{p} + \frac{1}{w}$.

(b) $T_N[\varphi] = T_N[\varphi_+]\,(T_{\varphi_+}^{-1}T_{\varphi_-}T_{\varphi_+}T_{\varphi_-}^{-1})\,T_N[\varphi_-]$.

Proof. (b) See Introduction.

(a) First note that $T_N[\varphi_+]^{-1}x \to T_{\varphi_+^{-1}}x$ in the L^s sense whenever $x \in H^r$ with

$$\frac{1}{s} < \frac{1}{r} + \frac{1}{2p}.$$

This is because $T_N[\varphi_+]^{-1}x = (T_{\varphi_+}P_N)^{-1}x = P_N T_{\varphi_+^{-1}}x$ and $(P_N - I) \to 0$ strongly in L^s. A similar statement holds for $T_N[\varphi_-]^{-1}$. Part (a) follows from this and part (b) provided we can show

$$P_N\{T_{\varphi_+^{-1}}, T_{\varphi_-}\}P_N^{-1} - P_N\{T_{\varphi_-}, T_{\varphi_+^{-1}}\}P_N \to 0\,,$$

in $\mathscr{L}(u, l)$ where $\frac{1}{u} = \frac{1}{2p} + \frac{1}{w}$ and $\frac{1}{l} = \frac{4}{p} + \frac{1}{u}$.

To prove the required convergence write

$$\{T_{\varphi_+^{-1}}, T_{\varphi_-}\} = I + [T_{\varphi_+^{-1}}, T_{\varphi_-}]T_{\varphi_+}T_{\varphi_-^{-1}} = I + A$$

where $A = -H_{\varphi_+^{-1}}H_{\tilde{\varphi}_-}T_{\varphi_+}T_{\varphi_-^{-1}}$ by Lemma 3.3(d). By Lemma 3.3(a)(b) the operator $A\colon H^m \to H^v$ is compact. The operator $\Gamma\colon H^v \to H^l$ defined by $\Gamma = \{T_{\varphi_-}, T_{\varphi_+}\}$ is bounded and has the property $\Gamma(I + A)\colon H^m \to H^l$ is the inclusion map. Here $\frac{1}{v} = \frac{1}{m} + \frac{1}{p}$ and $\frac{1}{l} = \frac{1}{v} + \frac{1}{p}\cdot$ Now

$$P_N\Gamma P_N(I + A)P_N = P_N + P_N\Gamma(P_N - I)(I + A)P_N =$$

$$= P_N + P_N\Gamma(P_N - I)A\, P_N,$$

and $(P_N - I)A\colon H^m \to H^v$ converges in norm to 0 by Lemma 3.2. So

$$\|P_N\Gamma P_N(I + A)P_N - P_N\|_{\mathscr{L}(m,\, l)} \to 0. \tag{3.2}$$

This implies $P_N(I + A)P_N^{-1}$ exists for sufficiently large N. Moreover, if

$$\|P_N(I + A)P_N^{-1}\|_{\mathscr{L}(u,\, m)} < K < \infty,$$

then we have

$$\|P_N\Gamma P_N - P_N(I + A)P_N^{-1}\|_{\mathscr{L}(u,\, l)} \to 0$$

which proves the lemma.

Set $P_N(1 + A)P_N = R_N$. We now prove $\|R_N^{-1}\|_{\mathscr{L}(u,\, m)} \leqslant K < \infty$ where $\frac{1}{m} = \frac{1}{u} + \frac{2}{p}\cdot$ Consider $\| R_N^{-1}g_N \|_m = \| P_Nf \|_m$ where $g_N = (I + A)P_Nf$. Since $\Gamma g_N = P_Nf$ and $\Gamma \in \mathscr{L}(u, m)$

$$\|\Gamma g_N\|_m \leqslant K\|g_N\|_u.$$

Thus

$$\|R_N^{-1}g_N\|_m = \|\Gamma g_N\|_m \leqslant K\|g_N\|_u.$$

The bounded on R_N follows from this because (3.2) implies the P_Ng_N are dense in range P_N.

Proof of Theorem 3.1. Begin by writing the analog of Lemma 3.3(d) for finite Toeplitz matrices. Let $\varphi \in L^p$, $\psi \in L^q$ then

$$T_N[\varphi\psi] = T_N[\varphi]T_N[\psi] + P_NH_\varphi H_{\tilde{\psi}}P_N + Q_NH_{\tilde{\varphi}}H_\psi Q_N\,, \tag{3.3}$$

where $Q_N(a_0, a_1, \ldots) = (a_N, a_{N-1}, \ldots, a_0)$. By Lemma 3.4(a) the operators $T_N[\varphi]$ and $T_N[\psi]$ are invertible and

$$T_N[\varphi\psi]T_N[\psi]^{-1}T_N[\varphi]^{-1} = I + A_N + B_N =$$

(3.4)

$$= (I + A_N)(I + B_N) - A_N B_N,$$

where

$$A_N = P_N H_\varphi H_{\tilde\psi} P_N T_N[\psi]^{-1} T_N[\varphi]^{-1}$$

$$B_N = Q_N H_{\tilde\varphi} H_\psi Q_N T_N[\psi]^{-1} T_N[\varphi]^{-1}.$$

Suppose φ and $\psi \in \mathscr{E}^{50}$ and have disjoint singular points. First we show that A_N converges to

$$A \overset{\Delta}{=} H_\varphi H_{\tilde\psi} T_{\psi_+^{-1}} T_{\psi_-^{-1}} T_{\varphi_+^{-1}} T_{\varphi_-^{-1}},$$

and that $Q_N B_N Q_N$ converges to

$$B = H_{\tilde\varphi} H_\psi T_{\tilde\psi_+^{-1}} T_{\tilde\psi_-^{-1}} T_{\tilde\varphi_+^{-1}} T_{\tilde\varphi_-^{-1}}$$

in the trace norm on H^2. By Lemma 3.3 (a, f) both A and B map H^2 into H^2 and are nuclear (which implies they are in the H^2 trace class). Now we prove $A_N \to A$. Lemma 3.4(a) applied to the adjoint of

$$T_N[\psi]^{-1} T_N[\varphi]^{-1} H^2 \to H^{10/9}$$

implies strong convergence to the adjoint of $T_{\psi_+^{-1}} T_{\psi_-^{-1}} T_{\varphi_+^{-1}} T_{\varphi_-^{-1}}$. By Lemma 3.3(f)

$$H_\varphi H_{\tilde\psi} : H^{\frac{10}{9}} \to H^{25}$$

is nuclear. So by Lemma 3.2 there is a sequence of bounds on

$$P_N H_\varphi H_{\tilde\psi}(T_N[\psi]^{-1} T_N[\varphi]^{-1} - T_{\psi_+^{-1}} T_{\psi_-^{-1}} T_{\varphi_+^{-1}} T_{\varphi_-^{-1}}),$$

which converges to 0. It is easy to check that any bound $s(w)$ on a trace class operator is bigger than the trace norm of the operator w (cf. Theorem 27I,[B]). So $A_N \to A$ in H^2 trace norm. The proof that $Q_N B_N Q_N \to B$ is similar once one observes $Q_N T_N[f] Q_N = T_N[\tilde f]$ and uses that to obtain

$$Q_N B Q_N = Q_N^2 H_{\tilde\varphi} H_\psi Q_N T_N[\psi]^{-1} Q_N Q_N T_N[\varphi]^{-1} Q_N$$

(3.4)′

$$Q_N B Q_N = H_{\tilde\varphi} H_\psi T_N[\tilde\psi]^{-1} T_N[\tilde\varphi]^{-1}.$$

Next we prove $A_N B_N \to 0$ in H^2 trace norm. First note that $Q_N\colon H^w \to H^w$ converges weakly to zero for any $w \neq \infty$. By Lemma 3.3(f) $H_{\tilde\varphi} H_\psi\colon H^{\frac{10}{9}} \to H^2$ is compact so its adjoint must be also. Thus $(Q_N H_{\tilde\varphi} H_\psi Q_N)^*$ and consequently the adjoint of $B_N\colon H^2 \to H^2$ converges strongly. That $(A_N - A)B_N + AB_N = A_N B_N \to 0$ follows from Lemma 3.2(b) and trace norm converge of A_N to A.

Use (3.4) to write

$$(3.5) \qquad \frac{D_N[\varphi\psi]}{D_N[\varphi]D_N[\psi]} = \det(I + A_N)(I + B_N)\det(I + E_N),$$

where $E_N = (I + A_N)^{-1} A_N B_N (I + B_N)^{-1}$. The $\det(I + E_N)$ term converges to 1 because $E_N\colon H^2 \to H^2$ converges to zero in trace norm. Consequently the limit as $N \to \infty$ of (3.5) is $\det(I + A)\det(I + B)$. To prove $E_N \to 0$ we need only show that $\|(I + A_N)^{-1}\|_{\mathscr{L}(2,2)}$ and $\|(I + B_N)^{-1}\|_{\mathscr{L}(2,2)} = \|(I + Q_N B_N Q_N)^{-1}\|_{\mathscr{L}(2,2)}$ are uniformly bounded. Since $A, B\colon H^2 \to H^2$ are trace class, uniform boundedness will hold if and only if $I + A$ and $I + B$ are invertible in $\mathscr{L}(2,2)$.

Now we analyze $I + A$ and $I + B$. If $x \in H^w$

$$\begin{aligned}(I + A)x &= I + [T_{\varphi\psi} - T_\varphi T_\psi](T_{\psi_+}^{-1} T_{\psi_-}^{-1} T_{\varphi_+}^{-1} T_{\psi_-}^{-1})x = \\ &= T_{\varphi\psi} T_{\psi_+}^{-1} T_{\psi_-}^{-1} T_{\varphi_+}^{-1} T_{\psi_-}^{-1} x = \\ &= T_{\varphi_-} T_{\psi_-} T_{\varphi_+} T_{\psi_-}^{-1} T_{\varphi_+}^{-1} T_{\psi_-}^{-1} x.\end{aligned}$$

These Toeplitz operators are viewed as bounded operators into successively larger H^r spaces provided $w > \frac{5}{4}$ to insure that these range spaces never get bigger than H^1. One consequence of the formula is that $(I + A)x$ is not 0 for any x in H^2. Since A is compact we conclude that $I + A$ is invertible in $\mathscr{L}(2,2)$ as required in the preceding paragraph. Another consequence is that $T_{\varphi_-}^{-1}(I + A)T_{\varphi_-} y$ equals the multiplicative commutator $\{T_{\psi_-}, T_{\varphi_+}\}y$ on y in H^l for $l > \frac{5}{4}$. The same argument which says that A is trace class implies that the operator $T_{\varphi_-}^{-1} A T_{\varphi_-}$ maps H^2 to H^2 and is also trace class. Thus

$$\{T_{\psi_-}, T_{\varphi_+}\} = I + \tau,$$

where τ is a H^2 trace class operator.

At last we compute

$$\begin{aligned}\det\{T_{\psi_-}, T_{\varphi_+}\} &= \lim_{N\to\infty} \det P_N T_{\varphi_-}^{-1}(I + A)T_{\varphi_-} P_N = \\ &= \lim_{N\to\infty} \det P_N(I + P_N T_{\varphi_-} P_N T_{\varphi_-}^{-1} A)P_N = \\ &= \lim_{N\to\infty} \det(I + A).\end{aligned}$$

The last equality holds because $P_N T_{\varphi_-} P_N T_{\varphi_-^{-1}}: H^3 \to H^2$ converges strongly and $A: H^2 \to H^3$ is nuclear, so the product converges in H^2 trace class to A.

The analysis of $\det(I + A)$ just completed applies to $\det(I + B)$. This proves the basic formula of Theorem 3.1.

Now we obtain the explicit formula for $\det\{T_{\psi_-}, T_{\varphi_+}\}$. To do this we approximate the H^p function $\varphi_+(e^{i\theta})$ by the function $\varphi_+(re^{i\theta})$ for $r < 1$. Likewise $\psi_-\left(\frac{1}{r}e^{i\theta}\right)$ converges in L^p norm to $\psi_-(e^{i\theta})$ as $r \uparrow 1$. There is a sequence $r_k \uparrow 1$ so that none of the functions $\varphi_+(r_k e^{i\theta})^{\pm 1}$ and $\psi_-(r_k e^{i\theta})^{\pm 1}$ vanish. Abbreviate $\varphi_+(r_k e^{i\theta})$ to $\varphi_k(e^{i\theta})$ and $\psi_-(r_k e^{i\theta})$ to $\psi_k(e^{i\theta})$. Then sections 1 and 10 of [H-H] apply directly to give

$$\det\{T_{\psi_k}, T_{\varphi_k}\} = \exp\frac{1}{2\pi i}\int_{\text{circle}} \frac{1}{\varphi_k}\frac{d\varphi_k}{d\theta}\ln\psi_k. \tag{3.6}$$

One can work through the proof that $\{T_{\psi_k}, T_{\varphi_k}\}$ is $I +$ trace class which we have just given and obtain

$$\{T_{\psi_k}, T_{\varphi_k}\} - \{T_\psi, T_\varphi\} \to 0$$

in trace norm. Thus

$$\det\{T_{\psi_k}, T_{\varphi_k}\} \to \det\{T_\psi, T_\varphi\}.$$

Next we must show that (3.6) converges to the integral in Theorem 3.1. Let f, g be a smooth partition of unity on the circle with $f\varphi_+$ and $g\psi_-$ in C^2. Insert $f + g$ in the integral, integrate appropriate terms by parts, and let $k \to \infty$ to get

$$\exp\frac{1}{2\pi i}\int_{\text{circle}} \frac{\varphi_+'}{\varphi_+} f \ln\psi_- - \frac{\psi_-'}{\psi_-} g \ln\varphi_+ + f'\ln\varphi_+ \ln\psi_- .$$

Let f and g approach functions which take only the values 0 and 1.

REMARK. The results of this paper apply to more general operators than Toeplitz and more general situations. Basically one needs an operator T and projections P_N so that T factors as $T = T_- T_+$ where $T_- P_N = P_N T_- P_N$ and $P_N T_+ = P_N T_+ P_N$ and $T_\pm^{\pm 1}$ are bounded. Then

$$\frac{\det P_N T P_N}{\det P_N T_+ P_N \det P_N T_- P_N} \quad \text{and} \quad \det P_N\{T_+^{-1}, T_-\}P_N$$

have the same asymptotics. Note that if T is a singular integral (order 0 pseudodifferential) operator and $H = \sum \lambda_n E_N$ is a first order selfadjoint elliptic operator on a manifold M, then $P_N = E_1 + E_2 + \dots E_N$ satisfies $P_N T_+ = P_N T_+ P_N$ if and only if the operator-valued function $e^{-iHx}T_+e^{iHx}$ has an analytic continuation to the upper half plane. In other words T_+ is what is commonly called an "analytic operator" for this automorphism group. So basically what one needs is an operator T which factors as a product of an "analytic and anti-analytic" operator.

Now if $T_+^{-1}T_- - T_-T_+^{-1}$ is trace class, $\{T_+^{-1}, T_-\}$ is determinant class so the problem reduces to computing $\det\{T_+^{-1}, T_-\}$. Whereas everything said so far is obvious we now make a point which is specialized enough not to be widely known. Namely, under many circumstances the results in [H-H] allow one to compute $\det\{T_+^{-1}, T_-\}$ explicitly even though $T_\pm$ are not Toeplitz. Also the techniques used in this computation are general enough that they might apply even when the trace class hypothesis isn't met.

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