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EQUALITY OF ESSENTIAL SPECTRA OF QUASISIMILAR QUASINORMAL OPERATORS

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In this paper all Hilbert spaces are complex. If $\mathscr{H}$ is a Hilbert space, we denote by $\mathfrak{L}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. (In this paper, we shall use the term *operator* to mean an element of $\mathfrak{L}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$.) An operator T is said to be *quasinormal* if T commutes with T^*T. If T is an operator and $TT^* \leqslant T^*T$, then T is said to be *hyponormal*. It is known that every quasinormal operator is hyponormal (cf. [1], [5]). If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces and $X: \mathscr{H} \to \mathscr{K}$ is a bounded linear transformation having trivial kernel and dense range, then X is called a *quasiaffinity*. If $T_1 \in \mathfrak{L}(\mathscr{H})$ and $T_2 \in \mathfrak{L}(\mathscr{K})$ and there exist quasiaffinities $X: \mathscr{H} \to \mathscr{K}$ and $Y: \mathscr{K} \to \mathscr{H}$ satisfying $XT_1 = T_2X$ and $T_1Y = YT_2$, then T_1 and T_2 are said to be *quasisimilar*. If T is an operator, let $\sigma(T)$ denote the spectrum of T. If $T \in \mathfrak{L}(\mathscr{H})$ and $\mathscr{H}$ is an infinite dimensional Hilbert space, let $\widetilde{T}$ denote the image of T in the Calkin algebra $\mathfrak{L}(\mathscr{H})/\mathfrak{C}$ under the natural quotient map and let $\sigma_e(T)$ denote the essential spectrum of T, i.e., the spectrum of $\widetilde{T}$ ($\mathfrak{C}$ denotes the norm-closed ideal of all compact operators in $\mathfrak{L}(\mathscr{H})$).

Let T_1 and T_2 be quasisimilar hyponormal operators on infinite dimensional Hilbert spaces. S. Clary proved in [3] that $\sigma(T_1) = \sigma(T_2)$. The present author showed in [9] that there are several cases which imply $\sigma_e(T_1) = \sigma_e(T_2)$. For example, if T_1 and T_2 are both biquasitriangular, if T_1 and T_2 are both weighted shifts (bilateral or unilateral), or if T_1 and T_2 are both partial isometries, then $\sigma_e(T_1) = \sigma_e(T_2)$. The purpose of this paper is to prove that if T_1 and T_2 are both quasinormal, then $\sigma_e(T_1) = \sigma_e(T_2)$.

Suppose that T is an operator. Then T is unitarily equivalent to $T_1 \oplus T_2$ where T_1 is normal and T_2 is *pure* (or completely non-normal), i.e., if $\mathscr{M}$ is a reducing subspace of T_2 and $T_2 \mid \mathscr{M}$ is normal, then $\mathscr{M} = (0)$. The operator T_1 is called the *normal part* of T and T_2 the *pure part* of T. (Note that either of the operators T_1 or T_2 may be the zero operator on the zero Hilbert space.)

Suppose that $\mathscr{H}$ is a Hilbert space. In this paper, we shall let

$$\hat{\mathscr{H}} = \sum_{n=1}^{\infty} \oplus \mathscr{H}_n ,$$

where $\mathscr{H}_n = \mathscr{H}$, $n = 1, 2, \ldots$. If $T \in \mathfrak{L}(\mathscr{H})$, we shall let

$$\hat{T} = \sum_{n=1}^{\infty} \oplus T_n,$$

where $T_n = T$, $n = 1, 2, \ldots$. We shall let $V_{\mathscr{H}}$ denote the unilateral shift on $\hat{\mathscr{H}}$, i.e., $V_{\mathscr{H}}(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ for each $(x_1, x_2, \ldots)$ in $\hat{\mathscr{H}}$. If $\mathscr{H}$ is one dimensional, we shall write simply V for the operator $V_{\mathscr{H}}$. Note that $\hat{T}$ and $V_{\mathscr{H}}$ commute, a fact that is used implicitly in our proofs. A. Brown proved in [2] that if T is quasinormal, then T is unitarily equivalent to $N \oplus V_{\mathscr{H}}\hat{P}$ where N is a normal operator and P is a positive definite operator on a Hilbert space $\mathscr{H}$, i.e., $(Px, x) > 0$ for all x in $\mathscr{H}$ except $x = 0$. It is known that quasinormal operators are subnormal (cf. [5], Problem 154) and it is clear that N is the normal part of T and $V_{\mathscr{H}}\hat{P}$ is the pure part of T. J. Conway proved in [4] that the normal parts of quasisimilar subnormal operators are unitarily equivalent. Thus, in order to prove that two quasisimilar quasinormal operators T_1 and T_2 have equal essential spectra, it suffices to study the pure parts of T_1 and T_2. Hence we shall begin by considering the pure parts of quasinormal operators.

We shall need the following additional notation and terminology. If T is an operator, let $\mathscr{K}(T)$ denote the kernel of T and $\mathscr{R}(T)$ the range of T. If P is a positive definite operator on an infinite dimensional Hilbert space, then let $\Lambda(P) = \sigma(P) \cap \cap \{t \in \mathbf{R}: t > \|\tilde{P}\|\}$. Let T be an operator on an infinite dimensional Hilbert space. Recall that T is *Fredholm* if $\mathscr{R}(T)$ is closed and both $\mathscr{K}(T)$ and $\mathscr{K}(T^*)$ are finite dimensional. If T is Fredholm, let

$$\mathrm{i}(T) = \dim(\mathscr{K}(T)) - \dim(\mathscr{K}(T^*))$$

denote the index of T. It is well-known that $\sigma_e(T) = \{\lambda \in \mathbf{C}: T - \lambda$ is not Fredholm$\}$. A *hole* in $\sigma_e(T)$ is a bounded component of $\mathbf{C} \setminus \sigma_e(T)$. It is also well-known that if H is a hole in $\sigma_e(T)$, then $\mathrm{i}(T - \lambda)$ is constant on H. (See [5] and [6] for facts about Fredholm operators and essential spectra.)

In the following (Lemmas 1—5 and Corollaries 1 and 2), let $\mathscr{H}$ be an infinite dimensional Hilbert space, let P be a positive definite operator on $\mathscr{H}$, and let $E(\cdot)$ be the spectral measure of P. Recall that a point c in $\sigma(P)$ belongs to $\sigma_e(P)$ if and only if c is an accumulation point of $\sigma(P)$ or an eigenvalue of P of infinite multiplicity (cf. [1], [6]). Thus if $c \in \sigma_e(P)$, then $\mathscr{R}(E([c - \varepsilon, \|P\|]))$ is infinite dimensional for each positive number ε. Since $\Lambda(P) \subseteq \sigma(P) \setminus \sigma_e(P)$, it also follows that if $c \in \Lambda(P)$, then c is an isolated eigenvalue of P of finite multiplicity. Thus $\Lambda(P)$ is countable and $\|\tilde{P}\|$ is its only possible accumulation point. Also $\Lambda(P) \neq \emptyset$ if and only if $\|\tilde{P}\| < \|P\|$. Furthermore, if $\Lambda(P) \neq \emptyset$, then $\|P\| \in \Lambda(P)$.

We shall first prove the following lemma.

LEMMA 1. $\|\tilde{P}\| = \sup\{t: 0 \leqslant t \leqslant \|P\|$ *and* $\mathscr{R}(E([t, \|P\|]))$ *is infinite dimensional*$\}$

Proof. Let $d = \sup\{t: 0 \leqslant t \leqslant \|P\| \text{ and } \mathscr{R}(E([t, \|P\|])) \text{ is infinite dimensional}\}$. The preceding discussion implies that if $t > \|\widetilde{P}\|$, then $\mathscr{R}(E([t, \|P\|]))$ is finite dimensional, whence $d \leqslant \|\widetilde{P}\|$. Moreover, if $t \in \sigma_e(P)$, then $\mathscr{R}(E([t-\varepsilon, \|P\|]))$ is infinite dimensional for each positive number ε, so $d \geqslant \sup\{t: t \in \sigma_e(P)\} = \|\widetilde{P}\|$. (Note that since P is positive, the spectral radius of $\widetilde{P}$ is equal to $\|\widetilde{P}\|$.)

LEMMA 2. *Suppose that $\|\widetilde{P}\| < \|P\|$. Let $\{c_n\}_{n\in A}$ be the sequence of eigenvalues of P in $A(P)$ repeated according to (finite) multiplicity. Then $V_{\mathscr{H}}\hat{P}$ is unitarily equivalent to $(\sum_{n\in A} \oplus c_n V) \otimes V_{\mathscr{M}}\hat{P}_0$, where P_0 is a positive definite operator on a subspace $\mathscr{M}$ of $\mathscr{H}$ and $\|P_0\| \leqslant \|\widetilde{P}\|$.*

Proof. Let $\mathscr{M} = \mathscr{R}(E([0, \|\widetilde{P}\|]))$ and, for each n in A, let $\mathscr{M}_n$ denote the one dimensional eigenspace corresponding to the eigenvalue c_n. Then $\mathscr{H} = (\sum_{n\in A} \oplus \mathscr{M}_n) \oplus \mathscr{M}$. Let $P_0 = P \mid \mathscr{M}$. Then

$$P = (\sum_{n\in A} \oplus c_n) \oplus P_0.$$

It is clear that $V_{\mathscr{H}}\hat{P}$ is unitarily equivalent to $(\sum_{n\in A} \oplus c_n V) \oplus V_{\mathscr{M}}\hat{P}_0$, that P_0 is a positive definite operator on $\mathscr{M}$, and that $\|P_0\| \leqslant \|\widetilde{P}\|$.

LEMMA 3. *If λ is a complex number satisfying $|\lambda| > \|\widetilde{P}\|$, then $\mathscr{K}(V_{\mathscr{H}}^*\hat{P} - \lambda)$ s finite dimensional.*

Proof. Since $\|V_{\mathscr{H}}^*\hat{P}\| = \|P\|$, the proof is complete if $\|\widetilde{P}\| = \|P\|$. Suppose that $\|\widetilde{P}\| < \|P\|$. Then, according to Lemma 2, $V_{\mathscr{H}}\hat{P}$ is unitarily equivalent to $(\sum_{n\in A} \oplus c_n V) \oplus V_{\mathscr{M}}\hat{P}_0$. Now $\lambda \notin \sigma(V_{\mathscr{M}}\hat{P}_0)$ since

$$|\lambda| > \|\widetilde{P}\| \geqslant \|P_0\| = \|V_{\mathscr{M}}\hat{P}_0\|;$$

moreover $\lambda \notin \sigma(c_n V)$ except for at most a finite number of n's. Thus $\mathscr{K}(V_{\mathscr{H}}^*\hat{P} - \lambda)$ $(\mathscr{K} = (V_{\mathscr{H}}\hat{P})^* - \lambda))$ is finite dimensional. (Here we have used the facts that $\mathscr{K}(V^* - \lambda)$ is one dimensional for $|\lambda| < 1$ and $\mathscr{K}(V^* - \lambda) = (0)$ for $|\lambda| \geqslant 1$.)

LEMMA 4. *If λ is a complex number satisfying $|\lambda| < \|\widetilde{P}\|$, then $\mathscr{K}(V_{\mathscr{H}}^*\hat{P} - \lambda)$ is infinite dimensional.*

Proof. Choose t such that $|\lambda| < t < \|\widetilde{P}\|$. Let

$$\mathscr{M} = \mathscr{R}(E([t, \|P\|]))$$

and let $P_1 = P \mid \mathscr{M}$. Now

$$\|P_1 x\| \geqslant t\|x\|$$

for each x in $\mathcal{M}$. Hence P_1 is invertible and

$$\|P_1^{-1}\| \leqslant 1/t.$$

Suppose that $x \in \mathcal{M}$. Let $x_1 = x$ and let

$$x_{n+1} = \lambda P_1^{-1} x_n, \qquad n = 1, 2, \ldots .$$

We have

$$\|x_{n+1}\| \leqslant (|\lambda|/t)^n \|x\|, \qquad n = 0, 1, 2, \ldots .$$

It follows that

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty.$$

Hence $\hat{x} = (x_1, x_2, \ldots) \in \hat{\mathcal{H}}$ and, since $Px_{n+1} = \lambda x_n$, $n = 1, 2, \ldots$, we have $V_{\mathcal{H}}^* \hat{P} \hat{x} = \lambda \hat{x}$. Thus $\hat{x} \in \mathcal{K}(V_{\mathcal{H}}^* \hat{P} - \lambda)$ for each x in $\mathcal{M}$. Therefore, $\mathcal{K}(V_{\mathcal{H}}^* \hat{P} - \lambda)$ is infinite dimensional since Lemma 1 implies that $\mathcal{M}$ is infinite dimensional.

COROLLARY 1. $\{\lambda \in \mathbf{C}: |\lambda| \leqslant \|\widetilde{P}\|\} \subseteq \sigma_e(V_{\mathcal{H}} \hat{P})$.

Proof. If $\|\widetilde{P}\| > 0$, then the result follows easily from Lemma 4. If we observe that $\mathcal{K}(V_{\mathcal{H}}^* \hat{P})$ is infinite dimensional, we see that the result also holds when $\|\widetilde{P}\| = 0$.

The next corollary follows from Lemma 2, Corollary 1, and from the fact that if $\|\widetilde{P}\| < \|P\|$, then $\Lambda(P)$ consists of isolated points whose only possible accumulation point is $\|\widetilde{P}\|$.

COROLLARY 2. $\sigma(V_{\mathcal{H}} \hat{P}) = \{\lambda \in \mathbf{C}: |\lambda| \leqslant \|P\|\}$ *and* $\sigma_e(V_{\mathcal{H}} \hat{P}) = (\bigcup_{c \in \Lambda(P)} \{\lambda \in \mathbf{C}: |\lambda| = c\}) \cup \cup \{\lambda \in \mathbf{C}: |\lambda| \leqslant \|\widetilde{P}\|\}$.

Note that in Corollary 2 we used the fact that $\sigma(V)$ is the closed unit disk and $\sigma_e(V)$ is the unit circle.

LEMMA 5. *If $c \in \Lambda(P)$ and λ_1 and λ_2 are complex numbers satisfying*

$$[|\lambda_1|, |\lambda_2|] \cap \sigma(P) = (|\lambda_1|, |\lambda_2|) \cap \sigma(P) = \{c\},$$

then

$$dim\,(\mathcal{K}(V_{\mathcal{H}}^* \hat{P} - \lambda_1)) = dim\,(\mathcal{K}(V_{\mathcal{H}}^* \hat{P} - \lambda_2)) + dim\,(\mathcal{K}(P - c)).$$

Proof. Since $\Lambda(P) \neq \emptyset$, we have $\|\widetilde{P}\| < \|P\|$. By Lemma 2, $V_{\mathcal{H}} \hat{P}$ is unitarily equivalent to $(\sum_{n \in \Lambda} \oplus c_n V) \oplus V_{\mathcal{M}} \hat{P}_0$, where $\{c_n\}_{n \in \Lambda}$ is the set of all eigenvalues of P

in $\Lambda(P)$ repeated according to multiplicity. Since

$$\|V_{\mathcal{M}}\hat{P}_0\| = \|P_0\| \leqslant \|\widetilde{P}\|,$$

it follows that both λ_1 and λ_2 do not belong to $\sigma(V_{\mathcal{M}}\hat{P}_0)$. Note also that $\lambda_1 \in \operatorname{int}(\sigma(c_n V))$ if and only if $c_n \geqslant c$ and $\lambda_2 \in \operatorname{int}(\sigma(c_n V))$ if and only if $c_n > c$. Thus

$$\dim(\mathcal{K}(V_{\mathcal{H}}^*\hat{P} - \lambda_1)) = \operatorname{card}\{n \in \Lambda : c_n \geqslant c\}$$

and

$$\dim(\mathcal{K}(V_{\mathcal{H}}^*\hat{P} - \lambda_2)) = \operatorname{card}\{n \in \Lambda : c_n > c\}.$$

Hence, the proof is complete.

We discuss next $V_{\mathcal{H}}\hat{P}$ where $\mathcal{H}$ is a finite dimensional Hilbert space. Thus, in Lemmas 6 and 7 and in Corollary 3, let $\mathcal{H}$ be a m-dimensional Hilbert space, where m is a positive integer, and let P be a positive definite operator on $\mathcal{H}$.

Lemma 6 and Corollary 3 are easy to verify.

LEMMA 6. *Let $\{c_n\}_{n=1}^m$ be the set of eigenvalues of P (repeated according to multiplicity). Then $V_{\mathcal{H}}\hat{P}$ is unitarily equivalent to $\sum_{n=1}^{m} \oplus c_n V$.*

COROLLARY 3. $\sigma(V_{\mathcal{H}}\hat{P}) = \{\lambda \in \mathbf{C} : |\lambda| \leqslant \|P\|\}$ *and* $\sigma_e(V_{\mathcal{H}}\hat{P}) = \bigcup_{c \in \sigma(P)} \{\lambda \in \mathbf{C} : |\lambda| = c\}$.

The proof of the following lemma is similar to that of Lemma 5.

LEMMA 7. *If $c \in \sigma(P)$ and λ_1 and λ_2 are complex numbers satisfying*

$$[|\lambda_1|, |\lambda_2|] \cap \sigma(P) = (|\lambda_1|, |\lambda_2|) \cap \sigma(P) = \{c\},$$

then

$$\dim(\mathcal{K}(V_{\mathcal{H}}^*\hat{P} - \lambda_1)) = \dim(\mathcal{K}(V_{\mathcal{H}}^*\hat{P} - \lambda_2) + \dim(\mathcal{K}(P - c)).$$

THEOREM 1. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are nonzero Hilbert spaces, that $X : \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_2$ and $Y : \hat{\mathcal{H}}_2 \to \hat{\mathcal{H}}_1$ are bounded linear transformations having dense ranges, and that P_1 and P_2 are positive definite operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.*

If

$$XV_{\mathcal{H}_1}\hat{P}_1 = V_{\mathcal{H}_2}\hat{P}_2 X \quad \text{and} \quad V_{\mathcal{H}_1}\hat{P}_1 Y = YV_{\mathcal{H}_2}\hat{P}_2,$$

then

$$\sigma_e(V_{\mathcal{H}_1}\hat{P}_1) = \sigma_e(V_{\mathcal{H}_2}\hat{P}_2).$$

Proof. It follows from Lemma B of [3] that

$$\sigma(V_{\mathscr{H}_1}\hat{P}_1) = \sigma(V_{\mathscr{H}_2}\hat{P}_2).$$

Since hyponormal operators are spectraloid, we have $\|P_1\| = \|P_2\|$ (see Corollary 2). Ba taking adjoints in the given equations and using the fact that X and Y have dense ranges, we can deduce that

$$\dim(\mathscr{K}(V^*_{\mathscr{H}_1}\hat{P}_1 - \lambda)) = \dim(\mathscr{K}(V^*_{\mathscr{H}_2}\hat{P}_2 - \lambda))$$

for each complex number λ. (cf. [5], Problem 42.) In particular, if we take $\lambda = 0$, we have $\dim \mathscr{H}_1 = \dim \mathscr{H}_2$. For definiteness, assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ are both infinite dimensional. (The proof is similar if $\dim(\mathscr{H}_1) = \dim(\mathscr{H}_2) = m$.) We know from Lemmas 3 and 4 that $\mathscr{K}(V^*_{\mathscr{H}_i}\hat{P}_i - \lambda)$ is finite dimensional for $|\lambda| > \|\widetilde{P}_i\|$ and infinite dimensional for $|\lambda| < \|\widetilde{P}_i\|$, $i = 1, 2$. Since

$$\dim(\mathscr{K}(V^*_{\mathscr{H}_1}\hat{P}_1 - \lambda)) = \dim(\mathscr{K}(V^*_{\mathscr{H}_2}\hat{P}_2 - \lambda))$$

for each λ, we have $\|\widetilde{P}_1\| = \|\widetilde{P}_2\|$. If $\|\widetilde{P}_1\| = \|P_1\|$, then it follows from Corollary 2 that

$$\sigma_e(V_{\mathscr{H}_1}\hat{P}_1) = \sigma_e(V_{\mathscr{H}_2}\hat{P}_2).$$

So assume that $\|\widetilde{P}_1\| < \|P_1\|$. Suppose that $\lambda \in \sigma(V_{\mathscr{H}_1}\hat{P}_1)\setminus\sigma_e(V_{\mathscr{H}_1}\hat{P}_1)$. Corollary 2 implies that λ belongs to a hole H in $\sigma_e(V_{\mathscr{H}_1}\hat{P}_1)$. We want to show that $\lambda \in \sigma(V_{\mathscr{H}_2}\hat{P}_2)\setminus$ $\setminus\sigma_e(V_{\mathscr{H}_2}\hat{P}_2)$. Assume, to the contrary, that $\lambda \in \sigma_e(V_{\mathscr{H}_2}\hat{P}_2)$. Since Corollary 1 implies that

$$|\lambda| > \|\widetilde{P}_1\| = \|\widetilde{P}_2\|,$$

we have $|\lambda| = c \in \Lambda(P_2)$. Choose complex numbers λ_1 and λ_2 in H such that

$$[|\lambda_1|, |\lambda_2|] \cap \sigma(P_2) = (|\lambda_1|, |\lambda_2|) \cap \sigma(P_2) = \{c\}.$$

Now $\mathrm{i}(V_{\mathscr{H}_1}\hat{P}_1 - \lambda_1) = \mathrm{i}(V_{\mathscr{H}_1}\hat{P}_1 - \lambda_2)$ implies that

$$\dim(\mathscr{K}(V^*_{\mathscr{H}_1}\hat{P}_1 - \bar{\lambda}_1)) = \dim(\mathscr{K}(V^*_{\mathscr{H}_1}\hat{P}_1 - \bar{\lambda}_2))$$

(since $\mathscr{K}(V_{\mathscr{H}_1}\hat{P}_1 - \lambda) = (0)$ for each complex number λ). This implies that

$$\dim(\mathscr{K}(V^*_{\mathscr{H}_2}\hat{P}_2 - \bar{\lambda}_1)) = \dim(\mathscr{K}(V^*_{\mathscr{H}_2}\hat{P}_2 - \bar{\lambda}_2)).$$

This last statement contradicts Lemma 5. Therefore, $\lambda \in \sigma(V_{\mathscr{H}_2}\hat{P}_2)\setminus\sigma_e(V_{\mathscr{H}_2}\hat{P}_2)$. We have shown that

$$\sigma_e(V_{\mathscr{H}_2}\hat{P}_2) \subseteq \sigma_e(V_{\mathscr{H}_1}\hat{P}_1).$$

Thus the proof is complete by symmetry.

The following corollary can be deduced easily from Theorem 1 and its proof, from Corollaries 2 and 3, and from Lemmas 5 and 7.

COROLLARY 4. *Suppose that $V_{\mathscr{H}_1}\hat{P}_1$ and $V_{\mathscr{H}_2}\hat{P}_2$ satisfy the hypotheses of Theorem 1.*

(1) *If $\mathscr{H}_1$ is infinite dimensional, then $dim(\mathscr{H}_2) = dim(\mathscr{H}_1)$, $\|\tilde{P}_1\| = \|\tilde{P}_2\|$,*

$$\Lambda(P_1) = \Lambda(P_2), \text{ and if } c \in \Lambda(P_1), \text{ then } dim(\mathscr{K}(P_1 - c)) = dim(\mathscr{K}(P_2 - c)).$$

(2) *If $dim(\mathscr{H}_1) = m$, where m is a positive integer, then $dim(\mathscr{H}_2) = m$, $\sigma(P_1) = \sigma(P_2)$, and if $c \in \sigma(P_1)$, then $dim(\mathscr{K}(P_1 - c)) = dim(\mathscr{K}(P_2 - c))$.*

Lemmas 2 and 6 and Corollary 4 imply the following corollary.

COROLLARY 5. *Suppose that $V_{\mathscr{H}_1}\hat{P}_1$ and $V_{\mathscr{H}_2}\hat{P}_2$ satisfy the hypotheses of Theorem 1. Then if P_1 is compact, then P_2 is also compact and $V_{\mathscr{H}_1}\hat{P}_1$ and $V_{\mathscr{H}_2}\hat{P}_2$ are unitarily equivalent.*

Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable Hilbert spaces, T_1^* is a hyponormal operator in $\mathfrak{L}(\mathscr{H}_1)$, $T_2 \in \mathfrak{L}(\mathscr{H}_2)$, $X\colon \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear transformation, and $XT_1 = T_2X$. J. G. Stampfli and B. L. Wadhwa proved in [8] that if T_2 is the unilateral shift of multiplicity one, then $X = 0$. It was proved in [9] that if T_2 is a non-normal hyponormal injective weighted shift (bilateral or unilateral), then $X = 0$. The following theorem shows that the above result is true also in the case that T_2 is a pure quasinormal operator. The proof of this theorem uses the ideas in the proofs of the above mentioned theorems.

THEOREM 2. *Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces (of arbitrary dimensions), T^* is a hyponormal operator in $\mathfrak{L}(\mathscr{H}_1)$, P is a positive definite operator in $\mathfrak{L}(\mathscr{H}_2)$, and $X\colon \mathscr{H}_1 \to \hat{\mathscr{H}}_2$ is a bounded linear transformation. If $XT = V_{\mathscr{H}_2}\hat{P}X$, then $X = 0$.*

In order to prove the above theorem, we shall need the following lemma.

LEMMA 8. *Suppose that P is a positive definite operator on a nonzero Hilbert space $\mathscr{H}$, $E(\cdot)$ is the spectral measure of P, m is a positive integer, and ε is a number that satisfies $0 < \varepsilon < \|P\|$. If $x \in \mathscr{R}(E([\varepsilon, \|P\|]))$, then there exists a bounded sequence $\{x_n\}_{n=1}^{\infty}$ of vectors in $\mathscr{H}$ such that $x_m = x$ and $Px_{n+1} = \varepsilon x_n$, $n = 1, 2, \ldots$.*

Proof. Let $P_1 = P|\mathscr{R}(E([\varepsilon, \|P\|]))$. We have

$$\|P_1 y\| \geqslant \varepsilon\|y\|$$

for each y in $\mathscr{R}(E([\varepsilon, \|P\|]))$. Hence P_1 is invertible and

$$\|P_1^{-1}\| \leqslant 1/\varepsilon.$$

Now let $x_m = x$, let $x_{n+1} = \varepsilon P_1^{-1} x_n$ for $n \geqslant m$, and let $x_n = (1/\varepsilon)P_1 x_{n+1}$ for $n < m$. Note that $\|x_n\| \leqslant \|x\|$ for $n \geqslant m$. Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence of vectors in $\mathscr{H}$ and clearly $Px_{n+1} = \varepsilon x_n$, $n = 1, 2, \ldots$.

Proof of Theorem 2. Without loss of generality assume that $\mathscr{H}_2 \neq (0)$. We first consider the case that $\mathscr{H}_1$ is separable. Let $E(\cdot)$ be the spectral measure of P. Suppose that ε satisfies $0 < \varepsilon < \|P\|$, $x \in \mathscr{R}(E([\varepsilon, \|P\|]))$, and m is a positive integer. According to Lemma 8, there exists a bounded sequence $\{x_n\}_{n=1}^{\infty}$ of vectors in $\mathscr{H}_2$ such that $x_m = x$ and $Px_{n+1} = \varepsilon x_n$, $n = 1, 2, \ldots$. For each λ in $D = \{\lambda \in \mathbf{C}\colon |\lambda| < 1\}$, let

$$f(\lambda, \varepsilon, x, m) = (\lambda x_1, \lambda^2 x_2, \lambda^3 x_3, \ldots).$$

Observe that $f(\lambda, \varepsilon, x, m) \in \hat{\mathscr{H}}_2$ and $V^*_{\mathscr{H}_2}\hat{P} f(\lambda, \varepsilon, x, m) = \varepsilon\lambda f(\lambda, \varepsilon, x, m)$. Rewriting our equation as

$$T^* X^* = X^* V^*_{\mathscr{H}_2} \hat{P},$$

we get

$$T^*(X^* f(\lambda, \varepsilon, x, m)) = \varepsilon\lambda X^* f(\lambda, \varepsilon, x, m).$$

Since T^* is hyponormal, $X^* f(\lambda, \varepsilon, x, m)$ is orthogonal to $X^* f(\mu, \varepsilon, x, m)$ for $\lambda \neq \mu$. Since $\mathscr{H}_1$ is separable, we have $X^* f(\lambda, \varepsilon, x, m) = 0$ for all but countably many λ's in D. Let $z = (z_1, z_2, z_3, \ldots)$ be a vector in $\hat{\mathscr{H}}_2$. Then

$$(f(\lambda, \varepsilon, x, m), z) = \sum_{n=1}^{\infty} (x_n, z_n)\, \lambda^n$$

is a power series in λ, and thus continuous on D. Hence the map $\lambda \to f(\lambda, \varepsilon, x, m)$ is continuous when $\hat{\mathscr{H}}_2$ is equipped with its weak topology. Thus the map $\lambda \to X^* f(\lambda, \varepsilon, x, m)$ is also weakly continuous. It follows that $X^* f(\lambda, \varepsilon, x, m) = 0$ for each λ in D and for each ε, x, and m.

We show next that the collection of all vectors of the form $f(\lambda, \varepsilon, x, m)$ spans $\hat{\mathscr{H}}_2$. Suppose that $z = (z_1, z_2, z_3, \ldots)$ is a vector in $\hat{\mathscr{H}}_2$ satisfying $(f(\lambda, \varepsilon, x, m), z) = 0$ for all $f(\lambda, \varepsilon, x, m)$ in $\hat{\mathscr{H}}_2$. Fix ε, x, and m. The power series

$$\sum_{n=1}^{\infty} (x_n, z_n)\, \lambda^n = (f(\lambda, \varepsilon, x, m), z)$$

is identically zero on D. Thus $(x_n, z_n) = 0$, $n = 1, 2, \ldots$. It follows that for each positive integer n, $(x, z_n) = 0$ for all x in $\mathscr{M} = \bigcup_{0<\varepsilon<\|P\|} \mathscr{R}(E([\varepsilon, \|P\|]))$. Since P is positive definite, we have $E(\{0\}) = 0_{\mathscr{H}_2}$; thus $\overline{\mathscr{M}} = \mathscr{H}_2$. Hence $z = 0$ and thus the collection of all vectors $f(\lambda, \varepsilon, x, m)$ spans $\hat{\mathscr{H}}_2$. Therefore, $X^* = 0$.

We now consider the general case. Let

$$\Gamma = \{(T^*)^{j_n} T^{k_n} \dots (T^*)^{j_1} T^{k_1} : n \text{ is a positive integer and } j_i \text{ and } k_i \text{ are nonnegative integers}, \ i = 1, 2, \dots, n\}.$$

Observe that Γ is a countable set. Suppose that $e \in \mathcal{H}_1$. Let $\mathcal{H}_0$ be the subspace of $\mathcal{H}_1$ that is spanned by $\{Se: S \in \Gamma\}$. Note that $\mathcal{H}_0$ reduces T. Let $T_0 = T|\mathcal{H}_0$ and let $X_0: \mathcal{H}_0 \to \hat{\mathcal{H}}_2$ be defined by $X_0 z = Xz$ for each z in $\mathcal{H}_0$. Then T_0^* is hyponormal in $\mathfrak{L}(\mathcal{H}_0)$ and $X_0 T_0 = V_{\mathcal{H}_2} \hat{P} X_0$. Thus, since $\mathcal{H}_0$ is separable, the above case implies that $X_0 = 0$. In particular, $X_0 e = Xe = 0$. Since e is an arbitrary element of $\mathcal{H}_1$, we have $X = 0$.

The following is the main theorem of this paper.

THEOREM 3. *Suppose that T_1 and T_2 are quasisimilar quasinormal operators on infinite dimensional Hilbert spaces. Then $\sigma_e(T_1) = \sigma_e(T_2)$.*

Proof. According to [2], T_i is unitarily equivalent to $N_i \oplus V_{\mathcal{H}_i} \hat{P}_i$ on $\mathcal{K}_i + \hat{\mathcal{H}}_i$ where N_i is a normal operator on the Hilbert space $\mathcal{K}_i$ and P_i is a positive definite operator on $\mathcal{H}_i$, $(i = 1, 2)$. Proposition 2.3 of [4] implies that N_1 and N_2 are unitarily equivalent. If either T_1 or T_2 is normal, then Lemma 2 of [7] implies that both T_1 and T_2 are normal. Thus, in this case, T_1 and T_2 are unitarily equivalent and $\sigma_e(T_1) = \sigma_e(T_2)$. Hence we may assume that both $\mathcal{H}_1$ and $\mathcal{H}_2$ are nonzero. In order to complete the proof, it suffices to show that $\sigma_e(V_{\mathcal{H}_1} \hat{P}_1) = \sigma_e(V_{\mathcal{H}_2} \hat{P}_2)$. There exist quasi-affinities X and Y such that

$$X(N_1 \oplus V_{\mathcal{H}_1} \hat{P}_1) = (N_2 \oplus V_{\mathcal{H}_2} \hat{P}_2) X$$

and

$$(N_1 \oplus V_{\mathcal{H}_1} \hat{P}_1) Y = Y(N_2 \oplus V_{\mathcal{H}_1} \hat{P}_2).$$

Let

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \text{ and } \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$$

be the matrices of X and Y, respectively, with respect to $\mathcal{K}_1 \oplus \hat{\mathcal{H}}_1$ and $\mathcal{K}_2 \oplus \hat{\mathcal{H}}_2$. A matrix calculation shows that $X_3 N_1 = V_{\mathcal{H}_2} \hat{P}_2 X_3$ and $Y_3 N_2 = V_{\mathcal{H}_1} \hat{P}_1 Y_3$. Theorem 2 implies that $X_3 = Y_3 = 0$. It follows that X_4 and Y_4 have dense ranges and a matrix calculation shows that $X_4 V_{\mathcal{H}_1} \hat{P}_1 = V_{\mathcal{H}_2} \hat{P}_2 X_4$ and $V_{\mathcal{H}_1} \hat{P}_1 Y_4 = Y_4 V_{\mathcal{H}_2} \hat{P}_2$. Hence, by Theorem 1, $\sigma_e(V_{\mathcal{H}_1} \hat{P}_1) = \sigma_e(V_{\mathcal{H}_2} \hat{P}_2)$.

As mentioned before, J. Conway proved in [4] that the normal parts of quasisimilar subnormal operators are unitarily equivalent. In that paper he also provided an example which showed that the pure parts of quasisimilar subnormal operators need not be quasisimilar. Close scrutiny of his example will reveal that one of the two quasisimilar subnormal operators is not quasinormal. However, a slight modification of his example will show that the pure parts of quasisimilar quasinormal operators need not be quasisimilar. We shall need the following lemma in the construction of the example. (The reader should notice the contrast between the following lemma and Theorem 2.)

Lemma 9. *Suppose that $\mathscr{H}$ is a Hilbert space and P is a positive definite operator on $\mathscr{H}$. Then there exist a quasiaffinity W and a positive definite operator R in $\mathfrak{L}(\hat{\mathscr{H}})$ such that $WV_{\mathscr{H}}\hat{P} = RW$.*

Proof. Suppose that $x = (x_1, x_2, \ldots) \in \hat{\mathscr{H}}$. Then, by the Hölder inequality, we have

$$\sum_{k=1}^{\infty} \|(1/2^{nk})\, x_k\| \leqslant \left(\sum_{k=1}^{\infty} (1/2^{2nk})\right)^{1/2} \left(\sum_{k=1}^{\infty} \|x_k\|^2\right)^{1/2} \leqslant (1/2^{n-1})\, \|x\|$$

for each positive integer n. Since absolute convergence implies convergence in a Banach space, we have $\sum_{k=1}^{\infty} (1/2^{nk})\, x_k \in \mathscr{H}$, $n = 1, 2, \ldots$. We also have

$$\sum_{n=1}^{\infty} \left\| \sum_{k=1}^{\infty} (1/2^{nk})\, x_k \right\|^2 \leqslant \sum_{n=1}^{\infty} (1/2^{n-1})^2\, \|x\|^2 \leqslant 2\|x\|^2.$$

Define W on $\hat{\mathscr{H}}$ by the matrix $[\alpha_{nk}]_{n,k=1}^{\infty}$, where $\alpha_{nk} = (1/2^{nk})\, 1_{\mathscr{H}}$, $n = 1, 2, \ldots, k = 1, 2, \ldots$. The above estimates show that W is a bounded operator on $\hat{\mathscr{H}}$. Suppose that $Wx = 0$. Then

$$\sum_{k=1}^{\infty} (1/2^{nk})\, x_k = 0, \; n = 1, 2, \ldots .$$

Suppose that $z \in \mathscr{H}$. Then

$$\sum_{k=1}^{\infty} (1/2^{nk})\, (x_k, z) = \left(\sum_{k=1}^{\infty} (1/2^{nk})\, x_k, z\right) = 0.$$

Hence the power series $\sum_{k=1}^{\infty} (x_k, z)\lambda^k$ converges on $D_0 = \{\lambda \in \mathbf{C} \colon |\lambda| < 1/2\}$ and thus represents an analytic function f there. But since $f(1/2^n) = 0$, $n = 1, 2, \ldots$, it follows that f is identically zero on D_0. Thus $(x_k, z) = 0$, $k = 1, 2, \ldots$, and since z is an arbitrary vector in $\mathscr{H}$, it follows that $x = 0$. We have shown that $\mathscr{K}(W) = (0)$. Since W is Hermitian, it follows that W is a quasiaffinity. Let $R = \sum_{k=1}^{\infty} \oplus\, (1/2^k)P$. An easy calculation shows that $WV_{\mathscr{H}}\hat{P} = RW$.

The following example is a modification of the above mentioned example of Conway.

EXAMPLE 1. Let $\mathscr{H}$ be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let V be the unilateral shift on $\mathscr{H}$ defined by

$$Ve_n = e_{n+1},\ n = 1, 2, \ldots .$$

According to Lemma 9, there exist a quasiaffinity W and a positive operator R in $\mathfrak{L}(\mathscr{H})$ such that $WV = RW$. Define U in $\mathfrak{L}(\mathscr{H})$ by $Ue_n = (1/2^n)e_n$, $n = 1, 2, \ldots$. The operator U is a quasiaffinity and $1/2VU = UV$. Let

$$T_1 = \hat{V} \oplus 1/2V \oplus 1/2\hat{R} \quad \text{on } \hat{\mathscr{H}} \oplus \mathscr{H} \oplus \hat{\mathscr{H}}$$

and

$$T_2 = \hat{V} \oplus 1/2\hat{R} \quad \text{on } \hat{\mathscr{H}} \oplus \hat{\mathscr{H}}.$$

The operators T_1 and T_2 are clearly quasinormal. Define $X: \hat{\mathscr{H}} \oplus \mathscr{H} \oplus \hat{\mathscr{H}} \to \hat{\mathscr{H}} \oplus \hat{\mathscr{H}}$ by

$$X((x_1, x_2, x_3, \ldots) \oplus x_0 \oplus (y_1, y_2, y_3, \ldots)) = (x_1, x_2, x_3, \ldots) \oplus$$
$$\oplus (Wx_0, y_1, y_2, y_3, \ldots)$$

and $Y: \hat{\mathscr{H}} \oplus \hat{\mathscr{H}} \to \hat{\mathscr{H}} \oplus \mathscr{H} \oplus \hat{\mathscr{H}}$ by

$$Y((x_1, x_2, x_3, \ldots) \oplus (y_1, y_2, y_3, \ldots)) = (x_2, x_3, \ldots) \oplus Ux_1 \oplus (y_1, y_2, y_3, \ldots).$$

It is clear that X and Y are quasiaffinities and a routine calculation shows that $XT_1 = T_2X$ and $T_1Y = YT_2$. Hence T_1 and T_2 are quasisimilar. Note that the pure part of T_1 is $\hat{V} \oplus 1/2V$ and the pure part of T_2 is $\hat{V}$. We shall show now that $\hat{V} \oplus 1/2V$ and $\hat{V}$ are not quasisimilar by using the same argument that Conway uses in his above mentioned example. Suppose that there exists a quasiaffinity $Z: \hat{\mathscr{H}} \oplus \mathscr{H} \to \hat{\mathscr{H}}$ such that

$$Z(\hat{V} \oplus 1/2V) = \hat{V}Z.$$

Define $W: \mathscr{H} \to \hat{\mathscr{H}}$ by

$$Wx = Z(0 \oplus x).$$

Let $\mathscr{M} = \overline{\mathscr{R}(W)}$. Then $\mathscr{M}$ is an invariant subspace for $\hat{V}$ and $W(1/2V) = (\hat{V}|\mathscr{M})W$ Note that

$$W^*((\hat{V} - \lambda)|\mathscr{M})^* = ((1/2)V - \lambda)^*W^*$$

and that $W^*: \mathscr{M} \to \mathscr{H}$ is injective. Since $\hat{V}$ is completely nonunitary, $\hat{V}/\mathscr{M}$ is a nonunitary isometry thus for $1/2 < |\lambda| < 1$, λ is an eigenvalue of $(\hat{V}|\mathscr{M})^*$ and thus also of $(1/2)V^*$. The last statement is clearly a contradiction. Therefore, the pure parts of T_1 and T_2 are not quasisimilar.

J. Conway also proved in [4] that subnormal operators are similar if and only if their normal parts are unitarily equivalent and their pure parts are similar. Hence the two quasisimilar quasinormal operators constructed in Example 1 are not similar. Thus the equality of the essential spectra of quasisimilar quasinormal operators is not a result of similarity. The following example shows that two quasisimilar quasinormal operators need not be similar even if both operators are pure.

EXAMPLE 2. Let $\mathscr{H}$ be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Let

$$c_1 = 1,\ d_1 = 1/2,\ d_{2n} = c_{2n} = 1/4, \quad n = 1, 2, \dots,$$

and

$$d_{2n+1} = c_{2n+1} = 1,\ n = 1, 2, \dots .$$

Define positive definite operators P_1 and P_2 on $\mathscr{H}$ by

$$P_1 e_n = c_n e_n \text{ and } P_2 e_n = d_n e_n,\ n = 1, 2, \dots .$$

Define an operator Y_1 on $\mathscr{H}$ by the following: let $Y_1 e_1 = e_2$ and for each positive integer n let $Y_1 e_{2n} = e_{2n+2}$ and $Y_1 e_{2n+1} = e_{2n-1}$. Let

$$X_n = P_2^n (P_1^{-1})^n \text{ and } Y_{n+1} = P_1^n Y_1 (P_2^{-1})^n,\ n = 1, 2, \dots .$$

Observe that, for each positive integer n, we have

$$\|X_n\| = \|Y_n\| = 1,\ X_{n+1} P_1 = P_2 X_n,$$

and

$$Y_{n+1} P_2 = P_1 Y_n.$$

Let

$$X = \sum_{n=1}^{\infty} \oplus X_n \text{ and } Y = \sum_{n=1}^{\infty} \oplus Y_n.$$

Then X and Y are quasiaffinities on $\hat{\mathscr{H}}$,

$$X V_{\mathscr{H}} \hat{P}_1 = V_{\mathscr{H}} \hat{P}_2 X,$$

and

$$V_{\mathscr{H}} \hat{P}_1 Y = Y V_{\mathscr{H}} \hat{P}_2.$$

Hence $T_1 = V_{\mathscr{H}} \hat{P}_1$ and $T_2 = V_{\mathscr{H}} \hat{P}_2$ are quasisimilar quasinormal operators. We show next that T_1 and T_2 are not similar. The operator T_1 is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus c_n V$ and T_2 is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus d_n V$. It follows that

$$\|(T_1 - 1/2)x\| \geqslant 1/4\|x\|$$

for each x in $\hat{\mathscr{H}}$. Thus $T_1 - 1/2$ has closed range. Since $1/2V$ is one of the direct summands of $\sum_{n=1}^{\infty} \oplus d_n V$ and $1/2V - 1/2$ does not have closed range, it follows that $T_2 - 1/2$ does not have closed range. Hence T_1 and T_2 are not similar.

In contrast to the above examples, we have the following theorem. (Note that if T is quasinormal, then both T and T^* commute with T^*T. Thus $\mathscr{K}(T^*) \ominus \mathscr{K}(T)$ reduces T^*T.)

THEOREM 4. *Suppose that T_1 and T_2 are two quasisimilar quasinormal operators. If $T_1^* T_1|\mathscr{K}(T_1^*)\ominus\mathscr{K}(T_1)$ is compact, then T_1 and T_2 are unitarily equivalent. In particular if $\mathscr{K}(T_1^*) \ominus \mathscr{K}(T_1)$ is finite dimensional, then T_1 and T_2 are unitarily equivalent.*

Proof. We know that the normal parts of T_1 and T_2 are unitarily equivalent. Let $V_{\mathscr{H}_1}\hat{P}_1$ and $V_{\mathscr{H}_2}\hat{P}_2$ be the pure parts of T_1 and T_2, respectively. We observed in the proof of Theorem 3 that there exist bounded linear transformations X and Y havin g dense ranges such that

$$XV_{\mathscr{H}_1}\hat{P}_1 = V_{\mathscr{H}_2}\hat{P}_2X \text{ and } V_{\mathscr{H}_1}\hat{P}_1Y = YV_{\mathscr{H}_2}\hat{P}_2.$$

Since $T_1^*T_1|\mathscr{K}(T_1^*) \ominus \mathscr{K}(T_1)$ is unitarily equivalent to P_1^2, it follows from Corollary 5 that the pure parts of T_1 and T_2 are also unitarily equivalent. Thus T_1 and T_2 are unitarily equivalent.

REMARK. Note that the only Hilbert spaces considered in [4] are separable ones. However, all of the results of [4] that are mentioned in this paper are valid for Hilbert spaces of arbitrary dimensions.

REFERENCES

1. BERBERIAN, S., The Weyl spectrum of an operator, *Indiana Univ. Math. J.*, **20** (1970), 529–544.
2. BROWN, A., On a class of operators, *Proc. Amer. Math. Soc.*, **4** (1953), 723–728.
3. CLARY, S., Equality of spectra of quasi-similar hyponormal operators, *Proc. Amer. Math. Soc.*, **53** (1975), 88–90.
4. CONWAY, J., On quasisimilarity for subnormal operators, preprint.
5. HALMOS, P. R., *A Hilbert Space Problem Book*, Van Nostrand, Princeton, 1967.
6. PEARCY, C., Some recent developments in operator theory, *NSF-CBMS Regional Conference Series in Mathematics*, **36** (1978).
7. RADJAVI, H.; ROSENTHAL, P., On roots of normal operators, *J. Math. Anal. Appl.*, **34** (1971), 653–664.
8. STAMPFLI, J. G.; WADHWA, B. L., An asymmetric Putnam-Fuglede theorem for dominant operators, *Indiana Univ. Math. J.*, **25** (1976), 359–365.
9. WILLIAMS, L. R., Equality of essential spectra of certain quasisimilar seminormal operators, *Proc. Amer. Math. Soc.*, to appear.

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