J. OPERATOR THEORY
3(1980), 89–113



# ELEMENTS OF SPECTRAL THEORY FOR GENERALIZED DERIVATIONS

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## 1. INTRODUCTION

Let $\mathscr{H}$ denote an infinite dimensional complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. For $A$ and $B$ in $\mathscr{L}(\mathscr{H})$, let $\mathscr{T}_{AB}$ (or simply $\mathscr{T}$) denote the operator on $\mathscr{L}(\mathscr{H})$ defined by $\mathscr{T}_{AB}(X) = AX - XB$. In the present note we characterize the Fredholm essential spectrum of $\mathscr{T}$ and we present several necessary or sufficient conditions for the range of $\mathscr{T}$ to be norm closed.

In order to state our results in detail and place them in perspective, we first recall some pertinent terminology and results from the literature. Let $\mathscr{X}$ denote an infinite dimensional complex Banach space and let $\mathscr{L}(\mathscr{X})$ denote the algebra of all bounded linear operators on $\mathscr{X}$. For $T \in \mathscr{L}(\mathscr{X})$, let $\sigma(T)$, $\sigma_l(T)$, and $\sigma_r(T)$ denote, respectively, the spectrum, left spectrum, and right spectrum of $T$. M. Rosenblum's fundamental result states that

$$\sigma(\mathscr{T}) \subset \sigma(A) - \sigma(B) \equiv \{\alpha - \beta : \alpha \in \sigma(A),\ \beta \in \sigma(B)\}$$

[22], and the identity $\sigma(\mathscr{T}) = \sigma(A) - \sigma(B)$ is contained in [17], Theorem 10. Thus $\mathscr{T}$ is invertible if and only if $A$ and $B$ have disjoint spectra.

Following [7], let $\sigma_\pi(T)$ and $\sigma_\delta(T)$ denote, respectively, the approximate point spectrum of $T$ and the approximate defect spectrum of $T$, i.e.,

$$\sigma_\delta(T) = \{\lambda \in \sigma(T) : T - \lambda \text{ is not surjective}\}.$$

(For $T \in \mathscr{L}(\mathscr{H})$, $\sigma_l(T) = \sigma_\pi(T)$ and $\sigma_r(T) = \sigma_\delta(T)$.) In [7], C. Davis and P. Rosenthal characterized the approximate point and defect spectra of $\mathscr{T}$ as follows:

$$\sigma_\pi(\mathscr{T}) = \sigma_\pi(A) - \sigma_\delta(B)$$

and

$$\sigma_\delta(\mathscr{T}) = \sigma_\delta(A) - \sigma_\pi(B).$$

Thus $\mathscr{T}$ is bounded below (resp. surjective) if and only if $\sigma_\pi(A) \cap \sigma_\delta(B) = \emptyset$ (resp. $\sigma_\delta(A) \cap \sigma_\pi(B) = \emptyset$). Subsequently, the author proved that $\sigma_\delta(\mathscr{T}) = \sigma_r(\mathscr{T})$ and $\sigma_\pi(\mathscr{T}) = \sigma_l(\mathscr{T})$ [10]. Furthermore, the range of $\mathscr{T}$ is norm dense in $\mathscr{L}(\mathscr{H})$ if and only if $\sigma_{rc}(A) \cap \sigma_{le}(B) = \emptyset$ and there exists no nonzero trace class operator $X$ such that $BX = XA$ [11], Theorem 1.1 (see below for notation).

For an operator $T \in \mathscr{L}(\mathscr{X})$, let ker $T$ and $\mathscr{R}(T)$ denote the kernel and range of $T$. Following [16], let nul $T =$ dim ker $T$ and let def $T = \dim(\mathscr{X}/\mathscr{R}(T)^-)$ (where $\mathscr{R}(T)^-$ denotes the norm closure of $\mathscr{R}(T)$). For $x \in \mathscr{X}$, let $[x]$ denote the image of $x$ in $\mathscr{X}/\mathscr{R}(T)^-$. An operator $T$ is *semi-Fredholm* if $\mathscr{R}(T)$ is closed and either nul $T < \infty$ or def $T < \infty$; in this case the *index* of $T$ is defined by

$$\text{ind } T = \text{nul } T - \text{def } T.$$

$T$ is *Fredholm* if $\mathscr{R}(T)$ is closed and both nul $T$ and def $T$ are finite. Let $\sigma_e(T)$ denote the Fredholm essential spectrum of $T$, i.e.

$$\sigma_e(T) = \{\lambda \in \mathbf{C}: T - \lambda \text{ is not Fredholm}\}.$$

Thus $\sigma_e(T)$ coincides with the spectrum of the image of $T$ in the quotient algebra $\mathscr{L}(\mathscr{X})/\mathscr{K}(\mathscr{X})$, where $\mathscr{K}(\mathscr{X})$ denotes the ideal of all compact operators on $\mathscr{X}$ (see [18], Chapter VII, Theorem 2, page 120).

Our principal result (Theorem 3.1) is the following characterization of the essential spectrum of $\mathscr{T}_{AB}$:

$$\sigma_e(\mathscr{T}) = (\sigma_e(A) - \sigma(B)) \cup (\sigma(A) - \sigma_e(B));$$

thus $\mathscr{T}$ is Fredholm if and only if

$$\sigma_e(A) \cap \sigma(B) = \sigma(A) \cap \sigma_e(B) = \emptyset$$

(Corollary 3.2). We prove this result in the setting where $A$ and $B$ act on possibly different Hilbert spaces, $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, and where $\mathscr{T}$ acts on $\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$. Moreover, the inclusion

$$\sigma_e(\mathscr{T}) \subset (\sigma_e(A) - \sigma(B)) \cup (\sigma(A) - \sigma_e(B))$$

is valid when $A$ and $B$ are Banach space operators (Theorem 3.7). Since our proof of the reverse inclusion in the Hilbert space case depends ultimately on Voiculescu's Theorem [25], Theorem 1.3 and its important consequence due to C. Apostol [3], Lemma 2.2 (see Lemma 2.10 below), we have not been able to extend this direction to the Banach space case. Section 2 contains a number of preliminary results used in the proof of Theorem 3.1; the reader may prefer to refer to section 2 only as needed in sections 3 and 4.

Several authors have studied the problem of characterizing when the range of $\mathscr{T}$ is norm closed in $\mathscr{L}(\mathscr{H})$. One type of result in this direction shows that $\mathscr{T}$ has closed range if the spectra of $A$ and $B$ are suitably separated. Rosenblum's

Theorem [22], and the results of C. Davis and P. Rosenthal [7] for the cases when $\mathcal{T}$ is bounded below or surjective, are of this flavor. Moreover, J. Anderson and C. Foiaş proved that if $A$ and $B$ are normal, then $\mathcal{T}$ has closed range if and only if $\sigma(A) \cap \sigma(B)$ contains no limit point of $\sigma(A) \cup \sigma(B)$ ([1], Theorem 3.3).

Considerable attention has also been devoted to the case $A = B$, in which case $\mathcal{T}$ is called the inner derivation induced by $A$, which we denote by $\delta_A$. J. G. Stampfli treated the case when $A$ is hyponormal [23], and C. Apostol and J.G. Stampfli studied the cases when $A$ is spectral, compact, nilpotent, or a weighted shift [5]. Subsequently, C. Apostol proved that $\mathcal{R}(\delta_A)$ is closed if and only if $A$ is algebraic and $\mathcal{R}(p(A))$ is closed for every polynomial $p(z)$; equivalently, $\mathcal{R}(\delta_A)$ is closed if and only if $A$ is similar to a Jordan operator ([3], Theorem 3.5).

In section 4 we obtain several general conditions which imply that $\mathcal{R}(\mathcal{T})$ is not closed. We apply these results to cases when $A$ and $B$ are compact, hyponormal, or nilpotent of order two, and we thereby recapture and partially extend analogous results of [1], [5], and [23].

We conclude this section with some additional terminology. Let $\mathcal{S}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ denote, respectively, the groups of all invertible and unitary operators in $\mathcal{L}(\mathcal{H})$. For $T$ in $\mathcal{L}(\mathcal{H})$, let

$$\mathcal{S}(T) \equiv \{X^{-1}TX: X \in \mathcal{S}(\mathcal{H})\}$$

and

$$\mathcal{U}(T) \equiv \{U^*TU: U \in \mathcal{U}(\mathcal{H})\}$$

denote, respectively, the similarity and unitary orbits of $T$. In the sequel we rely on certain extensions of similarity and unitary equivalence. Operators $T$ and $S$ in $\mathcal{L}(\mathcal{H})$ are *approximately similar* ($T \underset{a}{\sim} S$) if there exists a sequence $\{X_n\} \subset \mathcal{S}(\mathcal{H})$ such that $\sup_n \|X_n\| < \infty$, $\sup_n \|X_n^{-1}\| < \infty$, and $\lim \|X_n^{-1}TX_n - S\| = 0$ [13]. $T$ and $S$ are *approximately unitarily equivalent* ($T \underset{a}{\approx} S$) if $\mathcal{U}(T)^- = \mathcal{U}(S)^-$ [13], [25]. Approximate similarity and approximate unitary equivalence are equivalence relations in $\mathcal{L}(\mathcal{H})$. If an operator $S$ acts on a space $\mathcal{K}$ isomorphic to $\mathcal{H}$, the notation $S \in \mathcal{U}(T)^-$ means that there exists a sequence of isomorphisms $U_n: \mathcal{H} \to \mathcal{K}$ such that $U_nTU_n^* \to S$; equivalently, $S$ is unitarily equivalent to an operator $S' \in \mathcal{L}(\mathcal{H})$ such that $T \underset{a}{\approx} S'$ in the above sense. Thus, whenever $S \in \mathcal{U}(T)^-$, we may assume that $S$ and $T$ act on the same space.

For $k > 1$, let $\{e_1, \ldots, e_k\}$ denote the standard orthonormal basis of $\mathbf{C}^k$, and let $q_k$ denote the operator on $\mathbf{C}^k$ defined by the following relations:

$$q_k(e_i) = e_{i-1}\ (2 \leqslant i \leqslant k);\ q_k e_1 = 0.$$

Let $q_1$ denote the zero operator on $\mathbf{C}^1$. For a cardinal number $\alpha \geqslant 1$, let $q_k^{(\alpha)}$ denote the orthogonal direct sum of $\alpha$ copies of $q_k$. We denote $q_k^{(\aleph_0)}$ by $q_k^{(\infty)}$ ($\equiv q_k \oplus q_k \oplus \ldots$). A *Jordan nilpotent operator* is a finite direct sum of the form

$$q_{k_1}^{(\alpha_1)} \oplus \ldots \oplus q_{k_n}^{(\alpha_n)}.$$

A *Jordan operator* is a finite direct sum of the form $\sum_{i=1}^{n} \oplus (\lambda_i 1_{\mathscr{H}_i} + N_i)$, where $N_i$ is a Jordan nilpotent operator on the Hilbert space $\mathscr{H}_i$, and $\lambda_i$ is a complex scalar (see [3]).

For a Banach space operator $T$, let $\sigma_p(T)$ denote the point spectrum of $T$. In the Hilbert space case, $\lambda \in \sigma_p(T)$ is a *reducing eigenvalue* for $T$ if

$$\dim(\ker(T - \lambda) \cap \ker((T - \lambda)^*)) > 0.$$

A complex number $\lambda$ is a *reducing essential eigenvalue* for $T$ if there exists an orthonormal sequence $\{e_n\}_{n=1}^{\infty} \subset \mathscr{H}$ such that

$$\lim \|(T - \lambda)e_n\| = \lim \|(T - \lambda)^* e_n\| = 0$$

[24]. Following [24], let $R_e(T)$ denote the set of all reducing essential eigenvalues of $T$. For $T$ in $\mathscr{L}(\mathscr{H})$, let $\tilde{T}$ denote the image of $T$ in the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$. A *hole* in $\sigma_e(T)$ ($\equiv \sigma(\tilde{T})$) is a bounded component of $\mathbf{C} \setminus \sigma_e(T)$ ([19], page 2). Let $\sigma_{re}(T)$ and $\sigma_{le}(T)$ denote, respectively, the right and left essential spectra of $T$, i.e. $\sigma_{re}(T) = \sigma_r(\tilde{T})$ and $\sigma_{le}(T) = \sigma_l(\tilde{T})$. Thus $R_e(T) \subset \sigma_{re}(T) \cap \sigma_{le}(T)$; note, however, that $R_e(T)$ may be empty ([24], Theorem 5.3).

In the sequel we will use the following results about hyponormal operators without further reference. If $T \in \mathscr{L}(\mathscr{H})$ is hyponormal, then $\sigma_l(T) \subset \sigma_r(T)$ and $\sigma_{le}(T) \subset \sigma_{re}(T)$; in particular, $\sigma_{le}(T) = R_e(T)$ ([24], Theorem 3.10). Moreover, if $\lambda$ is an isolated point of $\sigma(T)$, then $\ker(T - \lambda)$ reduces $T$ and $\ker(T - \lambda)$ coincides with the Riesz subspace for $T$ corresponding to $\{\lambda\}$, i.e.

$$\ker(T - \lambda) = \{x \in \mathscr{H} : \|(T - \lambda)^n x\|^{1/n} \to 0\}$$

(see [9], Section 4 for a discussion of these facts and additional references).

*Acknowledgement*. The author wishes to thank Kung-Wei Yang and Domingo A. Herrero for several helpful discussions.

## 2. PRELIMINARIES

Let $\mathscr{X}$ denote a complex Banach space and let $T$ be in $\mathscr{L}(\mathscr{X})$. Following [3], we define

$$\gamma(T) = \inf\{\|Tx\| : x \in \mathscr{X} \text{ and } \operatorname{dist}(x, \ker T) \geqslant 1\};$$

for several equivalent formulations of $\gamma(T)$, see [16], Chapter IV, page 231. The range of $T$ is closed in $\mathscr{X}$ if and only if $\gamma(T) > 0$ ([16], Chapter IV, Theorem 5.2). Let $\mathscr{H}_1$ and $\mathscr{H}_2$ denote Hilbert spaces of arbitrary dimension and let $\mathscr{X} = \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$.

For $A \in \mathcal{L}(\mathcal{H}_1)$ and $B \in \mathcal{L}(\mathcal{H}_2)$, let $T = \mathcal{T}_{AB} \in \mathcal{L}(\mathcal{X})$. We begin this section by recording some facts about $\gamma(\mathcal{T}_{AB})$ that will be useful in the sequel.

LEMMA 2.1. *If* $J \in \mathcal{S}(\mathcal{H}_1)$, *then*

$$\gamma(\mathcal{T}_{J^{-1}AJ,\,B}) \leqslant \|J\|\,\|J^{-1}\|\,\gamma(\mathcal{T}_{AB}).$$

*Proof.* Suppose first that $\|J\| = 1$. Let $\{X_n\} \subset \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ be a sequence such that

$$\operatorname{dist}(X_n,\ \ker \mathcal{T}_{AB}) \geqslant 1$$

and

$$\|AX_n - X_nB\| < \gamma(\mathcal{T}_{AB}) + 1/n \qquad \text{for } n \geqslant 1.$$

For $Z \in \ker \mathcal{T}_{J^{-1}AJ,B}$, let $Y = JZ$; then $AY = YB$ and thus $\|X_n - Y\| \geqslant 1$. Now

$$\|J^{-1}X_n - Z\| = \|J^{-1}(X_n - Y)\| \geqslant (1/\|J\|)\,\|X_n - Y\| \geqslant 1,$$

and it follows that

$$\operatorname{dist}(J^{-1}X_n, \ker \mathcal{T}_{J^{-1}AJ,B}) \geqslant 1.$$

Thus

$$\gamma(\mathcal{T}_{J^{-1}AJ,\,B}) \leqslant \|J^{-1}AJJ^{-1}X_n - J^{-1}X_nB\| \leqslant$$

$$\leqslant \|J^{-1}\|\,\|AX_n - X_nB\| < \|J^{-1}\|\,(\gamma(\mathcal{T}_{AB}) + 1/n) \qquad \text{for } n \geqslant 1,$$

and so

$$\gamma(\mathcal{T}_{J^{-1}AJ,\,B}) \leqslant \|J^{-1}\|\gamma(\mathcal{T}_{AB}).$$

For the general case, the preceding inequality implies that

$$\gamma(\mathcal{T}_{J^{-1}AJ,B}) = \gamma(\mathcal{T}_{((\|J\|J^{-1})\,A(1/\|J\|)\,J,\,B)}) \leqslant \|J\|\,\|J^{-1}\|\,\gamma(\mathcal{T}_{AB}).$$

LEMMA 2.2. $\gamma(\mathcal{T}_{AB}) = \gamma(\mathcal{T}_{B^*A^*})$.

*Proof.* Let $\{X_n\} \subset \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ be such that

$$\operatorname{dist}(X_n, \ker \mathcal{T}_{AB}) \geqslant 1$$

and

$$\|AX_n - X_nB\| < \gamma(\mathcal{T}_{AB}) + 1/n \qquad \text{for } n \geqslant 1.$$

If $Y \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $B^*Y = YA^*$, then $AY^* = Y^*B$, so

$$\|X_n^* - Y\| = \|X_n - Y^*\| \geqslant 1,$$

and thus

$$\operatorname{dist}(X_n^*, \ker \mathcal{T}_{B^*A^*}) \geqslant 1.$$

Now

$$\|B^*X_n^* - X_n^*A^*\| = \|AX_n - X_nB\| < \gamma(\mathcal{T}_{AB}) + 1/n,$$

and thus

$$\gamma(\mathcal{T}_{B^*A^*}) \leqslant \gamma(\mathcal{T}_{AB}).$$

The result follows from the last inequality by replacing $B^*$ by $A$ and $A^*$ by $B$.

LEMMA 2.3. *If* $A \underset{a}{\sim} A'$ *and* $B \underset{a}{\sim} B'$, *then* $\mathcal{R}(\mathcal{T}_{AB})$ *is closed if and only if* $\mathcal{R}(\mathcal{T}_{A'B'})$ *is closed.*

*Proof.* We assume first that $B' = B$. For $\gamma \geqslant 0$, let

$$\mathcal{L}_\gamma = \{S \in \mathcal{L}(\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)) \colon \gamma(S) \geqslant \gamma\};$$

[3], Lemma 1.9 implies that $\mathcal{L}_\gamma$ is norm closed. Since $A \underset{a}{\sim} A'$, there exists $\{X_n\} \subset \subset \mathcal{S}(\mathcal{H}_1)$ and there exists $M > 0$, such that $\|X_n\| \leqslant M$ and $\|X_n^{-1}\| \leqslant M$ for $n \geqslant 1$, and such that $X_n A X_n^{-1} \to A'$. Lemma 2.1 implies that

$$\gamma(\mathcal{T}_{X_nAX_n^{-1},B}) \geqslant (1/\|X_n\|\,\|X_n^{-1}\|)\,\gamma(\mathcal{T}_{AB}) \geqslant (1/M^2)\,\gamma(\mathcal{T}_{AB});$$

thus $\mathcal{T}_{X_nAX_n^{-1},B} \in \mathcal{L}_\gamma$ for $\gamma = (1/M^2)\gamma(\mathcal{T}_{AB})$. Since $\mathcal{T}_{X_nAX_n^{-1},B} \to \mathcal{T}_{A'B}$ and $\mathcal{L}_\gamma$ is closed, it follows that if $\mathcal{R}(\mathcal{T}_{AB})$ is closed, then so is $\mathcal{R}(\mathcal{T}_{A'B})$, and the converse follows by symmetry. The general case follows from the preceding case and Lemma 2.2 via the following sequence of implications:

$$\gamma(\mathcal{T}_{AB}) > 0 \Leftrightarrow \gamma(\mathcal{T}_{A'B}) > 0 \Leftrightarrow \gamma(\mathcal{T}_{B^*A'^*}) > 0 \Leftrightarrow \gamma(\mathcal{T}_{B'^*A'^*}) > 0 \Leftrightarrow$$

$$\Leftrightarrow \gamma(\mathcal{T}_{A'B'}) > 0.$$

Formal repetition of the preceding argument in the case of approximate unitary equivalence yields the following result.

COROLLARY 2.4. *If* $A \underset{a}{\approx} A'$ *and* $B \underset{a}{\approx} B'$, *then* $\gamma(\mathcal{T}_{AB}) = \gamma(\mathcal{T}_{A'B'})$.

LEMMA 2.5. *If* $A' \in \mathcal{S}(A)$ *and* $B' \in \mathcal{S}(B)$, *then* $\operatorname{nul}\mathcal{T}_{AB} = \operatorname{nul}\mathcal{T}_{A'B'}$ *and* $\operatorname{def}\mathcal{T}_{AB} = \operatorname{def}\mathcal{T}_{A'B'}$.

*Proof.* Let $U \in \mathcal{S}(\mathcal{H}_1)$ and $V \in \mathcal{S}(\mathcal{H}_2)$ be invertible operators such that $A' = U^{-1}AU$ and $B' = V^{-1}BV$. The mapping $X \to UXV^{-1}$ is an isomorphism of $\ker\mathcal{T}_{A'B'}$ onto $\ker\mathcal{T}_{AB}$. Similarly, the mapping $[Y] \to [UYV^{-1}]$ is an isomorphism of corange $\mathcal{T}_{A'B'}$ onto corange $\mathcal{T}_{AB}$.

The following examples show that the preceding lemma cannot be extended to the case $A' \in \mathcal{U}(A)^-$ or $B' \in \mathcal{U}(B)^-$.

EXAMPLE 2.6. Let $B = 0_{\mathcal{H}_2}$ and let $A$ denote an injective, non-invertible normal operator; thus $0 \in R_c(A)$. Let $\mathcal{H}$ denote a separable, infinite dimensional Hilbert space and let $A' \in \mathscr{L}(\mathscr{H}_1)$ denote an operator unitarily equivalent to $A \oplus 0_{\mathscr{H}}$. Lemma 2.12 (below) implies that $A' \in \mathscr{U}(A)^-$. It follows readily that nul $\mathscr{T}_{AB} = 0$ and nul $\mathscr{T}_{A'B} = \infty$.

EXAMPLE 2.7. Let $A$ be as in the preceding example. Let $B$ denote a normal operator such that $0 \in \sigma_p(B)$, $\sigma_e(A) \cap \sigma_e(B) = \emptyset$, and $\sigma_p(A) \cap \sigma_p(B) = \emptyset$. It follows from [7], Theorem 5 and [11], Proposition 4.2 that the range of $\mathscr{T}_{AB}$ is proper but norm dense, so that def $\mathscr{T}_{AB} = 0$. On the other hand, $A \oplus 0_{\mathscr{H}} \approx A' \in \mathscr{U}(A)^-$, and since $0 \in \sigma_p(A') \cap \sigma_p(B)$, [11], Prop. 4.2 implies that the range of $\mathscr{T}_{A'B}$ is not dense. Thus def $\mathscr{T}_{AB} = 0 <$ def $\mathscr{T}_{A'B}$. (We note that [7], Theorem 5 implies that if $\mathscr{T}_{AB}$ is surjective and $A' \in \mathscr{U}(A)^-$, $B' \in \mathscr{U}(B)^-$, then $\mathscr{T}_{A'B'}$ is also surjective.)

Despite the preceding examples, we do have the following stability result.

PROPOSITION 2.8. *If $A' \underset{a}{\sim} A$ and $B' \underset{a}{\sim} B$, then $\mathscr{T}_{AB}$ is semi-Fredholm if and only if $\mathscr{T}_{A'B'}$ is semi-Fredholm, and in this case* nul $\mathscr{T}_{AB} =$ nul $\mathscr{T}_{A'B'}$, def $\mathscr{T}_{AB} =$ $=$ def $\mathscr{T}_{A'B'}$, *and* ind $\mathscr{T}_{AB} =$ ind $\mathscr{T}_{A'B'}$.

*Proof.* There exist uniformly bounded sequences $\{X_n\} \subset \mathscr{S}(\mathscr{H}_1)$ and $\{Y_n\} \subset$ $\subset \mathscr{S}(\mathscr{H}_2)$ such that $\{X_n^{-1}\}$ and $\{Y_n^{-1}\}$ are uniformly bounded, $X_n^{-1}AX_n \to A'$, and $Y_n^{-1}BY_n \to B'$. If $\mathscr{T}_{A'B'}$ is semi-Fredholm, then since $\mathscr{T}_{X_n^{-1}AX_n, Y_n^{-1}BY_n} \to \mathscr{T}_{A'B'}$, it follows that for a sufficiently large value of $n$, $\mathscr{T}_{X_n^{-1}AX_n, Y_n^{-1}BY_n}$ is semi-Fredholm and

$$\text{nul}\, \mathscr{T}_{X_n^{-1}AX_n, Y_n^{-1}BY_n} \leqslant \text{nul}\, \mathscr{T}_{A'B'},$$

$$\text{def}\, \mathscr{T}_{X_n^{-1}AX_n, Y_n^{-1}BY_n} \leqslant \text{def}\, \mathscr{T}_{A'B'}$$

and

$$\text{ind}\, \mathscr{T}_{X_n^{-1}AX_n, Y_n^{-1}BY_n} = \text{ind}\, \mathscr{T}_{A'B'}$$

([16], Chapter IV, Theorem 5.22, page 236). Lemma 2.3 and Lemma 2.5 now imply that $\mathscr{T}_{AB}$ is semi-Fredholm and that

$$\text{nul}\, \mathscr{T}_{AB} = \text{nul}\, \mathscr{T}_{X_n^{-1}AX_n, Y_n^{-1}BY_n} \leqslant \text{nul}\, \mathscr{T}_{A'B'},$$

$$\text{def}\, \mathscr{T}_{AB} = \text{def}\, \mathscr{T}_{X_n^{-1}AX_n, Y_n^{-1}BY_n} \leqslant \text{def}\, \mathscr{T}_{A'B'},$$

and

$$\text{ind}\, \mathscr{T}_{AB} = \text{ind}\, \mathscr{T}_{X_n^{-1}AX_n, Y_n^{-1}BY_n} = \text{ind}\, \mathscr{T}_{A'B'};$$

the result follows by symmetry.

LEMMA 2.9. nul $\mathscr{T}_{AB} =$ nul $\mathscr{T}_{B^*A^*}$ *and* def $\mathscr{T}_{AB} =$ def $\mathscr{T}_{B^*A^*}$. *Moreover, $\mathscr{T}_{AB}$ is semi-Fredholm if and only if $\mathscr{T}_{B^*A^*}$ is semi-Fredholm, and in this case* ind $\mathscr{T}_{AB} =$ $=$ ind $\mathscr{T}_{B^*A^*}$.

*Proof.* The mapping $X \to X^*$ is a conjugate linear isomorphism of ker $\mathscr{T}_{AB}$ onto ker $\mathscr{T}_{B^*A^*}$; similarly, the mapping $[Y] \to [Y^*]$ is a conjugate linear isomorphism of corange $\mathscr{T}_{AB}$ onto corange $\mathscr{T}_{B^*A^*}$. The result follows from these observations and Lemma 2.2.

The next lemma is essentially due to C. Apostol ([3], Lemma 2.2).

LEMMA 2.10. *Let $\mathscr{H}$ be an infinite dimensional Hilbert space and let $T$ be in $\mathscr{L}(\mathscr{H})$.*

i) *If* $0 \in \sigma_l(T)$, *then either* nul $T > 0$, *or* $0 \in \sigma_{le}(T)$; *if* $0 \in \sigma_{le}(T)$, *then there exists* $S \in \mathscr{U}(T)^-$ *such that* nul $S = \infty$;

ii) *If* $0 \in \sigma_r(T)$, *then either* nul $T^* > 0$, *or* $0 \in \sigma_{re}(T)$; *if* $0 \in \sigma_{re}(T)$, *then there exists* $S \in \mathscr{U}(T)^-$ *such that* nul $S^* = \infty$.

*Proof.* i) If $0 \in \sigma_l(T)$ and $T$ is injective, then $\mathscr{R}(T)$ is not closed. Thus $0 \in \sigma_{le}(T)$ and the result follows from [3], Lemma 2.2.

ii) Apply i) to $T^*$.

LEMMA 2.11. *Let $\mathscr{H}$ be an infinite dimensional Hilbert space, let $T$ be in $\mathscr{L}(\mathscr{H})$, and assume $R_e(T) \neq \varnothing$. Then there exists a separable subspace $\mathfrak{M} \subset \mathscr{H}$ such that $\mathfrak{M}$ reduces $T$ and $R_e(T|\mathfrak{M}) = R_e(T)$.*

*Proof.* Let $\{\lambda_i\}_{i \in \mathfrak{I}}$ denote a finite or denumerable sequence that is dense in $R_e(T)$. For each $i \in \mathfrak{I}$, let $\{e_{ik}\}_{k=1}^{\infty}$ denote an orthonormal sequence in $\mathscr{H}$ such that

$$\lim_{k\to\infty} \|(T - \lambda_i)e_{ik}\| = \lim_{k\to\infty} \|(T - \lambda_i)^* e_{ik}\| = 0.$$

Let $\mathfrak{M}_i$ denote the closed subspace spanned by all vectors of the form $p(T, T^*)e_{ik}$, where $k \geqslant 1$ and $p(x, y)$ is any non-commutative polynomial with rational coefficients. Let $\mathfrak{M} = \bigvee_{i \in \mathfrak{I}} \mathfrak{M}_i$; $\mathfrak{M}$ is a separable reducing subspace for $T$, and since $R_e(T|\mathfrak{M})$ is closed ([24], Theorem 4.3), it follows readily that $R_e(T|\mathfrak{M}) = R_e(T)$.

The following lemma extends a result of N.Salinas ([24], Corollary 4.9) from the case of separable Hilbert spaces to the non-separable case.

LEMMA 2.12. *Let $\mathscr{H}$ be an infinite dimensional Hilbert space, $T \in \mathscr{L}(\mathscr{H})$, and $R_e(T) \neq \varnothing$. If $N$ is a normal operator on a separable Hilbert space and $\sigma(N) \subset R_e(T)$, then $T \oplus N \in \mathscr{U}(T)^-$.*

*Proof.* Let $\mathfrak{M}$ be as in Lemma 2.11. It follows from [24], Corollary 4.9 that $(T|\mathfrak{M}) \oplus N \in \mathscr{U}(T|\mathfrak{M})^-$, and thus

$$T \oplus N = (T|\mathfrak{M}^\perp) \oplus (T|\mathfrak{M}) \oplus N \in \mathscr{U}((T|\mathfrak{M}^\perp) \oplus (T|\mathfrak{M}))^- = \mathscr{U}(T)^-.$$

COROLLARY 2.13 (cf. [24], Theorem 4.6) *Let $\mathscr{H}$ be an infinite dimensional Hilbert space, $T \in \mathscr{L}(\mathscr{H})$, and $\lambda \in R_e(T)$. Then there exists $S \in \mathscr{U}(T)^-$ such that*

$$\dim(\ker(S - \lambda)^* \cap \ker(S - \lambda)) = \infty.$$

We conclude this section with a known folk-type result; the proof, which is based on the Riesz decomposition theorem [21], ([20], Theorem 2.10), will be omitted (cf. the proof of [8], Theorem 7).

LEMMA 2.14. *Let $\mathscr{H}$ be a Hilbert space and let $T$ be in $\mathscr{L}(\mathscr{H})$. If $\sigma(T)=\bigcup_{i=1}^{n}\sigma_i$, where each $\sigma_i$ is a non-empty closed subset of $\sigma(T)$ and $\sigma_i \cap \sigma_j = \emptyset$ for $i \neq j$, then there exists an orthogonal decomposition $\mathscr{H}=\mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_n$ and operators $T_i \in \mathscr{L}(\mathscr{H}_i)$, such that $\sigma(T_i)=\sigma_i$ $(1 \leqslant i \leqslant n)$ and such that $T$ is similar to $T_1 \oplus \ldots \oplus T_n$.*

## 3. THE ESSENTIAL SPECTRUM OF $\mathscr{T}_{AB}$

Unless otherwise noted, $\mathscr{H}_1$ and $\mathscr{H}_2$ are infinite dimensional Hilbert spaces, $A \in \mathscr{L}(\mathscr{H}_1)$, $B \in \mathscr{L}(\mathscr{H}_2)$, and $\mathscr{T}_{AB}(X)=AX-XB$ for $X \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$. The main result of this section is the following description of the Fredholm essential spectrume of $\mathscr{T}_{AB}$.

THEOREM 3.1. $\sigma_e(\mathscr{T}_{AB})=(\sigma_e(A)-\sigma(B)) \cup (\sigma(A)-\sigma_e(B))$.

COROLLARY 3.2. *$\mathscr{T}_{AB}$ is Fredholm if and only if $\sigma_e(A) \cap \sigma(B)=\emptyset$ and $\sigma(A) \cap \sigma_e(B)=\emptyset$.*

Before proving Theorem 3.1, it is convenient to treat the special case when $A=0$ or $B=0$, and we now consider this case. In the following lemma $\mathscr{H}_2$ is an arbitrary Hilbert space.

LEMMA 3.3. *If $B=0_{\mathscr{H}_2}$, then $\mathscr{R}(\mathscr{T})$ is closed if and only if $\mathscr{R}(A)$ is closed.*

*Proof.* Suppose that $\mathscr{R}(A)$ is closed. If $P$ denotes the orthogonal projection of $\mathscr{H}_1$ onto $(\ker A)^{\perp}$, then $A=AP$, and there exists $L \in \mathscr{L}(\mathscr{H}_1)$ such that $LAP=P$. If $\{X_n\} \subset \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$, $Y \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$, and $AX_n \to Y$, then $PX_n = LAPX_n = LAX_n \to LY$, and so $AX_n = APX_n \to ALY$. Thus $Y=ALY$ and it follows that $\mathscr{R}(\mathscr{T})$ is closed.

For the converse, we assume that $\mathscr{R}(A)$ is not closed. Thus there exists $\{x_n\} \subset \mathscr{H}_1$ and $y \in \mathscr{H}_1$, $\|y\|=1$, such that $Ax_n \to y$ and $y \notin \mathscr{R}(A)$. Let $U \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ be an operator whose range contains $y$ and let $z_0 \in \mathscr{H}_2$ satisfy $Uz_0=y$. Define $X, X_n \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ by the formulas $Xz=(Uz,y)y$ $(z \in \mathscr{H}_2)$ and $X_nz=(Uz,y)x_n$ $(z \in \mathscr{H}_2)$ for $n \geqslant 1$. Since $Ax_n \to y$, it follows that $AX_n \to X$. If $R \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ satisfies $AR=X$, then

$$ARz_0 = Xz_0 = (Uz_0, y)y = \|y\|^2 y = y,$$

which contradicts the fact that $y \notin \mathscr{R}(A)$. Thus $X \notin \mathscr{R}(\mathscr{T})$, so the proof is complet

In the sequel, $\mathscr{H}_2$ is again infinite dimensional.

LEMMA 3.4. *If* $B = 0_{\mathscr{H}_2}$, *then* $\sigma_e(\mathscr{T}) = \sigma(A)$.

*Proof.* Since $\sigma_e(\mathscr{T}) \subset \sigma(\mathscr{T}) = \sigma(A)$, it suffices to prove that if $0 \in \sigma(A)$, then $0 \in \sigma_e(\mathscr{T})$. From Lemma 3.3, we may assume that $\mathscr{R}(A)$ is closed, and thus $\mathscr{R}(\mathscr{T})$ is closed and $\dim \ker A > 0$ or $\dim \ker A^* > 0$. Let $\{e_n\}_{n=1}^{\infty} \subset \mathscr{H}_2$ denote an orthonormal sequence. We consider first the case $\dim \ker A > 0$; let $x \in \mathscr{H}_1$ denote a nonzero vector in $\ker A$. For $n \geqslant 1$, define $X_n \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ by $X_n v = (v, e_n)x$ ($v \in \mathscr{H}_2$). It follows readily that $\{X_n\}$ is an independent sequence in $\ker \mathscr{T}$, so $\mathscr{T}$ is not Fredholm. For the remaining case, let $y \in \mathscr{H}_1 \ominus \mathscr{R}(A)$, $y \neq 0$. For $n \geqslant 1$define $Z_n \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ by $Z_n v = (v, e_n)y$ ($v \in \mathscr{H}_2$), and let $[Z_n]$ denote the image of $Z_n$ in $\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)/\mathscr{R}(\mathscr{T})$. It is easily verified that $\{[Z_n]\}$ is independent, so $\operatorname{def} \mathscr{T} = \infty$ and the proof is complete.

COROLLARY 3.5. *If* $A = 0$ *or* $B = 0$, *then*

$$\sigma_e(\mathscr{T}) = \sigma(\mathscr{T}) = \sigma(A) - \sigma(B).$$

*Proof.* The case when $B = 0$ follows from Lemma 3.4; the case when $A = 0$ follows from the first case by an application of Lemma 2.9.

*Proof of Theorem 3.1.* By virtue of Corollary 3.5 we may assume that $A$ and $B$ are nonzero. In the first part of the proof we prove that

$$(\sigma_e(A) - \sigma(B)) \cup (\sigma(A) - \sigma_e(B)) \subset \sigma_e(\mathscr{T}).$$

Note that

$$\mathscr{T}_{AB} - (\alpha - \beta) = \mathscr{T}_{A-\alpha,\, B-\beta};$$

it thus suffices to prove that if $0 \in (\sigma_e(A) \cap \sigma(B)) \cup (\sigma(A) \cap \sigma_e(B))$, then $0 \in \sigma_e(\mathscr{T}_{AB})$. We consider several special cases.

i) $0 \in (\sigma_{le}(A) \cap \sigma_l(B)) \cup (\sigma_l(A) \cap \sigma_{le}(B))$. Lemma 2.10 implies that there exists $A' \in \mathscr{U}(A)^-$ and $B' \in \mathscr{U}(B)^-$ such that $\dim \ker A' > 0$, $\dim \ker B' > 0$, and $\dim \ker A' = \infty$ or $\dim \ker B' = \infty$. From Proposition 2.8, it suffices to prove that $0 \in \sigma_e(\mathscr{T}')$, where $\mathscr{T}' = \mathscr{T}_{A'B'}$.

Relative to the decompositions $\mathscr{H}_1 = \ker A' \oplus (\ker A')^{\perp}$ and $\mathscr{H}_2 = \ker B' \oplus$ $\oplus (\ker B')^{\perp}$, the operator matrices of $A'$ and $B'$ are, respectively, of the form

$$\begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & B_{12} \\ 0 & B_{22} \end{pmatrix}.$$

For $V \in \mathscr{L}(\ker B', \ker A')$, let $M(V) \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ denote the operator whose matrix (relative to the above decompositions of $\mathscr{H}_2$ and $\mathscr{H}_1$) is of the form

$$\begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix};$$

let $\mathfrak{M}$ denote the subspace $\{M(V): V \in \mathscr{L}(\ker B', \ker A')\}$. Suppose there exists $X \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ such that $A'X - XB' = M(V)$; a matrix calculation shows that

$$\mathscr{T}'(A'X) = A'A'X - A'XB' = A'(A'X - XB') = A'M(V) = 0.$$

Since $A'X - XB' = M(V)$, a calculation implies that the matrix of $A'X (\in \ker \mathscr{T}')$ is of the form

$$\begin{pmatrix} V & * \\ 0 & * \end{pmatrix},$$

and it follows that $\operatorname{nul} \mathscr{T}' \geqslant \dim (\mathfrak{M} \cap \mathscr{R}(\mathscr{T}'))$.

We may thus assume that $n \equiv \dim (\mathfrak{M} \cap \mathscr{R}(\mathscr{T}')) < \infty$ and that $\mathscr{R}(\mathscr{T}')$ is closed, for otherwise $0 \in \sigma_e(\mathscr{T}')$. We claim that $\operatorname{def} \mathscr{T}' = \infty$. If $n > 0$, let $\{V_i\}_{i=1}^n \subset \mathscr{L}(\ker B', \ker A')$ be such that $\{M(V_i)\}_{i=1}^n$ is a basis for $\mathfrak{M} \cap \mathscr{R}(\mathscr{T}')$. Since $\ker A'$ or $\ker B'$ is infinite dimensional, so is $\mathscr{L}(\ker B', \ker A')$. Thus there exists $\{W_j\}_{j=1}^\infty \subset \mathscr{L}(\ker B', \ker A')$ such that $\{V_i\} \cup \{W_j\}$ is independent. For $S \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$, let $[S]$ denote the image of $S$ in $\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)/\mathscr{R}(\mathscr{T}')$, and note that $\{[M(W_j)]\}_{j=1}^\infty$ is independent. Indeed, if $c_1, \ldots, c_m$ are scalars such that $\sum_{j=1}^m c_j[M(W_j)] = 0$, then $M\left(\sum_{j=1}^m c_jW_j\right) \in \mathscr{R}(\mathscr{T}') \cap \mathfrak{M}$, and thus there exist scalars $d_1, \ldots, d_n$ such that

$$\sum_{j=1}^m c_jW_j = \sum_{i=1}^n d_iV_i.$$

Since $\{V_i\} \cup \{W_j\}$ is independent, each $c_j = 0$. Thus $\operatorname{def} \mathscr{T}' = \infty$ and $0 \in \sigma_e(\mathscr{T}')$. The case when $n = 0$ is treated similarly.

ii) $0 \in (\sigma_{le}(A) \cap \sigma_r(B)) \cup (\sigma_l(A) \cap \sigma_{re}(B))$. As in the preceding case, we may assume (via Lemma 2.10 and Proposition 2.8) that $\dim \ker A > 0$, $\dim \ker B^* > 0$, and $\dim \ker A = \infty$ or $\dim \ker B^* = \infty$. Relative to the decompositions $\mathscr{H}_1 = \ker A \oplus (\ker A)^\perp$ and $\mathscr{H}_2 = \ker B^* \oplus \mathscr{R}(B)^-$, the operator matrices of $A$, $B$, and $X \in \mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$ are of the form

$$A = \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix},$$

and

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

A matrix calculation shows that if $X_{12}$, $X_{21}$, $X_{22}$ are zero operators, then $AX - XB = 0$. Thus

$$\text{nul}\, \mathcal{T} \geqslant \dim \mathcal{L}(\ker B^*, \ker A).$$

Since the hypotheses on $A$ and $B$ imply that $\mathcal{L}(\ker B^*, \ker A)$ is infinite dimensional, so is $\ker \mathcal{T}$, and thus $0 \in \sigma_e(\mathcal{T})$.

iii) $0 \in (\sigma_{re}(A) \cap \sigma_l(B)) \cup (\sigma_r(A) \cap \sigma_{le}(B))$. From Lemma 2.10 and Proposition 2.8 we may assume that $\dim \ker A^* > 0$, $\dim \ker B > 0$, and $\dim \ker A^* = \infty$ or $\dim \ker B = \infty$. With respect to the decompositions $\mathcal{H}_1 = \ker A^* \oplus \mathcal{R}(A)^-$ and $\mathcal{H}_2 = \ker B \oplus (\ker B)^\perp$, the matrices of $A$, $B$, and $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ are of the form

$$\begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad \begin{pmatrix} 0 & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

and

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

The matrix of $AX - XB$ assumes the form

$$\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}.$$

It is clear that $\text{def}\, \mathcal{T} \geqslant \dim \mathcal{L}(\ker B, \ker A^*) = \infty$, so $0 \in \sigma_e(\mathcal{T})$.

If we apply Lemma 2.9 to case i) (above), the conclusion that $0 \in \sigma_e(\mathcal{T})$ may be extended to the following case:

iv) $0 \in (\sigma_r(A) \cap \sigma_{re}(B)) \cup (\sigma_{re}(A) \cap \sigma_r(B))$.

Cases i) — iv) imply that

$$(\sigma_e(A) - \sigma(B)) \cup (\sigma(A) - \sigma_e(B)) \subset \sigma_e(\mathcal{T}).$$

To prove the reverse inclusion it suffices to prove that if

$$\sigma_e(A) \cap \sigma(B) = \sigma(A) \cap \sigma_e(B) = \varnothing,$$

then $\mathcal{T}_{AB}$ is Fredholm. If the spectra of $A$ and $B$ are disjoint, then $\mathcal{T}$ is invertible [22], so we may assume that $K \equiv \sigma(A) \cap \sigma(B) \neq \varnothing$; moreover, the hypotheses imply that

$$K = (\sigma(A) \setminus \sigma_e(A)) \cap (\sigma(B) \setminus \sigma_e(B)).$$

We claim that if $\lambda \in K$, then $\lambda$ is an isolated eigenvalue of $A$ with finite multiplicity or $\lambda$ is an isolated eigenvalue of $B$ with finite multiplicity. Suppose the claim is false. Since $\lambda \in \sigma(A) \setminus \sigma_e(A)$ and $\lambda$ is not an isolated eigenvalue of $A$, there exists a hole $H$ in $\sigma_e(A)$ such that $\lambda \in H$ and $H \subset \sigma(A)$ ([19], Proposition 1.27). Similarly, since $\lambda \in \sigma(B) \setminus \sigma_e(B)$ and $\lambda$ is not isolated in $\sigma(B)$, there exists a hole $L$ in $\sigma_e(B)$ such that $\lambda \in L$ and $L \subset \sigma(B)$. $H$ and $L$ are bounded, open, connected sets and $H \cap L \neq \neq \varnothing$. If $L \not\subset H$, then bdry$(H) \cap L \neq \varnothing$. Since bdry$(H) \subset \sigma_e(A)$ and $L \subset \sigma(B)$, it follows that $\sigma_e(A) \cap \sigma(B) \neq \varnothing$, which is a contradiction. If $H \not\subset L$, then

$$\sigma_e(B) \cap \sigma(A) \supset \text{bdry}(L) \cap H \neq \varnothing,$$

which is impossible. In the remaining case, $H = L$; thus

$$\text{bdry}(H) = \text{bdry}(L) \subset \sigma_e(A) \cap \sigma_e(B),$$

which again contradicts the hypothesis. The proof of the claim is complete.

Since each point in $K$ is isolated in $\sigma(A)$ or $\sigma(B)$, it follows that $K$ is finite. Denote the distinct elements of $K$ by

$$K = \{\alpha_1, \ldots, \alpha_n\} \cup \{\beta_1, \ldots, \beta_p\},$$

where $\alpha_i \in \sigma(A) \setminus \sigma_e(A)$ is an isolated eigenvalue for $A$ with finite multiplicity and $\alpha_i \in \sigma(B) \setminus \sigma_e(B)$ $(1 \leqslant i \leqslant n)$, and $\beta_j \in \sigma(B) \setminus \sigma_e(B)$ is an isolated eigenvalue for $B$ with finite multiplicity and $\beta_j \in \sigma(A) \setminus \sigma_e(A)$ $(1 \leqslant j \leqslant p)$. (In the sequel we assume that both $\alpha_i$'s and $\beta_j$'s are present; if instead, $K$ consists entirely of $\alpha_i$'s or entirely of $\beta_j$'s it is necessary to make certain obvious modifications in the following argument.)

It follows from Lemma 2.14 that there exists an orthogonal decomposition

$$\mathscr{H}_1 = \mathfrak{M}_1 \oplus \ldots \oplus \mathfrak{M}_n \oplus \mathfrak{M}_{n+1},$$

and operators $A_i \in \mathscr{L}(\mathfrak{M}_i)$ $(1 \leqslant i \leqslant n+1)$ such that the following properties hold:

i) $\dim \mathfrak{M}_i < \infty \quad (1 \leqslant i \leqslant n)$;
ii) $\sigma(A_i) = \{\alpha_i\} \quad (1 \leqslant i \leqslant n)$;
iii) $\sigma(A_{n+1}) \cap \{\alpha_1, \ldots, \alpha_n\} = \varnothing$;
iv) $A$ is similar to $A' = A_1 \oplus \ldots \oplus A_{n+1}$.

(The finite dimensionality of $\mathfrak{M}_i$ in i) results from the fact that $\mathfrak{M}_i$ is the Riesz subspace for $A'$ corresponding to $\{\alpha_i\}$ and the fact that $\alpha_i \in \sigma(A) \setminus \sigma_e(A) = \sigma(A') \setminus \sigma_e(A')$.) Similarly, there exists an orthogonal decomposition

$$\mathscr{H}_2 = \mathscr{K}_1 \oplus \ldots \oplus \mathscr{K}_p \oplus \mathscr{K}_{p+1}$$

and operators $B_j \in \mathscr{L}(\mathscr{K}_j)$ $(1 \leqslant j \leqslant p+1)$, such that:

v) $\dim \mathscr{K}_j < \infty \quad (1 \leqslant j \leqslant p)$;
vi) $\sigma(B_j) = \{\beta_j\} \quad (1 \leqslant j \leqslant p)$;
vii) $\sigma(B_{p+1}) \cap \{\beta_1, \ldots, \beta_p\} = \varnothing$;
viii) $B$ is similar to $B' = B_1 \oplus \ldots \oplus B_{p+1}$.

Note that since $\sigma(A) \cap \sigma(B) = \sigma(A') \cap \sigma(B') = K$, iii) and vii) imply that $\sigma(A_{n+1}) \cap \cap \sigma(B_{p+1}) = \emptyset$.

By virtue of Proposition 2.8, it suffices to prove that $\mathcal{T}_{A'B'}$ is Fredholm. Let $X = (X_{ij})_{1 \leqslant i \leqslant n+1,\ 1 \leqslant j \leqslant p+1}$ denote the operator matrix of an element $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ relative to the above decompositions of $\mathcal{H}_2$ and $\mathcal{H}_1$. A matrix calculation shows that the operator matrix of $A'X - XB'$ is of the form $(A_i X_{ij} - X_{ij} B_j)_{1 \leqslant i \leqslant n+1,\ 1 \leqslant j \leqslant p+1}$. Thus the $ij$ entry of $\mathcal{T}_{A'B'}(X)$ is equal to $\mathcal{T}_{A_i B_j}(X_{ij})$.

It is straightforward to verify that if each $\mathcal{T}_{A_i B_j}$ is Fredholm (as an operator on $\mathcal{L}(\mathcal{K}_j, \mathfrak{M}_i)$), then $\mathcal{T}_{A'B'}$ is Fredholm. Since $\dim \mathfrak{M}_i < \infty$ $(1 \leqslant i \leqslant n)$ and $\dim \mathcal{K}_j < \infty$ $(1 \leqslant j \leqslant p)$, it is clear that $\mathcal{T}_{A_i B_j}$ is Fredholm for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant p$. Moreover, since $\sigma(A_{n+1}) \cap \sigma(B_{p+1}) = \emptyset$, Rosenblum's Theorem [22] implies that $\mathcal{T}_{A_{n+1} B_{p+1}}$ is invertible. For the remaining operators $\mathcal{T}_{A_i B_{p+1}}$ $(1 \leqslant i \leqslant n)$ and $\mathcal{T}_{A_{n+1} B_j}$ $(1 \leqslant j \leqslant p)$, we rely on the following lemma.

LEMMA 3.6. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\mathcal{H}_1$ be a finite dimensional Hilbert space. If $A$ is a Fredholm operator in $\mathcal{L}(\mathcal{H})$, then the operator on $\mathcal{L}(\mathcal{H}_1, \mathcal{H})$ defined by $\mathcal{T}(X) = AX$ is a Fredholm operator.*

*Proof.* Since $A$ has closed range, Lemma 3.3 implies that $\mathcal{R}(\mathcal{T})$ is closed. We next show that $\operatorname{def} \mathcal{T} < \infty$. Since $A$ is Fredholm, there exist operators $R, F \in \mathcal{L}(\mathcal{H})$, with $F$ finite rank, such that $AR = F + 1$ ([18], Th. 2, pg. 120). Let $P$ denote the (finite rank) projection onto $\mathcal{H} \ominus \ker F$, and let $\{X_1, \ldots, X_k\}$ denote a basis for the finite dimensional space $\mathcal{L}(\mathcal{H}_1, P\mathcal{H})$. For $X$ in $\mathcal{L}(\mathcal{H}_1, \mathcal{H})$ we have $ARX = FX + X = FPX + X$. Since $PX \in \mathcal{L}(\mathcal{H}_1, P\mathcal{H})$, there exist scalars $c_1, \ldots, c_k$ such that $PX = \sum_{i=1}^{k} c_i X_i$. Let $[X]$ denote the image of $X$ in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}) / \mathcal{R}(\mathcal{T})$. Thus $[X] = [ARX] - [FPX] = -\sum_{i=1}^{k} c_i [FX_i]$. It follows that $\{[FX_1], \ldots, [FX_k]\}$ spans $\mathcal{L}(\mathcal{H}_1, \mathcal{H}) / \mathcal{R}(\mathcal{T})$ and thus $\operatorname{def} \mathcal{T} < \infty$.

To complete the proof it suffices to show that $\operatorname{nul} \mathcal{T} < \infty$. Suppose to the contrary that $\operatorname{nul} \mathcal{T} = \infty$. Let $X_1 \in \ker \mathcal{T}$, $X_1 \neq 0$, and let $t_1 \in \mathcal{H}_1$ be such that $X_1 t_1 \neq 0$. Since $\ker \mathcal{T}$ is infinite dimensional and $\mathcal{L}(\mathcal{H}_1, \mathcal{R}(X_1))$ is finite dimensional, there exists $X_2 \in \ker \mathcal{T}$ such that $\mathcal{R}(X_2) \not\subset \mathcal{R}(X_1)$; let $t_2 \in \mathcal{H}_1$ be such that $X_2 t_2 \notin \mathcal{R}(X_1)$. Proceding inductively, it follows that there exist sequences $\{X_n\} \subset \ker \mathcal{T}$ and $\{t_n\} \subset \mathcal{H}_1$ such that $X_n t_n \notin \bigvee_{i=1}^{n-1} \mathcal{R}(X_i)$. Clearly $\{X_n t_n\}_{n=1}^{\infty}$ is independent and $AX_n t_n = 0$ for each $n$. Thus $\ker A$ is infinite dimensional, which is a contradiction.

Returning to the proof of Theorem 3.1, we now consider the operators $\mathcal{T}_{A_{n+1} B_j}$ $(1 \leqslant j \leqslant p)$. Since $A - \beta_j$ is Fredholm, then $A' - \beta_j$ is Fredholm, and thus $A_{n+1} - \beta_j$ is also Fredholm. Since $\dim \mathcal{K}_j < \infty$, Lemma 3.6 implies that

$\mathscr{T}_{A_{n+1}-\beta_j, 0}$ is a Fredholm operator on $\mathscr{L}(\mathscr{K}_j, \mathfrak{M}_{n+1})$. Thus there exists $\delta_j > 0$ such that if $S \in \mathscr{L}(\mathscr{K}_j, \mathfrak{M}_{n+1})$ and

$$\|S - \mathscr{T}_{A_{n+1}-\beta_j, 0}\| < \delta_j,$$

then $S$ is Fredholm ([16], Ch. IV, Th. 5.22, pg. 236). Since $\sigma(B_j) = \{\beta_j\}$, $B_j - \beta_j$ is nilpotent; thus there exists an invertible operator $X_j \in \mathscr{L}(\mathscr{K}_j)$ such that

$$\|X_j^{-1}(B_j - \beta_j)X_j\| < \delta_j.$$

It follows that $S = \mathscr{T}_{A_{n+1}-\beta_j, X_j^{-1}(B_j-\beta_j)X_j}$ is Fredholm, and Proposition 2.8 implies that $\mathscr{T}_{A_{n+1}, B_j} = \mathscr{T}_{A_{n+1}-\beta_j, B_j-\beta_j}$ is Fredholm.

Finally, we consider the operators $\mathscr{T}_{A_i, B_{p+1}} = \mathscr{T}_{A_i-\alpha_i, B_{p+1}-\alpha_i}$ $(1 \leqslant i \leqslant n)$. Since $B - \alpha_i$ is Fredholm, so is $B_{p+1} - \alpha_i$ and thus also $(B_{p+1} - \alpha_i)^*$. Lemma 3.6 implies that $\mathscr{T}_{(B_{p+1}-\alpha_i)^*, 0}$ (acting on $\mathscr{L}(\mathfrak{M}_i, \mathscr{K}_{p+1})$) is Fredholm, and since $(A_i - \alpha_i)^*$ is nilpotent, the preceding argument implies that $\mathscr{T}_{(B_{p+1}-\alpha_i)^*, (A_i-\alpha_i)^*}$ is Fredholm. An application of Lemma 2.9 implies that $\mathscr{T}_{A_i-\alpha_i, B_{p+1}-\alpha_i}$ is Fredholm, which completes the proof.

Although we are primarily concerned with the case when $\mathscr{T}$ acts on $\mathscr{L}(\mathscr{H}_2, \mathscr{H}_1)$, we note that part of Theorem 3.1 has an analogue for the case when the underlying spaces are Banach spaces. Let $\mathscr{X}_1$ and $\mathscr{X}_2$ denote infinite dimensional complex Banach spaces, and for $A \in \mathscr{L}(\mathscr{X}_1)$ and $B \in \mathscr{L}(\mathscr{X}_2)$, define $\mathscr{T} \in \mathscr{L}(\mathscr{L}(\mathscr{X}_2, \mathscr{X}_1))$ by $\mathscr{T}(X) = AX - XB$.

THEOREM 3.7. *$\sigma_e(\mathscr{T}) \subset (\sigma(A) - \sigma_e(B)) \cup (\sigma_e(A) - \sigma(B))$; in particular, $\mathscr{T}$ is Fredholm if $\sigma(A) \cap \sigma_e(B) = \sigma_e(A) \cap \sigma(B) = \varnothing$.*

Rather than prove Theorem 3.7 in detail, we will merely sketch the appropriate modifications of the proof of Theorem 3.1. As in Theorem 3.1, it suffices to assume that

$$\sigma_e(A) \cap \sigma(B) = \sigma(A) \cap \sigma_e(B) = \varnothing,$$

and to prove that $\mathscr{T}$ is Fredholm. Let $K = \sigma(A) \cap \sigma(B)$. The proof that each point in $K$ is an isolated eigenvalue of $A$ or $B$ with finite multiplicity depends on the following fact: if $\mathscr{X}$ is a Banach space, $T \in \mathscr{L}(\mathscr{X})$, and $\lambda$ is a non-isolated point of $\sigma(T) \setminus \sigma_e(T)$, then there is a hole $H$ in $\sigma_e(T)$ such that $\lambda \in H$ and $H \subset \sigma(T)$. The proof of this fact follows the same method as in the Hilbert space case ([19], Proposition 1.27), but using [16], Ch. IV, Th. 5.31, page 241, which is the Banach space analogue of [19], Proposition 1.25.

Instead of orthogonal decompositions, we use the Riesz functional calculus to obtain Banach space direct sum decompositions of $\mathscr{X}_1$ and $\mathscr{X}_2$ corresponding to the $\alpha_i$'s and $\beta_j$'s. (In fact, even in the Hilbert space case it is not necessary to use orthogonal decompositions based on Lemma 2.14, although it seems convenient to do so.)

There is a technical difficulty near the conclusion of the proof due to the fact that $\mathscr{T}_{B^*A^*}$ acts on $\mathscr{L}(\mathscr{X}_1^*, \mathscr{X}_2^*)$; moreover, we are unable to prove that $\gamma(\mathscr{T}_{AB}) = \gamma(\mathscr{T}_{B^*A^*})$. Instead of relying on an analogue Lemma 2.2, we give separate arguments to prove that $\mathscr{T}_{A_{n+1}, B_j}$ and $\mathscr{T}_{A_i, B_{p+1}}$ are Fredholm. To do this, we rely on the following analogue of Lemma 3.6: if $\mathscr{X}$ is an infinite dimensional Banach space, $\mathscr{X}_1$ is a finite dimensional Banach space, and $A \in \mathscr{L}(\mathscr{X})$ is Fredholm, then the operators $\mathscr{T}_1(X) = AX\ (X \in \mathscr{L}(\mathscr{X}_1, \mathscr{X}))$ and $\mathscr{T}_2(X) = XA\ (X \in \mathscr{L}(\mathscr{X}, \mathscr{X}_1))$ are Fredholm operators on $\mathscr{L}(\mathscr{X}_1, \mathscr{X})$ and $\mathscr{L}(\mathscr{X}, \mathscr{X}_1)$ respectively.

Finally, the direct analogue of Proposition 2.8 for Banach spaces is valid and its proof depends on analogues of Lemma 2.3 and Lemma 2.5. Since the reductions based on Lemma 2.2 are not available, these results can be proved directly, using the following extension of Lemma 2.1:

$$\gamma(\mathscr{T}_{J^{-1}AJ,\, K^{-1}BK}) \leqslant \|J\|\, \|J^{-1}\|\, \|K\|\, \|K^{-1}\|\gamma(\mathscr{T}_{AB}).$$

QUESTION 3.8. In the Banach space case, is

$$(\sigma_e(A) - \sigma(B)) \cup (\sigma(A) - \sigma_e(B)) \subset \sigma_e(\mathscr{T})?$$

## 4. CLOSURE PROPERTIES OF THE RANGE OF $\mathscr{T}_{AB}$

In this section we give several necessary or sufficient conditions for $\mathscr{T}_{AB}$ to have closed range. Except as noted, $\mathscr{H}_1$ and $\mathscr{H}_2$ denote arbitrary Hilbert spaces.

LEMMA 4.1. *$\mathscr{R}(\delta_{A\oplus B})$ is closed if and only if $\mathscr{R}(\delta_A)$, $\mathscr{R}(\delta_B)$, $\mathscr{R}(\mathscr{T}_{AB})$, and $\mathscr{R}(\mathscr{T}_{BA})$ are closed.*

*Proof.* Let $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ and denote the operator matrix of $X \in \mathscr{L}(\mathscr{H})$ relative to this decomposition by $(X_{ij})_{1\leqslant i,\, j\leqslant 2}$. A matrix calculation shows that $\delta_{A\oplus B}(X)$ has the form

$$\begin{pmatrix} \delta_A(X_{11}) & \mathscr{T}_{AB}(X_{12}) \\ \mathscr{T}_{BA}(X_{21}) & \delta_B(X_{22}) \end{pmatrix};$$

the result follows directly.

LEMMA 4.2. *If $\mathscr{R}(\delta_A)$ and $\mathscr{R}(\delta_B)$ are closed, then $\mathscr{R}(\mathscr{T}_{AB})$ is closed.*

*Proof.* C. Apostol's theorem ([3], Theorem 3.5) implies that $A$ and $B$ are similar to Jordan operators. Thus $A \oplus B$ is similar to a Jordan operator, and [3], Theorem 3.5 implies that $\delta_{A\oplus B}$ has closed range. The result now follows from Lemma 4.1.

COROLLARY 4.3. *If $A$ and $B$ are similar to Jordan operators, then $\mathscr{R}(\mathscr{T}_{AB})$ is closed.*

The next two results will be cited frequently in the sequel in cases where $\mathcal{T}_{AB}$ does not have closed range.

LEMMA 4.4. i) *If there exists* $\lambda \in \mathbf{C}$ *such that* $\mathcal{R}(A-\lambda)$ *is not closed and* $\ker(B-\lambda) \cap \ker((B-\lambda)^*) \neq \{0\}$, *then* $\mathcal{R}(\mathcal{T}_{AB})$ *is not closed.*

ii) *If there exists* $\lambda \in \mathbf{C}$ *such that* $\ker(A-\lambda) \cap \ker((A-\lambda)^*) \neq \{0\}$ *and* $\mathcal{R}(B-\lambda)$ *is not closed, then* $\mathcal{R}(\mathcal{T}_{AB})$ *is not closed.*

*Proof.* Since ii) follows from i) by an application of Lemma 2.2, it suffices to prove i). Since $\mathcal{R}(A-\lambda)$ is not closed, there exists $\{z_n\} \subset (\ker(A-\lambda))^\perp$ and $y \in \mathcal{H}_1$ such that $(A-\lambda)z_n \to y$ and $y \notin \mathcal{R}(A-\lambda)$. Let $e \in \mathcal{H}_2$ denote a unit vector such that $(B-\lambda)e = (B-\lambda)^*e = 0$. For $n \geqslant 1$, define $U_n \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ by the following relations: $U_n e = z_n$; $U_n z = 0$ for each $z$ in $\mathcal{H}_2$ such that $(z, e) = 0$. Define $W \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ as follows: $We = y$; $Wz = 0$ whenever $z \in \mathcal{H}_2$ and $(z, e) = 0$.

We will show that $W \in \mathcal{R}(\mathcal{T})^- \setminus \mathcal{R}(\mathcal{T})$. Let $h \in \mathcal{H}_2$, $\|h\| = 1$; thus $h = \alpha e + f$ with $(e, f) = 0$ and $|\alpha|^2 + \|f\|^2 = 1$. Now

$$(AU_n - U_nB - W)h = A(\alpha z_n) - U_nB(\alpha e + f) - \alpha y =$$
$$= \alpha((A-\lambda)z_n - y) - U_nBf.$$

Since

$$(Bf, e) = (f, B^*e) = \lambda(f, e) = 0,$$

then $U_nBf = 0$, and so

$$\|(AU_n - U_nB - W)h\| \leqslant \|(A-\lambda)z_n - y\| \to 0.$$

Thus $W \in \mathcal{R}(\mathcal{T})^-$; suppose there exists $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that $AX - XB = W$. It follows that $y = We = AXe - XBe = (A-\lambda)Xe$, which contradicts the fact that $y \notin \mathcal{R}(A-\lambda)$. Thus $\mathcal{R}(\mathcal{T})$ is not closed.

COROLLARY 4.5. i) *If there exists* $\lambda \in \mathbf{C}$ *such that* $\mathcal{R}(A-\lambda)$ *is not closed and* $\lambda \in R_e(B)$, *then* $\mathcal{R}(\mathcal{T}_{AB})$ *is not closed.*

ii) *If there exists* $\lambda \in \mathbf{C}$ *such that* $\lambda \in R_e(A)$ *and* $\mathcal{R}(B-\lambda)$ *is not closed, then* $\mathcal{R}(\mathcal{T}_{AB})$ *is not closed.*

*Proof.* i) Since $\lambda \in R_e(B)$, Corollary 2.13 implies that there exists $B' \in \mathcal{U}(B)^-$ such that $\dim(\ker(B'-\lambda) \cap \ker((B'-\lambda)^*)) = \infty$. Thus Lemma 4.4 i) implies that $\mathcal{R}(\mathcal{T}_{AB'})$ is not closed, and the result follows from Lemma 2.3.

ii) Apply the preceding case to $\mathcal{T}_{B^*A^*}$ and then apply Lemma 2.2.

REMARK. If we assume that $\mathcal{R}(A)$ is not closed but only assume that nul $B =$ nul $B^* = \infty$, then it may happen that $\mathcal{R}(\mathcal{T}_{AB})$ is closed. For example, let $A$ be an operator such that $A^2 = 0$ and $\mathcal{R}(A)$ is not closed, and let $B = q_2^{(\infty)}$. Although $\gamma(A) = 0$ and nul $B =$ nul $B^* = \infty$, $\mathcal{T}_{AB}$ does have closed range (see Proposition 4.18).

In [5], Theorem 2, C. Apostol and J. G. Stampfli proved that if $A$ is compact, then $\mathscr{R}(\delta_A)$ is closed if and only if $A$ has finite rank (i.e. $A$ has closed range). We extend this result as follows.

THEOREM 4.6. *If $A$ and $B$ are compact, then $\mathscr{R}(\mathscr{T}_{AB})$ is closed if and only if $A$ and $B$ are finite rank operators.*

*Proof.* If $A$ and $B$ are finite rank operators, the conclusion that $\mathscr{T}$ has closed range follows from Corollary 4.3. Suppose $A$ is not finite rank; since $A$ is compact, then $\mathscr{R}(A)$ is not closed. Since $B$ is compact, $0 \in R_e(B)$, and thus Corollary 4.5 implies that $\mathscr{T}$ does not have closed range. If $B$ is not finite rank, then the preceding argument and Lemma 2.2 imply that

$$\gamma(\mathscr{T}_{AB}) = \gamma(\mathscr{T}_{B^*A^*}) = 0,$$

so $\mathscr{R}(\mathscr{T})$ is not closed in this case either.

We next consider the case when $A$ and $B$ are hyponormal operators.

LEMMA 4.7. *If $A$ is hyponormal and $\lambda$ is a limit point of $\sigma_l(A)$, then $\mathscr{R}(A - \lambda)$ is not closed.*

*Proof.* We may assume that $\lambda = 0$. Let $\{\lambda_n\} \subset \sigma_l(A)$ be a sequence of distinct nonzero points such that $\lambda_n \to 0$. If for some $n$, $\lambda_n \notin \sigma_{le}(A)$, then $\operatorname{nul}(A - \lambda_n) > 0$ and $\ker(A - \lambda_n) \subset (\ker A)^\perp$. Thus, if $\lambda_n \notin \sigma_{le}(A)$ for infinitely many values of $n$, it follows that $A|(\ker A)^\perp$ is not bounded below, and thus $\mathscr{R}(A)$ is not closed. We may therefore assume that $\{\lambda_n\} \subset \sigma_{le}(A) = R_e(A)$. Let $D$ denote a diagonalizable normal operator whose eigenvalues are precisely the $\lambda_n$'s (each with multiplicity one). Lemma 2.12 implies that $A' = A \oplus D \in \mathscr{U}(A)^-$, and $\gamma(A') = 0$ since $\lambda_n \to 0$. Corollary 2.4 and Lemma 3.3 imply that $\gamma(A) = \gamma(A') = 0$, so the proof is complete.

REMARK. Elementary considerations with the adjoint of the unilateral shift show that the analogue of Lemma 4.7 for co-hyponormal operators is false. The example of the shift also shows that in the hypothesis of Lemma 4.7, $\sigma_l(A)$ cannot be replaced by $\sigma_r(A)$.

THEOREM 4.8. *If $A$ and $B$ are hyponormal and $\sigma_l(A) \cap \sigma_l(B)$ contains a limit point of $\sigma_l(A) \cup \sigma_l(B)$, then $\mathscr{R}(\mathscr{T}_{AB})$ is not closed.*

*Proof.* We first consider the case when there exists a point $\lambda$ such that $\lambda$ is a limit point of $\sigma_l(A)$ and $\lambda \in \sigma_l(B)$. If $\lambda \in \sigma_p(B)$, then $\lambda$ is a reducing eigenvalue for $B$. Since Lemma 4.7 implies that $\mathscr{R}(A - \lambda)$ is not closed, it follows from Lemma 4.4 that $\mathscr{R}(\mathscr{T})$ is not closed. If $\lambda \notin \sigma_p(B)$, then $\lambda \in \sigma_l(B) \setminus \sigma_p(B)$, so $\mathscr{R}(B - \lambda)$ is not closed and $\lambda \in \sigma_{le}(B) = R_e(B)$. Since $\mathscr{R}(A - \lambda)$ is not closed, Corollary 4.5 implies that the range of $\mathscr{T}$ is not closed.

In the remaining case, there exists a limit point $\lambda$ of $\sigma_l(B)$ such that $\lambda \in \sigma_l(A)$. Lemma 4.7 implies that $\mathscr{R}(B - \lambda)$ is not closed. Since $\lambda \in \sigma_l(A) = \sigma_p(A) \cup R_e(A)$, the result follows from Lemma 4.4 ii) and Corollary 4.5 ii).

The following example, which is due to D. A. Herrero [15], shows that the converse of Theorem 4.8 is false.

EXAMPLE 4.9. Let $\{e_n\}_{n=-\infty}^{\infty}$ denote an orthonormal basis for a Hilbert spaee $\mathscr{H}$. Let $T \in \mathscr{L}(\mathscr{H})$ be the bilateral weighted shift defined by the following relations: $Te_{-n} = (1/2^n)e_{-n+1}$ $(n \geqslant 1)$; $Te_n = e_{n+1}$ $(n \geqslant 0)$. Let $S = (1/2)T$. Familiar results about shifts imply that $T$ and $S$ are hyponormal,

$$\sigma_1(T) = \{0\} \cup \{z: |z| = 1\},$$

and

$$\sigma_1(S) = \{0\} \cup \{z: |z| = 1/2\}.$$

Thus

$$\sigma_1(T) \cap \sigma_1(S) = \{0\}$$

and 0 is isolated in $\sigma_1(T) \cup \sigma_1(S)$. Nevertheless, $\mathscr{R}(\mathscr{T}_{TS})$ is not closed, because $\mathscr{R}(T)$ is not closed and $0 \in R_e(S)$ (Corollary 4.5).

In contrast to the preceding example, we next exhibit hyponormal operators $A$ and $B$ such that $\sigma_1(A) \cap \sigma_1(B) = \varnothing$ and $\mathscr{R}(\mathscr{T})$ is closed, although $\mathscr{T}$ is neither bounded below nor surjective.

EXAMPLE 4.10. Let $U \in \mathscr{L}(\mathscr{H})$ denote the unilateral shift of multiplicity one, let $A = (1/2)U \oplus 2U$, and let $B = U \oplus U$. $A$ and $B$ are hyponormal and

$$\sigma_r(A) = \{z: |z| \leqslant 2\},$$

$$\sigma_1(A) = \{z: |z| = 1/2\} \cup \{z: |z| = 2\},$$

$$\sigma_r(B) = \{z : |z| \leqslant 1\},$$

and

$$\sigma_1(B) = \{z: |z| = 1\}.$$

Since

$$\sigma_r(A) \cap \sigma_1(B) = \{z: |z| = 1\}$$

and

$$\sigma_1(A) \cap \sigma_r(B) = \{z: |z| = 1/2\},$$

it follows from [7] that $\mathscr{T}_{AB}$ is neither surjective nor bounded below; moreover,

$$\sigma_r(A) \cap \sigma_r(B) = \{z: |z| \leqslant 1\}.$$

However,

$$\sigma_1(A) \cap \sigma_1(B) = \varnothing,$$

and we will show that $\mathcal{T}$ has closed range. Let $X = (X_{ij})_{1 \leq i, j \leq 2}$ denote the matrix of an operator $X$ on $\mathcal{H} \oplus \mathcal{H}$. A calculation shows that the matrix of $\mathcal{T}_{AB}(X)$ is of the form

$$\begin{pmatrix} \mathcal{T}_{(1/2)U, U}(X_{11}) & \mathcal{T}_{(1/2)U, U}(X_{12}) \\ \mathcal{T}_{2U, U}(X_{21}) & \mathcal{T}_{2U, U}(X_{22}) \end{pmatrix}.$$

The results of [7] imply that $\mathcal{T}_{(1/2)U, U}$ is surjective and that $\mathcal{T}_{2U, U}$ is bounded below, from which it follows that $\mathcal{T}_{AB}$ has closed range.

The preceding examples suggest the following questions.

QUESTION 4.11. Suppose $A$ and $B$ are hyponormal operators such that $\sigma_1(A) \cap \sigma_1(B) = \{\lambda_1, \ldots, \lambda_n\}$ and such that each $\lambda_i$ does not satisfy the hypothesis of either Lemma 4.4 or of Corollary 4.5. Is the range of $\mathcal{T}_{AB}$ closed?

QUESTION 4.12. If $A$ and $B$ are hyponormal and $\sigma_1(A) \cap \sigma_1(B) = \varnothing$, is the range of $\mathcal{T}_{AB}$ closed?

Using a spectral decomposition similar to that used in the following result, it can be shown that the preceding questions are actually equivalent; however, we omit the details of the proof of this equivalence. The next result provides a partial (affirmative) answer to Question 4.12.

PROPOSITION 4.13. *If $A$ and $B$ are hyponormal and $\sigma(A) \cap \sigma(B)$ contains no limit point of $\sigma(A) \cup \sigma(B)$, then $\mathcal{R}(\mathcal{T}_{AB})$ is closed.*

*Proof.* From Rosenblum's Theorem [22] we may assume that $K \equiv \sigma(A) \cap \sigma(B) \neq \varnothing$, and it follows that $K$ is finite; we denote the distinct elements of $K$ by $\{\lambda_1, \ldots, \lambda_n\}$. Since $A$ and $B$ are hyponormal, each $\lambda_i$ satisfies the following properties:

i) $\lambda_i$ is an isolated point in $\sigma(A)$ and $\ker(A - \lambda_i)$ is a reducing subspace for $A$ which coincides with the Riesz subspace for $A$ corresponding to $\{\lambda_i\}$;

ii) $\lambda_i$ is an isolated point in $\sigma(B)$ and $\ker(B - \lambda_i)$ is a reducing subspace for $B$ which coincides with the Riesz subspace for $B$ corresponding to $\{\lambda_i\}$.

Let $\mathcal{L}_1 = \bigvee_{i=1}^{n} \ker(A - \lambda_i)$, $\mathcal{K}_1 = \mathcal{H}_1 \ominus \mathcal{L}_1$, $\mathcal{L}_2 = \bigvee_{i=1}^{n} \ker(B - \lambda_i)$, and $\mathcal{K}_2 = \mathcal{H}_2 \ominus \mathcal{L}_2$. Relative to the decomposition $\mathcal{H}_1 = \mathcal{L}_1 \oplus \mathcal{K}_1$, $A = A_1 \oplus A_2$, where $\sigma(A_1) = K$ and $\sigma(A_1) \cap \sigma(A_2) = \varnothing$. Similarly, relative to $\mathcal{H}_2 = \mathcal{L}_2 \oplus \mathcal{K}_2$, $B = B_1 \oplus B_2$, $\sigma(B_1) = K$, and $\sigma(B_1) \cap \sigma(B_2) = \varnothing$. It follows that $\sigma(A_2) \cap \sigma(B_2) = \varnothing$. For $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$, let $(X_{ij})_{1 \leq i, j \leq 2}$ denote the matrix of $X$ relative to the above decompositions. The matrix of $AX - XB$ is of the form

$$\begin{pmatrix} A_1X_{11} - X_{11}B_1 & A_1X_{12} - X_{12}B_2 \\ A_2X_{21} - X_{21}B_1 & A_2X_{22} - X_{22}B_2 \end{pmatrix}.$$

Note that

$$\sigma(A_1) \cap \sigma(B_2) = \sigma(A_2) \cap \sigma(B_1) = \sigma(A_2) \cap \sigma(B_2) = \varnothing.$$

These identities, together with the fact that $A_1$ and $B_1$ are normal operators with finite spectra, readily imply that $\mathcal{R}(\mathcal{T})$ is closed (see [22] and Corollary 4.3).

REMARK. The reader will note similarities between the preceding proof and the proof of one direction of [1], Theorem 3.3. Assume that $A$ and $B$ satisfy the hypothesis of Proposition 4.13, so that $\mathscr{R}(\mathscr{T})$ is closed. A slight elaboration of the above argument allows us to determine when $\mathscr{T}$ is semi-Fredholm and to calculate the index. Let $\mathfrak{M}_i = \ker(A - \lambda_i)$ and $\mathfrak{N}_i = \ker(B - \lambda_i)$ $(1 \leqslant i \leqslant n)$. Relative to the decompositions $\mathscr{L}_1 = \mathfrak{M}_1 \oplus \ldots \oplus \mathfrak{M}_n$ and $\mathscr{L}_2 = \mathfrak{N}_1 \oplus \ldots \oplus \mathfrak{N}_n$, the matrices of $A_1$ and $B_1$ are of the form

$$A_1 = \lambda_1 1_{\mathfrak{M}_1} \oplus \ldots \oplus \lambda_n 1_{\mathfrak{M}_n}$$

and

$$B_1 = \lambda_1 1_{\mathfrak{N}_1} \oplus \ldots \oplus \lambda_n 1_{\mathfrak{N}_n}.$$

If $(Y_{ij})_{1 \leqslant i,j \leqslant n}$ denotes the matrix of an operator $X_{11} \in \mathscr{L}(\mathscr{L}_2, \mathscr{L}_1)$, then the matrix of $A_1 X_{11} - X_{11} B_1$ is of the form $((\lambda_i - \lambda_j) Y_{ij})_{1 \leqslant i,j \leqslant n}$. It is now clear that

$$\operatorname{nul} \mathscr{T} = \operatorname{nul} \mathscr{T}_{A_1 B_1} = \sum_{i=1}^{n} \dim(\mathscr{L}(\mathfrak{N}_i, \mathfrak{M}_i)) = \operatorname{def} \mathscr{T}_{A_1 B_1} = \operatorname{def} \mathscr{T}.$$

Since

$$\dim(\mathscr{L}(\mathfrak{N}_i, \mathfrak{M}_i)) = \operatorname{nul}(A - \lambda_i) \operatorname{nul}(B - \lambda_i),$$

it follows that $\mathscr{T}$ is semi-Fredholm if and only if $\operatorname{nul}(A - \lambda_i) < \infty$ and $\operatorname{nul}(B - \lambda_i) < < \infty$ $(1 \leqslant i \leqslant n)$; moreover, in this case, $\operatorname{ind}(\mathscr{T}) = 0$.

The next corollary recaptures a result of J. Stampfli, whose proof in [23] differs from that below.

COROLLARY 4.14. ([23], Theorem 1) *If $A$ is hyponormal, then $\mathscr{R}(\delta_A)$ is closed if and only if $\sigma(A)$ is finite.*

*Proof.* Recall that bdry $(\sigma(A)) \subset \sigma_l(A)$. Thus, if $\sigma(A)$ is infinite, then it follows from Theorem 4.8 that $\mathscr{R}(\delta_A)$ is not closed. Conversely, if $\sigma(A)$ is finite, then Proposition 4.13 implies that $\mathscr{R}(\delta_A)$ is closed. (Alternately, since $A$ is a normal Jordan operator in this case, Corollary 4.3 applies directly.)

Corollary 4.15. *If $A$ and $B$ are hyponormal operators such that $\sigma(A) = \sigma_l(A)$ and $\sigma(B) = \sigma_l(B)$, then $\mathscr{R}(\mathscr{T}_{AB})$ is closed if and only if $\sigma(A) \cap \sigma(B)$ contains no limit point of $\sigma(A) \cup \sigma(B)$.*

*Proof.* The result follows from Theorem 4.8 and Proposition 4.13.

Recall that $T \in \mathscr{L}(\mathscr{H})$ is non-quasitriangular if there exists $\lambda \in \mathbf{C}$ such that $T - \lambda$ is semi-Fredholm and $\operatorname{ind}(T - \lambda) < 0$ [8]. (The converse is also true [4].) Thus the hypothesis of Corollary 4.15 is satisfied if $A$ and $B$ are quasitriangular hyponormal operators, and, in particular, if $A$ and $B$ are normal [14].

In [1], Theorem 3.3, J. Anderson and C. Foiaş proved that if $A$ and $B$ are scalar operators on a Banach space, then $\mathscr{R}(\mathscr{T})$ is closed if and only if $\sigma(A) \cap \sigma(B)$ contains no limit point of $\sigma(A) \cup \sigma(B)$. In the Hilbert space case, the scalar operators of [1] are similar to normal operators (see [1]), so in the Hilbert space case the result of [1], Theorem 3.3 is equivalent to the following result.

COROLLARY 4.16. *If $A$ and $B$ are similar to normal operators, then $\mathscr{R}(\mathscr{T}_{AB})$ is closed if and only if $\sigma(A) \cap \sigma(B)$ contains no limit point of $\sigma(A) \cup \sigma(B)$.*

*Proof.* Apply Lemma 2.3 and Corollary 4.15.

Proposition 4.13 and the Remark following it, together with Corollary 4.15, immediately imply the following result.

COROLLARY 4.17. *If $A$ and $B$ are hyponormal, $\sigma(A) = \sigma_1(A)$, and $\sigma(B)=\sigma_1(B)$, then the following are equivalent :*

i) *$\mathscr{T}$ is semi-Fredholm;*

ii) *$\mathscr{T}$ is Fredholm and* $\operatorname{ind} \mathscr{T} = 0$;

iii) *$\mathscr{T}$ is Fredholm;*

iv) *$\sigma(A) \cap \sigma(B)$ contains no limit point of $\sigma(A) \cup \sigma(B)$, and for each $\lambda \in \sigma(A) \cap \sigma(B)$,* $\operatorname{nul}(A - \lambda) < \infty$ *and* $\operatorname{nul}(B - \lambda) < \infty$.

One of the themes of [1] is that if $A$ and $B$ are normal, then $\mathscr{T}_{AB}$ enjoys several properties of a normal operator on Hilbert space. Suppose $A$ and $B$ are quasitriangular hyponormal operators; the same is true for $A - \lambda$ ($\lambda \in \mathbb{C}$). Thus Corollary 4.17 implies that there exists no $\lambda \in \mathbb{C}$ such that $\mathscr{T}_{AB} - \lambda$ ($= \mathscr{T}_{A-\lambda,B}$) is semi-Fredholm with negative index. If $A$ and $B$ are normal, then $\mathscr{T}_{AB}$ is normal in the Banach space sense (see the proof of [12], Corollary 6).

We conclude by briefly considering the case when $A$ and $B$ are nilpotent operators. For the case when $A = B$ and $A$ is nilpotent, it follows from [3], Corollary 3.4 that $\mathscr{R}(\delta_A)$ is closed if and only if $A^k$ has closed range for each $k \geqslant 1$ (cf. [5], Theorem 6). Suppose that $A$ and $B$ are nonzero operators such that $A^2 = B^2 = 0$, and consider the decompositions $\mathscr{H}_1 = \ker A \oplus (\ker A)^\perp$ and $\mathscr{H}_2 = \ker B \oplus (\ker B)^\perp$. Let $P$ denote the projection of $\mathscr{H}_1$ onto $\ker A$ and let $Q$ denote the projection of $\mathscr{H}_2$ onto $\ker B$. Let $A_1 = PA|(1 - P)\mathscr{H}_1 \in \mathscr{L}((\ker A)^\perp, \ker A)$ and let $B_1 = QB|(1 - Q)\mathscr{H}_2 \in \mathscr{L}((\ker B)^\perp, \ker B)$.

PROPOSITION 4.18. *$\mathscr{T}_{AB}$ has closed range if and only if at least one of the following properties is satisfied:*

i) *$A_1$ is invertible;*

ii) *$B_1$ is invertible;*

iii) *$A_1$ and $B_1$ have closed range.*

*Proof.* If iii) holds, then it follows from [5], Theorem 6 that $\delta_A$ and $\delta_B$ have closed range. Thus, Lemma 4.2 implies that $\mathscr{T}$ has closed range. Let $(X_{ij})_{1 \leqslant i,j \leqslant 2}$

denote the matrix of an operator $X \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ relative to the above decompositions of $\mathcal{H}_1$ and $\mathcal{H}_2$. A calculation shows that the matrix of $\mathcal{T}(X)$ is equal to

$$\begin{pmatrix} A_1X_{21} & A_1X_{22} - X_{11}B_1 \\ 0 & -X_{21}B_1 \end{pmatrix};$$

it now follows easily that $\mathcal{T}$ has closed range if $A_1$ or $B_1$ is invertible.

For the converse, we assume that both $A_1$ and $B_1$ are non-invertible and that either $\mathcal{R}(A_1)$ or $\mathcal{R}(B_1)$ is not closed. We seek to show that $\mathcal{R}(\mathcal{T})$ is not closed, and we first consider the case when $\mathcal{R}(A_1)$ is dense. Since $A_1$ is injective and non-invertible, it follows that $\mathcal{R}(A_1)$ is not closed; thus $\mathcal{R}(A)$ is not closed. We distinguish two subcases.

i) $\mathcal{R}(B_1)$ is not dense. Relative to the decomposition $\mathcal{H}_2 = (Q\mathcal{H}_2 \ominus \mathcal{R}(B_1)) \oplus \oplus (Q\mathcal{H}_2 \cap \mathcal{R}(B_1)^-) \oplus (1 - Q)\mathcal{H}_2$, the matrix of $B$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $B$ has a nontrivial reducing kernel, and since $\mathcal{R}(A)$ is not closed, Lemma 4.4 implies that $\mathcal{R}(\mathcal{T})$ is not closed.

ii) $\mathcal{R}(B_1)$ is dense. Since $B_1$ is non-invertible and injective, it follows that $\mathcal{R}(B_1)$ is not closed. Thus $\mathcal{R}(B_1^*)$ is not closed, and [6], Corollary 2.4 implies that there exists an infinite dimensional subspace $\mathfrak{M} \subset Q\mathcal{H}_2$, having an infinite dimensional complement $\mathfrak{M}^\perp = Q\mathcal{H}_2 \ominus \mathfrak{M}$, such that $K \equiv B_1^* \mid \mathfrak{M}: \mathfrak{M} \to (1 - Q)\mathcal{H}_2$ is a compact operator. Relative to the decomposition $\mathcal{H}_2 = \mathfrak{M} \oplus \mathfrak{M}^\perp \oplus (1 - Q)\mathcal{H}_2$, the matrix of $B$ is of the form

$$\begin{pmatrix} 0 & 0 & K^* \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $K^*$ is compact, $0 \in R_e(B)$, and since $\mathcal{R}(A)$ is not closed, Corollary 4.5 implies that $\mathcal{R}(\mathcal{T})$ is not closed.

We next consider the case when $\mathcal{R}(A_1)$ is not dense, and it follows as in i) above that $A$ has a nontrivial reducing kernel. We again examine two subcases.

i′) $\mathcal{R}(B_1)$ is not closed. Since $A$ has a nontrivial reducing kernel and $\mathcal{R}(B)$ is not closed, Lemma 4.4 ii) implies that $\mathcal{R}(\mathcal{T})$ is not closed.

ii′) $\mathcal{R}(B_1)$ is closed. Thus $\mathcal{R}(A_1)$ is not closed; moreover, since $B_1$ is injective and non-invertible, then $\mathcal{R}(B_1)$ is not dense. It follows as above that $B$ has a nontrivial reducing kernel, and since $\mathcal{R}(A)$ is not closed, the result follows from Lemma 4.4. The proof is now complete.

In contrast to the close relationship between Theorem 4.6 and [5], Theorem 2 for the case of compact operators, the preceding result suggests that there is no direct analogue of [3], Corollary 3.4 for the case when $A$ and $B$ are nilpotent. Further progress in this direction seems to await a characterization of the nilpotent operators $T$ for which $0 \in R_e(T)$. In this connection, we note that if $A^2 = 0$, then $0 \in R_e(A)$ if and only if the reducing kernel of $A$ is infinite dimensional or the range of $A$ is not closed; the proof of this fact is largely implicit in the proof of Proposition 4.18.

*Research supported by N.S.F. Grant MCS76—07537.*

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Received July 18, 1979.

*Added in proof.* In a forthcoming sequel to this paper, we determine the semi-Fredholm domain of $\mathcal{T}_{AB}$. We prove that $\mathcal{R}(\mathcal{T}_{AB})$ is closed and $\operatorname{def}(\mathcal{T}_{AB}) < \infty$ if and only if $\sigma_{re}(A) \cap \sigma_l(B) = \sigma_r(A) \cap \sigma_{le}(B) = \emptyset$; $\mathcal{R}(\mathcal{T}_{AB})$ is closed and $\operatorname{nul}(\mathcal{T}_{AB}) < \infty$ if and only if $\sigma_{le}(A) \cap \sigma_r(B) = \sigma_l(A) \cap \sigma_{re}(B) = \emptyset$. As a consequence we prove that $\sigma_{SF}(\mathcal{T}_{AB}) \equiv \{\lambda \in C\colon \mathcal{T}_{AB} - \lambda$ is not semi-Fredholm$\} = [(\sigma_{le}(A) - \sigma_r(B)) \cup (\sigma_l(A) - \sigma_{re}(B))] \cap [(\sigma_{re}(A) - \sigma_l(B)) \cup (\sigma_r(A) - \sigma_{le}(B))]$.