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CHARACTERISTIC FUNCTIONS AND DILATIONS OF NONCONTRACTIONS

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1. INTRODUCTION

In a series of papers in *Acta Sci. Math.* between 1953 and 1966, B. Sz.-Nagy and C. Foiaş developed a theory of contractions on Hilbert space. This theory is presented in the book [19], where references to these papers can be found. The original paper [18] by Sz.-Nagy proved the existence of a unitary dilation of a contraction, and this forms the basis of the Sz.-Nagy and Foiaş theory.

In 1970, Ch. Davis [8] proved that every closed operator T has a dilation which is unitary with respect to an indefinite inner product (see Sec. 2 below), and in [9] Davis and Foiaş study the relationship between this dilation and the characteristic function (see Sec. 6 below). We continue this study in this paper, generalizing some of the work of Sz.-Nagy and Foiaş for contractions.

2. KREĬN SPACES. DILATIONS

Here is a summary of some of the notation and results that will be used in this paper (see [3], [13], [14], [15]).

An *indefinite inner product space* is a complex vector space $\mathscr{K}$ on which is defined an inner product $[.,.]$ that is not assumed to be positive, i.e., it is possible for $[h, h]$ to be negative for some $h \in \mathscr{K}$. We call $\mathscr{K}$ a *Kreĭn space* if there is an operator J on $\mathscr{K}$ such that $J^2 = I$, $J = J^*$ (i.e., $[Jh, k] = [h, Jk]$), and the *J-inner product*

$$(h, k) = [Jh, k] \tag{2.1}$$

makes $\mathscr{K}$ a Hilbert space. Such an operator J is called a *fundamental symmetry.* (See [3], Chapter V.)

In Kreĭn spaces, the emphasis is always on the indefinite inner product, with the *J-norm* $\|h\|_J = [Jh, h]^{1/2}$ serving mainly to define the topology (the *strong* topology). Accordingly, if A is a continuous operator between Kreĭn spaces $\mathscr{K}$ and $\mathscr{K}'$, we use A^* to denote the adjoint of A with respect to the indefinite inner products.

Different fundamental symmetries J on a Kreĭn space define different J-norms, but the strong topologies obtained coincide (see [13], Sec. I.4; [3], Corollary IV.6.3, Theorem V.1.1). Thus we can talk about *the* strong topology on a Kreĭn space.

If $[h, k] = 0$ then we write $h \perp k$. If $\mathscr{A}$ and $\mathscr{B}$ are two subsets of $\mathscr{K}$, then we write $h \perp \mathscr{B}$ if $h \perp k$ for all $k \in \mathscr{B}$, and $\mathscr{A} \perp \mathscr{B}$ if $h \perp \mathscr{B}$ for all $h \in \mathscr{A}$. If $\mathscr{L}$ is a subspace of a Kreĭn space $\mathscr{K}$, and if

$$\mathscr{L}^{\perp} = \{h \in \mathscr{K} : h \perp \mathscr{L}\},$$

then $\mathscr{L}$ is called *non-degenerate* if $\mathscr{L} \cap \mathscr{L}^{\perp} = \{0\}$ and *regular* if $\mathscr{L} \oplus \mathscr{L}^{\perp} = \mathscr{K}$ (where $\oplus$ denotes an orthogonal direct sum).

A *projection* on a Kreĭn space is a strongly continuous operator P satisfying $P^2 = P^* = P$. Associated with every regular subspace $\mathscr{L}$ of a Kreĭn space $\mathscr{K}$ is a projection $P_{\mathscr{L}}$ *(the projection of* $\mathscr{K}$ *onto* $\mathscr{L}$*)* which annihilates $\mathscr{L}^{\perp}$ and has range $\mathscr{L}$. In fact, the regular subspaces are precisely those that are the ranges of projections. See [15], Sec. 4.)

An operator U from $\mathscr{K}$ to $\mathscr{K}'$ is called an *isometry* if it is continuous and $[Uh, Uk] = [h, k]$ for each $h, k \in \mathscr{K}$. The condition that a continuous operator U be an isometry is equivalent to $U^*U = I$. An isometry is called *unitary* if it is surjective. As in Hilbert space, the unitary operators U are characterized by the relations $U^*U = I$ and $UU^* = I$. (See [15], Sec. 5.)

Let T be a bounded operator on a Hilbert space $\mathscr{H}$. Then there exist a Kreĭn space $\mathscr{K}$, containing $\mathscr{H}$ as a subspace, and a unitary operator U on $\mathscr{K}$ such that

$$T^n = P_{\mathscr{H}} U^n | \mathscr{H} \qquad (n = 1, 2, \ldots).$$

($\mathscr{H}$ is necessarily regular, since $P_{\mathscr{H}}$, the projection of $\mathscr{K}$ onto $\mathscr{H}$, is just the adjoint of the injection map of $\mathscr{H}$ into $\mathscr{K}$.) Also,

$$\bigvee_{n=-\infty}^{\infty} U^n \mathscr{H} = \mathscr{K}, \tag{2.2}$$

where $\bigvee$ denotes closed linear span. (See [8].) We call U a *minimal unitary dilation* of T. Note that the strong topology and inner product $[.,.]$ on $\mathscr{K}$ must restrict to the strong topology and inner product $(.,.)$ on the Hilbert space $\mathscr{H}$. Thus we have

$$[U^n h, k] = (T^n h, k) \tag{2.3}$$

for all $h, k \in \mathscr{H}$, $n = 0, 1, 2, \ldots$.

3. THE GEOMETRY OF THE DILATION SPACE

Let $\mathscr{H}$ be a Hilbert space with inner product $(.,.)$, and let T be a bounded operator on $\mathscr{H}$.

As in [9] we make the following definitions:

$$J_T = \operatorname{sgn}(I - T^*T),\ Q_T = |I - T^*T|^{1/2},$$

$$J_{T^*} = \operatorname{sgn}(I - TT^*),\ Q_{T^*} = |I - TT^*|^{1/2},$$

$$\mathscr{D}_T = J_T\mathscr{H},\ \mathscr{D}_{T^*} = J_{T^*}\mathscr{H}.$$

As well as considering $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ as subspaces of the Hilbert space $\mathscr{H}$, we will be considering them as Kreĭn spaces with the inner products $[.,.] = (J_T.,.)$ and $[.,.] = (J_{T^*}.,.)$, respectively. Note that J_T and J_{T^*} are fundamental symmetries on the Kreĭn spaces $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$, respectively.

Let U be the minimal unitary dilation of T constructed in [8], acting on the Kreĭn space $\mathscr{K}$. Then there is a fundamental symmetry J on $\mathscr{K}$ which satisfies, for $h \in \mathscr{H}$ and $n = 0, 1, 2, \ldots$, $Jh = h$, $JU^n(U - T)h = U^n(U - T)J_Th$, and $JU^{*n}(U^* - T^*)h = U^{*n}(U^* - T^*)J_{T^*}h$. It is not difficult to show (see [13], Theorem III.3.3), using techniques similar to those used in [19], Theorem I.4.1, that these conditions (with the minimality condition (2.2)) uniquely determine the dilation (up to isomorphism: cf. [19], Sec. I.4.1; [13], Sec. III.1). *In this paper we will be considering only this dilation.*

Let us define the subspaces

$$\mathscr{L} = \overline{(U - T)\mathscr{H}},\quad \mathscr{L}^* = \overline{(U^* - T^*)\mathscr{H}},\quad \text{and } \mathscr{L}_* = U\mathscr{L}^*.$$

Then (see [8]) $\mathscr{L}$ and $\mathscr{L}^*$ are regular subspaces which are *wandering* for U, i.e., $U^p\mathscr{L} \perp U^q\mathscr{L}$ and $U^p\mathscr{L}_* \perp U^q\mathscr{L}_*$ for all integers p and q, $p \neq q$. There is a unitary operator $\varphi\colon \mathscr{L} \to \mathscr{D}_T$ such that

$$\varphi(U - T)h = Q_Th\ \ (h \in \mathscr{H}), \tag{3.1}$$

$$\varphi J \,|\, \mathscr{L} = J_T\varphi,$$

and

$$\|\varphi l\| = \|l\| \quad (l \in \mathscr{L}).$$

Similarly, $\mathscr{L}^*$ is isomorphic to $\mathscr{D}_{T^*}$, with the isomorphism intertwining $J\,|\,\mathscr{L}^*$ and J_{T^*}, but it is more convenient to define the unitary operator from $\mathscr{L}_*(= U\mathscr{L}^*)$ to $\mathscr{D}_{T^*}$: There is a unitary operator $\varphi_*\colon \mathscr{L}_* \to \mathscr{D}_{T^*}$ such that

$$\varphi_*(I - UT^*)h = J_{T^*}Q_{T^*}h\ \ (h \in \mathscr{H}),$$

$$\varphi_*UJU^* \,|\, \mathscr{L}_* = J_{T^*}\varphi_*,$$

and

$$\|\varphi_*l_*\| = \|l_*\|\ \ (l_* \in \mathscr{L}_*).$$

(See [8]; [13], Sec. III.8.)

Note that $\mathscr{L}$ is a Kreĭn space with fundamental symmetry $J|\mathscr{L}$, and $\mathscr{L}_*$ is a Kreĭn space with fundamental symmetry $UJU^*|\mathscr{L}_*$. In general, $\mathscr{L}_*$ is not invariant for J.

Let us make the definitions

$$M(\mathscr{L}) = \bigvee_{n=-\infty}^{\infty} U^n\mathscr{L},$$

$$M_+(\mathscr{L}) = \bigvee_{n=0}^{\infty} U^n\mathscr{L}, \quad \text{and} \quad M_-(\mathscr{L}) = \bigvee_{n=-\infty}^{-1} U^n\mathscr{L}.$$

We define $M(\mathscr{L}_*)$, $M_+(\mathscr{L}_*)$, and $M_-(\mathscr{L}_*)$ similarly. The dilation constructed in [8] has the property that the space $\mathscr{K}$ can be decomposed into the orthogonal direct sum

$$\mathscr{K} = M_-(\mathscr{L}_*) \oplus \mathscr{H} \oplus M_+(\mathscr{L}), \tag{3.2}$$

and thus $M_-(\mathscr{L}_*)$ and $M_+(\mathscr{L})$ are regular.

If $\mathscr{K}_+ = \bigvee_{n=0}^{\infty} U^n\mathscr{H}$, then we also have

$$\mathscr{K}_+ = \mathscr{H} \oplus M_+(\mathscr{L}). \tag{3.3}$$

Let $\mathscr{M}$ denote any one of the subspaces $M_+(\mathscr{L})$, $M(\mathscr{L})$, or $M_-(\mathscr{L})$. If $h \in \mathscr{M}$, then the *Fourier coefficients of h in* $\mathscr{M}$ are

$$l_n = PU^{*n}h, \tag{3.4}$$

where P is the projection of $\mathscr{K}$ onto $\mathscr{L}$ (see [15]). In (3.4), n is an integer satisfying $0 \leqslant n < \infty$, $-\infty < n < \infty$, or $-\infty < n \leqslant -1$, according to whether $\mathscr{M}$ is $M_+(\mathscr{L})$, $M(\mathscr{L})$, or $M_-(\mathscr{L})$, respectively. The Fourier coefficients for $M_+(\mathscr{L}_*)$, $M(\mathscr{L}_*)$, and $M_-(\mathscr{L}_*)$ are defined similarly, using (3.4) with $P = P_{\mathscr{L}_*}$.

For any bounded operator T there is a maximal subspace $\mathscr{H}_0$ in $\mathscr{H}$ reducing T to a unitary operator (see [1] and [10]), and this can be given explicitly in terms of the dilation:

$$\mathscr{H}_0 = \bigcap_{n=-\infty}^{\infty} U^n\mathscr{N}, \tag{3.5}$$

where $\mathscr{N} = \mathscr{H} \ominus (\mathscr{D}_T \vee \mathscr{D}_{T^*})$. (See [8], Sec. 4.) If $\mathscr{H}_1 = \mathscr{H} \ominus \mathscr{H}_0$, then $T|\mathscr{H}_1$ is completely non-unitary, i.e. there is no non-zero subspace of $\mathscr{H}_1$ which reduces T to a unitary operator.

We have the following extension of [19], Proposition II.1.4:

THEOREM 3.1. $M(\mathscr{L}) \vee M(\mathscr{L}_*) = \mathscr{K} \ominus \mathscr{H}_0$.

Proof. Let $\mathscr{H}_0' = (M(\mathscr{L}) \vee M(\mathscr{L}_*))^\perp$. From (3.2) it follows that $\mathscr{H}_0' \subseteq \mathscr{H}$. Suppose $h \in \mathscr{H}_0'$. Then, since $h \perp U^{-1}\mathscr{L}$, we have for all $h' \in \mathscr{H}$ (using (2.3))

$$0 = [h, U^{-1}(U - T)h'] = (h, (I - T^*T)h') =$$

$$= ((I - T^*T)h, h').$$

Hence $Q_T h = 0$, and so, by (3.1), $Uh = Th$. Since $M(\mathscr{L})$ and $M(\mathscr{L}_*)$ reduce U, so does $\mathscr{H}_0'$, and the above calculation shows that $\mathscr{H}_0'$ reduces T to a unitary. Hence $\mathscr{H}_0' \subseteq \mathscr{H}_0$.

Conversely, we know by (3.5) that $\mathscr{H}_0$ reduces U and, by (3.2) (since $\mathscr{H}_0 \subseteq \mathscr{H}$), that $\mathscr{H}_0 \perp \mathscr{L}$ and $\mathscr{H}_0 \perp U^{-1}\mathscr{L}_*$. Hence $\mathscr{H}_0 \perp (M(\mathscr{L}) \vee M(\mathscr{L}_*))$, i.e., $\mathscr{H}_0 \subseteq \mathscr{H}_0'$.

We conclude that $\mathscr{H}_0 = \mathscr{H}_0'$, and the theorem is proved. ▨

COROLLARY 3.2. *If T is completely non-unitary, then $M(\mathscr{L}) \vee M(\mathscr{L}_*) = \mathscr{K}$.* ▨

4. THE RESIDUAL AND DUAL RESIDUAL SPACES

The *residual space* and *dual residual space* are defined by

$$\mathscr{R} = M(\mathscr{L}_*)^\perp \text{ and } \mathscr{R}_* = M(\mathscr{L})^\perp,$$

respectively. Since $M(\mathscr{L}_*)$ and $M(\mathscr{L})$ reduce U, so do $\mathscr{R}$ and $\mathscr{R}_*$. Note that, by (3.2) and (3.3), $\mathscr{R} \subseteq \mathscr{K}_+$, and so $\mathscr{R}$ may be written as the space $M_+(\mathscr{L}_*)^\perp$, considered as a subspace of $\mathscr{K}_+$. (We could also make the obvious dual comments about $\mathscr{R}_*$.)

(3.2) implies that $M_-(\mathscr{L}_*)$ is regular. If $M_+(\mathscr{L}_*)$ is also regular, then so is $M(\mathscr{L}_*)$ ([15], Theorem 10.1 and Theorem 4.6). Consequently, in this case we have

$$\mathscr{K} = M(\mathscr{L}_*) \oplus \mathscr{R} \text{ and } \mathscr{K}_+ = M_+(\mathscr{L}_*) \oplus \mathscr{R}.$$

However, as the following example shows, $M_+(\mathscr{L}_*)$ is not always regular and the geometry of the dilation space can be quite different from that described above.

EXAMPLE 4.1. Let $\mathscr{H}$ be the one-dimensional space of complex numbers, and let T be multiplication by the complex number α, $|\alpha| > 1$. Then a vector in $\mathscr{K}_+$ may be represented as a sequence $h = \{h_n\}_{n \geq 0}$, where $h_n \in \mathscr{H}$ for all $n \geq 0$, and

$$\|h\|^2 = \sum_{n=0}^{\infty} |h_n|^2 < \infty.$$

The inner product on $\mathscr{K}_+$ is given by

$$[h, k] = h_0\bar{k}_0 - \sum_{n=1}^{\infty} h_n\bar{k}_n ,$$

and the dilation U satisfies (for $h \in \mathscr{K}_+$)

$$(Uh)_n = \begin{cases} \alpha h_0 & (n = 0), \\ (|\alpha|^2 - 1)^{1/2} h_0 & (n = 1), \\ h_{n-1} & (n > 1). \end{cases}$$

$\mathscr{L}_*$ is spanned by the vector l, with $l_0 = (|\alpha|^2 - 1)^{1/2}$, $l_1 = \bar{\alpha}$, and $l_n = 0$ for $n > 1$. We then argue, as in [15], Example 6.4, that $M_+(\mathscr{L}_*)^\perp$ is spanned by the vector r, with $r_0 = (|\alpha|^2 - 1)^{-1/2}$, and $r_n = \alpha^{-n}$ for $n \geqslant 1$, and that $r \in M_+(\mathscr{L}_*)$. Hence, $M_+(\mathscr{L}_*)$ is degenerate.

(Note: [15], Example 6.4 is the case $\alpha = 2$.) ▨

We have the following useful representation of the residual space. (The dual residual space has the obvious dual representation, but this will not be needed.) Observe that, since $\mathscr{R} \subseteq \mathscr{K}_+$, it suffices to consider only vectors $k \in \mathscr{K}_+$ in the following theorem.

THEOREM 4.2. *A vector $k \in \mathscr{K}_+$ is in $\mathscr{R}$ if and only if there is a sequence $\{h_n\}_{n \geq 0}$ of vectors in $\mathscr{H}$ such that*

(i) *h_0 is the projection of k into $\mathscr{H}$,*

(ii) *$Th_{n+1} = h_n$ $(n \geqslant 0)$, and*

(iii) *$\{(U - T)h_{n+1}\}_{n \geq 0}$ is the sequence $\{l_n\}_{n \geqslant 0}$ of Fourier coefficients in $M_+(\mathscr{L})$ of the projection of k into $M_+(\mathscr{L})$.*

The sequence $\{h_n\}_{n \geq 0}$ and k uniquely determine each other.

Proof. By (3.3), every $k \in \mathscr{K}_+$ has a unique representation of the form $k = h_0 + m$, where $m \in M_+(\mathscr{L})$ and h_0 is the vector in $\mathscr{H}$ satisfying (i). Suppose $k \in \mathscr{K}_+$, and assume that conditions (ii) and (iii) are also satisfied for some sequence $\{h_n\}_{n \geqslant 0}$. We know (by (3.2)) that $M_-(\mathscr{L}_*) \perp M_+(\mathscr{L})$, and thus for $N \geqslant 0$ we have $U^N \mathscr{L}_* \perp U^{N+1} M_+(\mathscr{L})$. Also, since $\{l_n\}_{n \geqslant 0}$ is the sequence of Fourier coefficients of m in $M_+(\mathscr{L})$, we have for $N \geqslant 0$

$$m - \sum_{n=0}^{N} U^n l_n \in U^{N+1} M_+(\mathscr{L})$$

see [15], Sec. 7), and thus we deduce, for all $l_* \in \mathscr{L}_*$ and $N \geqslant 0$, the equation

$$(4.1) \qquad [k, U^N l_*] = [h_0 + m, U^N l_*] =$$

$$= [h_0, U^N l_*] + \sum_{n=0}^{N} [U^n l_n, U^N l_*].$$

Let us compute this for a dense set of l_* in $\mathscr{L}_*$, namely for $l_* = (I - UT^*)h$, where $h \in \mathscr{H}$. Using (iii), we then obtain, for $0 \leqslant n \leqslant N - 1$,

$$[U^n l_n, U^N l_*] = [U^n(U - T)h_{n+1}, U^N(I - UT^*)h] =$$

$$= [h_{n+1}, U^{N-n-1}(I - UT^*)h] - [Th_{n+1}, U^{N-n}(I - UT^*)h] =$$

$$= (h_{n+1}, T^{N-n-1}(I - TT^*)h) - (Th_{n+1}, T^{N-n}(I - TT^*)h).$$

By successive applications of (ii), we can write $h_{n+1} = T^{N-n-1}h_N$, and it follows that

$$[U^n l_n, U^N l_*)] = (h_N, (T^{*N-n-1}T^{N-n-1} - T^{*N-n}\, T^{N-n})\, (I - TT^*)\, h).$$

We therefore have a telescoping series, and can deduce the formula

$$\sum_{n=0}^{N-1} [U^n l_n, U^N l_*] = (h_N, (I - TT^*)h) - (h_N, T^{*N}T^N(I - TT^*)h). \tag{4.2}$$

We also have the following (using the fact that $\mathscr{H} \perp U^{-1}\mathscr{L}_*$ (from (3.2)), and thus $U\mathscr{H} \perp \mathscr{L}_*$):

$$\begin{aligned}[U^N l_N, U^N l_*] &= [l_N, l_*] = [(U - T)h_{N+1}\,, (I - UT^*)h] = \\ &= -(Th_{N+1}, (I - TT^*)h) = \\ &= -(h_N, (I - TT^*)\, h).\end{aligned} \tag{4.3}$$

Computing the final term in the expression in (4.1) gives us

$$\begin{aligned}[h_0, U^N l_*] &= [h_0, U^N(I - UT^*)h] = (h_0, T^N(I - TT^*)h) = \\ &= (h_n, T^{*N}T^N(I - TT^*)\, h).\end{aligned} \tag{4.4}$$

Therefore, we have from equations (4.1) — (4.4) the result that $[k, U^N l_*] = 0$ for all $N \geqslant 0$ and for a dense set of vectors l_* in $\mathscr{L}_*$. Hence, $k \perp M_+(\mathscr{L}_*)$ and (since we have assumed $k \in \mathscr{K}_+$) it follows that $k \in \mathscr{R}$.

Conversely, suppose $k \in \mathscr{R}$. We will define the sequence $\{h_n\}_{n \geqslant 0}$ inductively. Let h_0 be defined by (i) and suppose that, for some $N \geqslant 0$, $h_0, h_1, \ldots, h_N$ have been defined so that (ii) and (iii) are satisfied. Equations (4.1), (4.2), and (4.4) remain valid, and thus we have (since $k \in \mathscr{R}$),

$$0 = [k, U^N l_*] = (h_N, (I - TT^*)h) + [l_N, l_*]\,, \tag{4.5}$$

where $l_* = (I - UT^*)h$. From (3.2) we obtain $[l_N, (I - TT^*)h] = 0$, and from this it follows that

$$[l_N, l_*] = - [l_N, (U - T)T^*h].$$

Consequently, we obtain from (4.5)

$$(h_N, (I - TT^*)\, h) = [l_N, (U - T)T^*h].$$

Consider the continuous operator $Q: \mathscr{H} \to \mathscr{L}$ defined by $Qh = (U - T)h$ $(h \in \mathscr{H})$. We can then write the previous result in the form

$$((I - TT^*)\, h_N,\ h) = [l_N, QT^*h] = (TQ^*l_N, h),$$

and hence we have

$$(I - TT^*)\, h_N = TQ^*l_N\,.$$

Therefore, if we define h_{N+1} by

$$h_{N+1} = T^*h_N + Q^*l_N\ ,$$

then we conclude that $Th_{N+1} = h_N$, and (ii) is satisfied. We also have

$$(I - T^*T)h_{N+1} = h_{N+1} - T^*h_N = Q^*l_N\,,$$

and it then follows, with the help of (2.3), that for all $h \in \mathscr{H}$

$$[l_N, (U - T)\, h] = [l_N, Qh] = (Q^*l_N, h) =$$
$$= ((I - T^*T)h_{N+1}, h) =$$
$$= [(U - T)h_{N+1}, (U - T)\, h].$$

Therefore, $l_N = (U - T)h_{N+1}$, and (iii) is satisfied. This completes the inductive definition of $\{h_n\}$, and it remains to prove the uniqueness assertion.

Suppose $h_n = 0$ for all $n \geqslant 0$. In particular, $h_0 = 0$ and thus $k \in M_+(\mathscr{L})$. $\{l_n\}_{n\geqslant 0}$ is the sequence of Fourier coefficients of k in $M_+(\mathscr{L})$ and, by (iii), $l_n = 0$ for all $n \geqslant 0$. Since $M_+(\mathscr{L})$ is regular, [15], Theorem 7.2 shows that $k = 0$, and thus $\{h_n\}_{n\geqslant 0}$ uniquely determines k.

The sequence $\{l_n\}_{n\geqslant 0}$ is uniquely determined by k, so by (i) and the recurrence relation $h_{n+1} = U^*(l_n + h_n)$ (easily derived from (ii) and (iii)), k uniquely determines $\{h_n\}_{n\geqslant 0}$. ∎

5. POSITIVITY OF THE RESIDUAL SPACE. A SUFFICIENT CONDITION FOR $\mathscr{R} = \{0\}$

If $[k, k] \geqslant 0$ ($[k, k] > 0$) for all nonzero k in a subspace, then that subspace is called *positive (positive definite)*.

THEOREM 5.1. *$\mathscr{R}$ is a positive subspace. If T is power bounded, then $\mathscr{R}$ is positive definite.*

Proof. Let k be a vector in $\mathscr{R}$, and let $\{h_n\}_{n\geqslant 0}$ be the sequence of vectors in $\mathscr{H}$ corresponding to k, defined by Theorem 4.2. (We will use throughout this proof the notation of Theorem 4.2.) Then we have, by the definition of the inner product in [8] and by (2.3), the following:

$$[k, k] = \|h_0\|^2 + \sum_{n=0}^{\infty} [l_n, l_n] =$$

$$= \|h_0\|^2 + \sum_{n=0}^{\infty} [(U - T)h_{n+1}, (U - T)h_{n+1}] =$$

$$= \|h_0\|^2 + \sum_{n=0}^{\infty} ((I - T^*T)h_{n+1}, h_{n+1}).$$

Theorem 4.2(ii) implies that, for $0 \leqslant n \leqslant N - 1$, $h_{n+1} = T^{N-n-1}h_N$. We therefore obtain the chain of equalities

$$[k, k] = \|h_0\|^2 + \lim_{N\to\infty} \sum_{n=0}^{N-1} ((I - T^*T)T^{N-n-1}h_N, T^{N-n-1}h_N) =$$

$$= \|h_0\|^2 + \lim_{N\to\infty} \left(\sum_{n=0}^{N-1} T^{*N-n-1}(I - T^*T)T^{N-n-1}h_N, h_N\right) =$$

$$= \|h_0\|^2 + \lim_{N\to\infty} (h_N - T^{*N}T^N h_N, h_N) =$$

$$= \|h_0\|^2 + \lim_{N\to\infty} (\|h_N\|^2 - \|T^N h_N\|^2).$$

But $T^N h_N = h_0$ (Theorem 4.2(ii)), and thus we have

$$[k, k] = \lim_{N\to\infty} \|h_N\|^2 \geqslant 0. \tag{5.1}$$

Hence $\mathscr{R}$ is positive.

Now let us suppose that $[k, k] = 0$; it follows from (5.1) that $\lim_{N\to\infty} h_N = 0$. For each $n \geqslant 0$ and $N \geqslant n$, we have

$$\|h_n\| = \|T^{N-n}h_N\| \leqslant \|T^{N-n}\| \, \|h_N\|.$$

Therefore, if T is power bounded, $h_n = 0$ for each $n \geqslant 0$, and so $k = 0$. Hence we conclude that if T is power bounded, $\mathscr{R}$ is positive definite. ▨

COROLLARY 5.2. *$\mathscr{R}_*$ is a positive subspace. If T is power bounded, then $\mathscr{R}_*$ is positive definite.* ▨

COROLLARY 5.3. *If T is power bounded, then $M(\mathscr{L})$ and $M(\mathscr{L}_*)$ are non-degenerate.* ▨

COROLLARY 5.4. *If $k \in \mathscr{R}$, then the sequence $\{h_n\}_{n\geq 0}$ of vectors in $\mathscr{H}$ defined by Theorem 4.2 is bounded.*

Proof. By (5.1), $\lim_{n\to\infty} \|h_n\|^2$ exists. ▨

THEOREM 5.5. *If $\lim_{n\to\infty} T^{*n} = 0$, then $M(\mathscr{L}_*) = \mathscr{K}$.*

Proof. Suppose $k \in \mathscr{R}$ and let $\{h_n\}_{n\geq 0}$ be the sequence of vectors in $\mathscr{H}$ defined by Theorem 4.2. For each $n \geq 0$, $N \geq n$, and $h \in \mathscr{H}$, we have

$$|(h_n, h)| = |(T^{N-n}h_N, h)| = |(h_N, T^{*N-n}h)| \leqslant$$
$$\leqslant \|h_N\| \, \|T^{*N-n}h\|.$$

By Corollary 5.4, the sequence $\{h_n\}_{n\geq 0}$ is bounded. Hence if $\lim_{N\to\infty} T^{*N} = 0$, then we obtain $(h_n, h) = 0$ for all $n \geq 0$ and for all $h \in \mathscr{H}$. Consequently, $h_n = 0$ for all $n \geqslant 0$, and so, by Theorem 4.2, $k=0$. We therefore conclude that $\mathscr{R} = \{0\}$, i.e., $M(\mathscr{L}_*) = \mathscr{K}$. ▨

COROLLARY 5.6. *If $\lim_{n\to\infty} T^n = 0$, then $M(\mathscr{L}) = \mathscr{K}$.* ▨

COROLLARY 5.7. *If a vector $h \in \mathscr{H}$ satisfies $\lim_{n\to\infty} T^{*n}h = 0$, then $h \in M_+(\mathscr{L}_*)$.*

Proof. Suppose $k \in \mathscr{R}$. Then, as above, we have (since h_0 is the projection of k into $\mathscr{H}$)

$$[k, h] = (h_0, h) = 0.$$

Thus $h \perp \mathscr{R}$, and using (3.2) we can deduce that $h \in M_+(\mathscr{L}_*)$. ▨

REMARK. We could, if we wished, deduce Theorem 5.5 from Corollary 5.7, since if $\mathscr{H} \subseteq M(\mathscr{L}_*)$ then the minimality of the dilation implies $M(\mathscr{L}_*) = \mathscr{K}$.

COROLLARY 5.8. *If a vector $h \in \mathscr{H}$ satisfies $\lim_{n\to\infty} T^n h = 0$, then $h \in M_-(\mathscr{L})$.* ▨

When T is a contraction, the condition $\lim_{n\to\infty} T^{*n} = 0$ is equivalent to $M(\mathscr{L}_*) = \mathscr{K}$ ([19], Theorem II.1.2). The following example shows that the converse to Theorem 5.5 is not valid for a general bounded operator T. However, in Sec. 8 it will be shown that an extra condition on T (namely the boundedness of its characteristic function) enables us to obtain the converse.

EXAMPLE 5.9. Let $\mathscr{H}$ be a two-dimensional space and define T on $\mathscr{H}$ by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Suppose $k \in \mathscr{R}$; then the sequence $\{h_n\}_{n\geq 0}$ of vectors in $\mathscr{H}$ defined by Theorem 4.2 is a constant sequence, since we have $T^2 = T$ and $h_n = Th_{n+1}$ for all $n \geqslant 0$. Consequently, the sequence $\{l_n\}_{n\geq 0}$ in Theorem 4.2 is also constant ($l_n = (U - T)h_{n+1}$), and since, by the definition of the norm in [8], $\sum_{n=0}^{\infty} \|l_n\|^2 < \infty$, we conclude that $l_n = 0$ for each $n \geqslant 0$. We have $Q_T = I$, and hence, by (3.1), $U - T$ is injective. It therefore follows that $\{h_n\}_{n \geqslant 0}$ is the zero sequence, and hence $k = 0$. Thus $\mathscr{R} = \{0\}$, while $\lim_{n\to\infty} T^{*n} = T^* \neq 0$. ▨

6. THE CHARACTERISTIC FUNCTION. FOURIER REPRESENTATIONS

The *characteristic function* of T is the operator valued analytic function

$$\Theta_T(\lambda) = [-TJ_T + \lambda J_{T^*} Q_{T^*}(I - \lambda T^*)^{-1} J_T Q_T] | \mathscr{D}_T. \tag{6.1}$$

$\Theta_T(\lambda)$ is defined for those complex numbers λ for which $I - \lambda T^*$ is boundedly invertible, and takes values which are continuous operators from $\mathscr{D}_T$ to $\mathscr{D}_{T^*}$. (See, for example, the following: [2], [4], [5], [6], [7], [9], [11], [12], [13], [14], [16], and [19].)

We will be assuming for the remainder of this paper that $\mathscr{H}$ is separable and that T has bounded characteristic function, i.e.,

$$\sup \{\|\Theta_T(\lambda)\| : |\lambda| < 1\} = C < \infty. \tag{6.2}$$

Suppose $h \in M(\mathscr{L}_*)$, and let $\{l_n\}$ be the sequence of Fourier coefficients (3.4) of h in $M(\mathscr{L}_*)$. It is a consequence of the definition of the norm in [8] that $\sum_{n=-\infty}^{-1} \|l_n\|^2 < \infty$. Also, it is shown in [9] that if Θ_T is bounded then $\sum_{n=0}^{\infty} \|l_n\|^2 < \infty$. Therefore, when Θ_T is bounded, we can define $\Phi_{\mathscr{L}_*}$, the *Fourier representation of* $M(\mathscr{L}_*)$, by

$$(\Phi_{\mathscr{L}_*} h)(t) = \sum_{n=-\infty}^{\infty} e^{int} \varphi_* l_n,$$

where φ_* is the unitary map from $\mathscr{L}_*$ to $\mathscr{D}_{T^*}$ discussed in Sec. 3. $\Phi_{\mathscr{L}_*}$ is a unitary operator from $M(\mathscr{L}_*)$ to $L^2(\mathscr{D}_{T^*})$, the Kreĭn space of square integrable $\mathscr{D}_{T^*}$-valued functions with inner product

$$[u, v] = 1/2\pi \int_0^{2\pi} [u(t), v(t)] \mathrm{d}t, \quad (u, v \in L^2(\mathscr{D}_{T^*}))$$

(cf. [19], Sec. V.1; [13], Sec. IV.1).

Similarly, if $h \in M(\mathscr{L})$ and $\{l_n\}$ is the sequence of Fourier coefficients of h in $M(\mathscr{L})$, then $\sum_{n=-\infty}^{\infty} \|l_n\|^2 < \infty$ whenever Θ_T is bounded. We define $\Phi_{\mathscr{L}}$, the *Fourier representation of* $M(\mathscr{L})$, by

$$(\Phi_{\mathscr{L}} h)(t) = \sum_{n=-\infty}^{\infty} e^{int} \varphi\, l_n.$$

$\Phi_{\mathscr{L}}$ is a unitary operator from $M(\mathscr{L})$ to $L^2(\mathscr{D}_T)$.

Note that the Fourier representations take their values in $\mathscr{D}_T$ (for $M(\mathscr{L})$) or $\mathscr{D}_{T^*}$ (for $M(\mathscr{L}_*)$), and not in $\mathscr{L}$ or $\mathscr{L}_*$. In this respect we are following [9] instead of [19]. Also note that from [9], p. 135 we can deduce that the Fourier representations and their inverses have norms less than or equal to $(2C^2 + 1)^{1/2}$, where C is given by (6.2).

When $\mathscr{H}$ is separable and Θ_T is bounded, then the strong limit

$$\Theta_T(e^{it}) = \lim_{r \to 1-} \Theta_T(re^{it})$$

exists almost everywhere, and we obtain a bounded operator $\Theta_T: L^2(\mathscr{D}_T) \to L^2(\mathscr{D}_{T^*})$ defined by

$$(\Theta_T v)(t) = \Theta_T(e^{it})v(t) \text{ a.e.} \quad (v \in L^2(\mathscr{D}_T))$$

(cf. [19], Sec. V.2). With our definition of $\Theta_T(\lambda)$, we then have, for $h \in M(\mathscr{L})$ and $h_* \in M(\mathscr{L}_*)$,

$$[h, h_*] = [\Theta_T \Phi_{\mathscr{L}} h, \Phi_{\mathscr{L}_*} h_*]. \tag{6.3}$$

This result appears in [9], Sec. III.1, with a slightly different definition of Θ_T. Also, in [9], (6.3) is proved for $M_+(\mathscr{L})$ and $M_+(\mathscr{L}_*)$ only, but it is not difficult to generalize the arguments in [9] to establish (6.3) for all $h \in M(\mathscr{L})$ and $h_* \in M(\mathscr{L}_*)$ (see [13], Sec. IV.6]).

An alternative formulation of (6.3) is

$$\Theta_T \Phi_{\mathscr{L}} = \Phi_{\mathscr{L}_*} P | M(\mathscr{L}), \tag{6.4}$$

where P is the projection of $\mathscr{K}$ onto $M(\mathscr{L}_*)$.

Finally, note that when Θ_T is bounded, $M(\mathscr{L})$ and $M(\mathscr{L}_*)$ are the ranges of the unitary operators $\Phi^*_{\mathscr{L}}$ and $\Phi^*_{\mathscr{L}_*}$, respectively. Thus, by [15], Theorem 5.2, $M(\mathscr{L})$ and $M(\mathscr{L}_*)$ are regular (cf. [9], Sec. III.2).

7. THE RESIDUAL AND DUAL RESIDUAL SPACES AS HILBERT SPACES. SOME SIMILARITY RESULTS

We again consider the spaces $\mathscr{R}$ and $\mathscr{R}_*$ introduced in Sec. 4. When Θ_T is bounded, $M(\mathscr{L})$ and $M(\mathscr{L}_*)$ are regular, and thus $\mathscr{R}$ and $\mathscr{R}_*$ are also regular. We can then strengthen Theorem 5.1 and Corollary 5.2:

THEOREM 7.1. *If Θ_T is bounded then, with the inner product $[.,.]$, $\mathscr{R}$ and $\mathscr{R}_*$ are Hilbert spaces. The intrinsic topologies on $\mathscr{R}$ and $\mathscr{R}_*$ (defined by $\|h\| = [h, h]^{1/2}$) coincide with the strong topologies (defined by $\|h\| = [Jh, h]^{1/2}$).*

Proof. Since $\mathscr{R}$ and $\mathscr{R}_*$ are regular, [3], Theorem V.3.4 implies that they are Kreĭn spaces. But $\mathscr{R}$ and $\mathscr{R}_*$ are both positive subspaces (Theorem 5.1 and Corollary 5.2), and so they are Hilbert spaces.

The intrinsic and strong topologies coincide by virtue of [3], Theorem V.5.2. ▨

If $\mathscr{K}$ is considered as a Hilbert space with the J-inner product (2.1), then the operators $U|\mathscr{R}$ and $U|\mathscr{R}_*$ are not unitary, but they are similar to unitary operators when Θ_T is bounded. This is proved in [9], p. 137, using the theorem of B. Sz.-Nagy [17], but we have here an explicit realization of this result. Indeed, simply renorming $\mathscr{R}$ and $\mathscr{R}_*$ with the equivalent norm $\|h\| = [h, h]^{1/2}$ makes the operators unitary. This observation leads us to a simple geometric interpretation of the similarity results of Sahnovič [16] and Davis and Foiaş [9].

THEOREM 7.2. ([16], Theorem 1) *Consider $\mathscr{K}$ as a Hilbert space with the J-inner product. Then, if Θ_T is bounded, U is similar to a unitary operator (on a Hilbert space).*

Proof. Since $M(\mathscr{L}_*)$ is regular, every vector $h \in \mathscr{K}$ is of the form $h = m + r$ ($m \in M(\mathscr{L}_*)$, $r \in \mathscr{R}$). We can thus define a norm on $\mathscr{K}$ by

$$\|h\|^2 = \|\Phi_{\mathscr{L}_*} m\|^2 + [r, r].$$

By Theorem 7.1 and the continuity of $\Phi_{\mathscr{L}_*}$ and $\Phi_{\mathscr{L}_*}^{-1}$, this norm is equivalent to the J-norm. Since $\Phi_{\mathscr{L}_*} U \Phi_{\mathscr{L}_*}^{-1}$ is multiplication by e^{it} on $L^2(\mathscr{D}_{T^*})$, we clearly have

$$\|Uh\| = \|h\|$$

for all $h \in \mathscr{K}$. Therefore, with this norm, U is unitary. ▨

COROLLARY 7.3. ([9]) *If Θ_T is bounded, T is similar to a contraction.*

Proof. (cf. [16]) The operator $U_+ = U|\mathscr{K}_+$ is similar to an isometry, and $T^* = U_+^* |\mathscr{H}$. ▨

8. THE OPERATOR F. SOME RESULTS ON RESIDUAL SPACES

Let us denote by G the operator $\Phi_{\mathscr{L}_*}^*$, considered as mapping $L^2(\mathscr{D}_{T^*})$ to $\mathscr{K}$, and let F be the operator G^*, mapping $\mathscr{K}$ to $L^2(\mathscr{D}_{T^*})$. Then, if P is the projection of $\mathscr{K}$ onto $M(\mathscr{L}_*)$, we have

$$F = \Phi_{\mathscr{L}_*} P,$$

and hence $FG = I$ and $GF = P$.

Theorem 3.1 shows that $\mathscr{K}$ is spanned by $M(\mathscr{L})$, $M(\mathscr{L}_*)$, and $\mathscr{H}_0$, where $\mathscr{H}_0$ is the maximal subspace of $\mathscr{H}$ reducing T to a unitary operator. Since F is continuous, it is determined by its values on these three subspaces. This representation is simple to write down, using (6.4) and noting that $\mathscr{H}_0 \subseteq \mathscr{R}$ (Theorem 3.1):

$$F|M(\mathscr{L}) = \Theta_T \Phi_{\mathscr{L}},$$

$$F|M(\mathscr{L}_*) = \Phi_{\mathscr{L}_*},$$

and

$$F|\mathscr{H}_0 = 0.$$

We also have the following explicit representation of F.

THEOREM 8.1. *For $h \in \mathscr{K}$, the function $Fh \in L^2(\mathscr{D}_{T^*})$ has Fourier series*

$$(Fh)(t) = \sum_{n=-\infty}^{\infty} e^{int} \varphi_* P_{\mathscr{L}_*} U^{*n} h,$$

where $P_{\mathscr{L}_*}$ denotes the projection of $\mathscr{K}$ onto $\mathscr{L}_*$.

Proof. If $h \in M(\mathscr{L}_*)$, then the vectors $P_{\mathscr{L}_*} U^{*n} h$ ($n=0, \pm 1, \pm 2, \ldots$) are the Fourier coefficients of h in $M(\mathscr{L}_*)$. It therefore follows from the definitions of $\Phi_{\mathscr{L}_*}$ and F that

$$\sum_{n=-\infty}^{\infty} e^{int} \varphi_* P_{\mathscr{L}_*} U^{*n} h = (\Phi_{\mathscr{L}_*} h)(t) = (Fh)(t).$$

Now suppose that $h \in \mathscr{R}$. Since $\mathscr{R}$ reduces U, we also have, for all n, $U^{*n} h \in \mathscr{R}$. But $\mathscr{R}$ is orthogonal to $M(\mathscr{L}_*)$, and hence also to $\mathscr{L}_*$, and thus $P_{\mathscr{L}_*} U^{*n} h = 0$. Consequently we have

$$\sum_{n=-\infty}^{\infty} e^{int} \varphi_* P_{\mathscr{L}_*} U^{*n} h = 0 = (Fh)(t).$$

The required result follows from the fact that $\mathscr{K} = M(\mathscr{L}_*) \oplus \mathscr{R}$. ▨

COROLLARY 8.2. *For $h \in \mathscr{H}$, the function $Fh \in L^2(\mathscr{D}_{T_*})$ has Fourier series*

$$(Fh)(t) = \sum_{n=0}^{\infty} e^{int} J_{T^*} Q_{T^*} T^{*n} h.$$

Proof. Since $\mathscr{H} \perp M_-(\mathscr{L}_*)$ (by (3.2)), we deduce that $U^{*n} \mathscr{H} \perp \mathscr{L}_*$ for $n < 0$, and thus

$$P_{\mathscr{L}^*} U^{*n} h = 0 \quad (h \in \mathscr{H},\ n < 0).$$

When $n \geqslant 0$, we can write

$$U^{*n} h = U T^{*n+1} h + \sum_{k=0}^{n} U^{*k} (I - UT^*) T^{*n-k} h.$$

Since $U\mathscr{H} \perp \mathscr{L}_*$ and $U^{*k} \mathscr{L}_* \perp \mathscr{L}_*$ for $k = 1, 2, \ldots, n$, we deduce that

$$P_{\mathscr{L}_*} U^{*n} h = (I - UT^*) T^{*n} h.$$

Hence we have $\varphi_* P_{\mathscr{L}_*} U^{*n} h = J_{T^*} Q_{T^*} T^{*n} h$ (see Sec. 3), and the corollary is proved. ▨

COROLLARY 8.3. *When Θ_T is bounded we have, for all $h \in \mathscr{H}$,*

$$\sum_{n=0}^{\infty} \|J_{T^*} Q_{T^*} T^{*n} h\|^2 < \infty. \text{ ▨}$$

The following extends [19], Proposition II.3.1 from contractions to all operators T with bounded characteristic function.

THEOREM 8.4. *Suppose T has bounded characteristic function. Then, for $h \in \mathscr{H}$,*

$$P_{\mathscr{R}} h = \lim_{n \to \infty} U^n T^{*n} h, \quad P_{\mathscr{R}_*} h = \lim_{n \to \infty} U^{*n} T^n h,$$

and hence

$$[P_{\mathscr{R}} h, P_{\mathscr{R}} h] = \lim_{n \to \infty} \|T^{*n} h\|^2, \; [P_{\mathscr{R}_*} h, P_{\mathscr{R}_*} h] = \lim_{n \to \infty} \|T^n h\|^2,$$

$$P_{\mathscr{H}} P_{\mathscr{R}} h = \lim_{n \to \infty} T^n T^{*n} h, \quad P_{\mathscr{H}} P_{\mathscr{R}_*} h = \lim_{n \to \infty} T^{*n} T^n h.$$

Proof. Suppose the function v in $L^2(\mathscr{D}_{T^*})$ has Fourier series

$$v(t) = \sum_{n=-\infty}^{\infty} e^{int} a_n.$$

Since $\Phi_{\mathscr{L}_*}^{-1} = \Phi_{\mathscr{L}_*}^*$ is continuous, and since for each M and N (with $M \leqslant N$) we have

$$\Phi_{\mathscr{L}_*}^* \sum_{n=M}^{N} e^{int} a_n = \sum_{n=M}^{N} U^n \varphi_*^{-1} a_n,$$

we deduce that

$$G v = \Phi_{\mathscr{L}_*}^* v = \sum_{n=-\infty}^{\infty} U^n \varphi_*^{-1} a_n.$$

Hence, with P denoting the projection of $\mathscr{K}$ onto $M(\mathscr{L}_*)$, Corollary 8.2 gives us

$$P h = G F h = \sum_{n=0}^{\infty} U^n \varphi_*^{-1} J_{T^*} Q_{T^*} T^{*n} h =$$

$$= \sum_{n=0}^{\infty} U^n (I - U T^*) T^{*n} h =$$

$$= \sum_{n=0}^{\infty} (U^n T^{*n} h - U^{n+1} T^{*n+1} h) =$$

$$= \lim_{n \to \infty} (h - U^n T^{*n} h).$$

Consequently we obtain

$$P_{\mathscr{R}}h = (I - P)h = \lim_{n\to\infty} U^n T^{*n} h.$$

Similarly, we also have $P_{\mathscr{R}_*}h = \lim_{n\to\infty} U^{*n}T^n h$, and the remaining assertions of the theorem follow immediately. ▨

We can now prove the result referred to in Sec. 5, immediately prior to Example 5.9. For the contraction case, see [19], Theorem II.1.2.

COROLLARY 8.5. *If Θ_T is bounded, then*

(i) $M(\mathscr{L}_*) = \mathscr{K}$ *if and only if* $\lim_{n\to\infty} T^{*n} = 0$, *and*

(ii) $M(\mathscr{L}) = \mathscr{K}$ *if and only if* $\lim_{n\to\infty} T^n = 0$.

Proof. It suffices to prove (i). If $M(\mathscr{L}_*) = \mathscr{K}$, then $P_{\mathscr{R}} = 0$. Hence, for $h \in \mathscr{H}$, we have

$$\lim_{n\to\infty} T^n T^{*n} h = P_{\mathscr{H}} P_{\mathscr{R}} h = 0,$$

and it therefore follows that

$$\lim_{n\to\infty} \|T^{*n}h\|^2 = \lim_{n\to\infty} (T^n T^{*n} h, h) = 0.$$

Thus $\lim_{n\to\infty} T^{*n}h = 0$ for all $h \in \mathscr{H}$, i.e., $\lim_{n\to\infty} T^{*n} = 0$.

The converse is Theorem 5.5. ▨

COROLLARY 8.6. *If Θ_T is bounded, then a vector $h \in \mathscr{H}$ satisfies $\lim_{n\to\infty} T^{*n}h = 0$ if and only if $h \in M_+(\mathscr{L}_*)$, and $\lim_{n\to\infty} T^n h = 0$ if and only if $h \in M_-(\mathscr{L})$.*

Proof. It suffices to prove the first assertion. If $h \in M_+(\mathscr{L}_*)$, then $P_{\mathscr{R}}h = 0$ and the rest of the proof is the same as in Corollary 8.5. The converse is Corollary 5.7. ▨

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