J. OPERATOR THEORY
3(1980), 115–142

SOME INDEX THEOREMS FOR SUBNORMAL OPERATORS

ROBERT F. OLIN and JAMES E. THOMSON

I. INTRODUCTION

Let N be a normal operator on a separable Hilbert space $\mathscr{K}$. An operator is pure if it has no reducing subspace on which it is normal. The set $\mathscr{S}(N)$ will denote the collection of subnormal operators that have N as their minimal normal extension (m.n.e.); $\mathscr{S}_p(N)$ denotes the pure operators in $\mathscr{S}(N)$. (See [3] for the basic results concerning subnormal operators.)

An operator T is semi-Fredholm if the range of T is closed and either ker T or ker T^* is finite dimensional. If T is semi-Fredholm then the index of T, denoted $\mathrm{i}(T)$, is defined to be the integer (possibly $\pm\infty$) dim ker T — dim ker T^*.

If S belongs to $\mathscr{S}(N)$ then $\sigma(S)$ (the spectrum of S) contains $\sigma(N)$. In fact, $\sigma(S)\setminus\sigma(N)$ is either empty or equals the union of some of the bounded components of the complement of $\sigma(N)$. We refer to these latter components as the holes of $\sigma(N)$. There has been considerable investigation of problems related to "hole filling". (See [8] and [13] for a history and some results concerning this subject.) The major purpose of this paper is to describe an intrinsic relationship between each normal operator N in a certain class and the index theory for the collection $\mathscr{S}_p(N)$.

Before we discuss this matter further, we present a result that is well-known. Suppose $S\in\mathscr{S}(N)$ and let Ω be a component of $\sigma(S)\setminus\sigma(N)$. Then for $\alpha,\ \beta\in\Omega$ it follows that

(a) $\dim(\mathscr{H}\ominus(S-\alpha)\mathscr{H})\neq 0$, and

(b) $\dim(\mathscr{H}\ominus(S-\alpha)\mathscr{H})=\dim(\mathscr{H}\ominus(S-\beta)\mathscr{H})$.

(Here $\mathscr{H}$ is the space on which S is defined.) To see this, observe that $\alpha,\beta\in\Omega$ implies that $S-\alpha$ and $S-\beta$ are bounded below but not invertible. Therefore, $(S-\alpha)\mathscr{H}$ and $(S-\beta)\mathscr{H}$ are proper closed subspaces of $\mathscr{H}$. That establishes (a). It also follows that $S-\alpha$ and $S-\beta$ are semi-Fredholm operators, because $\ker(S-\alpha)$ and $\ker(S-\beta)$ are trivial. Hence, to obtain (b) it suffices to show that $\mathrm{i}(S-\alpha)=\mathrm{i}(S-\beta)$. The latter now follows from the fact that the index is a continuous function from the semi-Fredholm operators to $\mathbf{Z}\cup\{+\infty,-\infty\}$ and the fact that

$\{S - \lambda : \lambda \in \Omega\}$ is a connected collection of semi-Fredholm operators. In other words, $S - \lambda$ is semi-Fredholm and $\mathrm{i}(S - \lambda)$ is negative-valued and constant for λ in Ω.

It seems reasonable to guess that $\mathrm{i}(S - \lambda)$ for λ in one of these holes is directly related to the multiplicity of the normal operator and "the number of times the spectrum of N surrounds the hole". The major part of this paper is to give some instances where the last statement can be rigorously defined and this relationship is valid.

Due to the technical arguments involved, the authors feel that the reader will be better equipped to read the rest of the paper if a particular example is discussed in some detail.

Let m_1 and m_2 denote normalized Lebesgue measure on the circles $\{|z| = 1\}$ and $\{|z - 1/2| = 1/4\}$, respectively. Let $\mu = m_1 + m_2$ and let N be M_z (multiplication by z) on $L^2(\mu)$. Let D denote the open unit disc and let Ω denote the outer component of $D \setminus \operatorname{spt} \mu$.

PROPOSITION 1. *If $S \in \mathscr{S}_p(N)$, then $\mathrm{i}(S - \lambda) = -1$ for all λ in Ω.*

REMARKS. The assumption that S is pure is only needed to guarantee that $\sigma(S) \supset \Omega$. That the normal operator has multiplicity one is crucial to our arguments. The authors do not know what happens in any other case.

Even this special case of Theorem 1 is a new result. The Sarason theory [16] for the annulus and the Abrahamse-Douglas theory [1] give the result only for the case where $\sigma(S) = \bar{\Omega}$.

Suppose that $\sigma(S) = \bar{D}$. It is an open question as to whether the only possibilities for $\mathrm{i}(S - \lambda)$ are -1 and -2 for λ in $D \setminus \bar{\Omega}$. That these numbers are possible follows from looking at $H^2(\mu)$ (the closure of the polynomials in $L^2(\mu)$) and $H^2(m_1) \oplus \oplus H^2(m_2)$.

It is crucial that Lebesgue measure on $\{|z - 1/2| = 1/4\}$ be absolutely continuous with respect to m_2 in the last question. If not, but still keeping the assumption $\operatorname{spt} m_2 = \{|z - 1/2| = 1/4\}$, we shall see later that $\mathrm{i}(S - \lambda) = -1$ for all λ in $D \setminus (\operatorname{spt} \mu)$.

We shall use the notation and results from section II in the following proof of Proposition 1.

Proof. Since $S \in \mathscr{S}(N)$, it follows that $\sigma(S) \supset \sigma(N)$. A theorem of Clancey and Putnam [7] shows that the spectrum of a pure subnormal operator cannot be the union of $\{|z| = 1\}$ and a subset of $\{|z - 1/2| \leqslant 1/4\}$. Thus $\sigma(S) \supset \Omega$ and $\mathrm{i}(S - \lambda) \leqslant -1$ for all λ in Ω.

Because $\mathrm{i}(S - \lambda)$ is a constant function of λ in Ω, it now suffices to show that $\mathrm{i}(S) \geqslant -1$. Suppose not. Then there exist K_1, K_2 in $\ker S^*$ such that $(K_1, K_2) = 0$ and $\|K_1\| = \|K_2\| = 1$. A simple computation shows that $(z^n K_i, K_j) = 0$ for all $n \geqslant 1$ and $i, j = 1, 2$. The function $(z|K_i|^2 \mathrm{d}\mu)^\wedge$ is analytic in Ω and has an analytic continuation across $\{|z| = 1\}$ by Lemmas 1 and 3. Note that the values of this analytic function on $\{|z| = 1\}$ are equal to $|K_i|^2$ a.e. with respect to m_1. Since $(\mathrm{Re}\, K_i\overline{K_j})\, \mathrm{d}\mu$ and $(\mathrm{Im}\, K_i\overline{K_j})\mathrm{d}\mu$ are annihilating measures for $i \neq j$, it also follows from the lemmas that $(zK_i\overline{K_j}\mathrm{d}\mu)^\wedge$ has an analytic continuation across $\{|z| = 1\}$ and its values on $\{|z| = 1\}$ are equal to $K_i\overline{K_j}$ a.e. (m_1). Thus the analytic continuations of $(z|K_1|^2\mathrm{d}\mu)^\wedge \cdot \cdot (z|K_2|^2\mathrm{d}\mu)^\wedge$ and $(zK_1\overline{K_2}\mathrm{d}\mu)^\wedge\, (zK_2\overline{K_1}\mathrm{d}\mu)^\wedge$ are equal on $\{|z| = 1\}$ and hence in Ω. But this is a contradiction, for the first product equals one at zero and the second vanishes at zero.

Before we state the main theorems, we need some notation. The word measure will always denote a finite, positive, regular Borel measure with compact support. If μ is a measure in the plane then $P^\infty(\mu)$ will denote the weak-star closure of the polynomials in $L^\infty(\mu)$. (Recall the dual of $L^1(\mu)$ is $L^\infty(\mu)$.) We will draw heavily on the facts set forth about $P^\infty(\mu)$ in [8] and [17].

Let N be a normal operator with a cyclic vector. Then there exists a measure μ in the plane such that N is unitarily equivalent to M_z, multiplication by z, on $L^2(\mu)$. A necessary and sufficient condition that $\mathscr{S}_p(M_z)$ is nonempty is that $P^\infty(\mu)$ has no L^∞ direct summand ([8], Prop. 9.6). (If the assumption N has a cyclic vector is left out this last result is not longer true. We will provide an example of this in the last part of the paper, where we will go into this problem in further detail.)

If one wants to study the index theory for $S \in \mathscr{S}_p(M_z)$, there are certain assumptions that can be made. By using the decomposition techniques set forth in [8], we can assume $P^\infty(\mu)$ is antisymmetric. Furthermore, by using a conformal map, we can assume $P^\infty(\mu)$ is isometrically isomorphic to the Hardy space $H^\infty(D)$, where D is the open unit disc and $H^\infty(D)$ is the algebra of bounded analytic functions on the disc. (We give a function $f \in H^\infty(D)$ its standard boundary values on ∂D, the boundary of the disc.)

Our results lead to a necessary and sufficient condition that there exists a hole Ω of $\sigma(N)$ such that for every $S \in \mathscr{S}_p(N)$ we have $\sigma(S) \supset \Omega$ and the index of $S - \lambda$ is -1 for every $\lambda \in \Omega$. If μ is a measure and $f \in L^\infty$ then $\|f\|_\mu$ will denote the essential supremum of f. If $F \subset \mathrm{spt}\, \mu$ then $\mu|_F$ will denote the restriction of μ to F. Our first result is the following:

THEOREM 1. *Let $N = M_z$ on $L^2(\mu)$ and assume $P^\infty(\mu) = H^\infty(D)$. If there exist $\lambda \in D$ and $f \in P^\infty(\mu)$ for which*

$$(*) \qquad |f(\lambda)| > \|f\|_{\mu|_D},$$

then $\mathrm{i}(S - \lambda) = -1$ *for each* $S \in \mathscr{S}_p(N)$.

REMARKS a. If Ω is a component of $D \setminus \mathrm{spt}\, \mu$ for which there exist a point $\lambda \in \Omega$ and a function $f_0 \in P^\infty(\mu)$ satisfying (*), we say that Ω is an *outer component (hole)* of the support of μ. Theorem 1 says every $S \in \mathscr{S}_p(N)$ fills in the outer holes and $i(S-\lambda)=-1$ in these components. (These last two remarks have some relevance to the recent work of S. Brown [5], Lemma 3.1.)

b. The example done in the first part of the introduction satisfies the hypothesis of the theorem. To see this let $f_0(z)=z$ for all $z\in D$. Then $\|f_0\|_\mu=1$ while $\|f_0\|_{\mu|_D}=$ $=\frac{3}{4}<1$. However, what the authors believe is the canonical example is the following example due to D. Sarason [17]. (It also shows why much more work is needed for the proof of the theorem than that for the example given in the first part of the introduction.)

Let C be a perfect, nowhere dense set of positive Lebesgue measure on ∂D. Join each of the endpoints of a component of $\partial D\setminus C$ by a straight line, denoted J_n. Then $J=C\cup\left(\bigcup_1^\infty J_n\right)$ is a rectifiable Jordan curve, and C has positive arc length measure on J. By a theorem of F. and M. Riesz, C has positive harmonic measure with respect to int J.

Let $\mu=\mathrm{m}|_C+\mathrm{d}A|_{D\setminus\mathrm{int}\,J}$ where m denotes normalized Lebesgue measure on ∂D and $\mathrm{d}A$ denotes planar Lebesgue measure. It is easy to see that $P^\infty(\mu)=H^\infty(D)$. Our next task is to show that int J is an outer component of the support of μ.

Let u be the solution of the Dirichlet problem in int J with boundary values 1 on C and 0 on $J\setminus C$. Let v be the harmonic conjugate of u and set $f_0=e^{u+iv}$. Clearly $|f_0(z)|>1$ for all z in int J. Using the Schwarz reflection principle, we extend f_0 to a function (still called f_0) belonging to $H^\infty(D)$. The extended f_0 has modulus 1 everywhere on $\bigcup_1^\infty J_n$ and modulus less than 1 everywhere on $D\setminus\overline{\mathrm{int}\,J}$. Hence $\|f_0\|_\mu=e$ while $\|f_0\|_{\mu|_D}=1$. Recalling that for any $\lambda\in J$, we have $|f_0(\lambda)|>1$; we see that int J is an outer component of spt μ.

c. We wish to note that each point λ in an outer component Ω of the support of μ is "surrounded only once by μ" in the following technical sense: If $\mu=\mu_1+\mu_2$ and $\mu_1\perp\mu_2$ (i.e., μ_1 is singular to μ_2), then a point $\lambda\in\Omega$ can be in at most one Sarason hull for μ_i, $i=1,2$. That is, if $\widetilde{K}_i$ is the set used by Sarason [17] to describe $P^\infty(\mu_i)$, $i=1,2$ then

$$\Omega\cap\mathrm{int}\,\widetilde{K}_1\cap\mathrm{int}\,\widetilde{K}_2=\emptyset.$$

The proof of this will come later in the paper.

THEOREM 2. *Let $N=M_z$ on $L^2(\mu)$ and assume $P^\infty(\mu)=H^\infty(D)$. Assume further that for every $f\in P^\infty(\mu)$ we have*

$$\|f\|_\mu=\|f\|_{\mu|_D}.$$

Then if Ω is any component of $D \setminus \operatorname{spt} \mu$ and $n \in \mathbf{Z}^+ \cup \{\infty\}$, there exists an operator $S \in \mathscr{S}_p(N)$ such that

$$\mathrm{i}(S - \lambda) = -n$$

for all $\lambda \in \Omega$.

d. If we combine Theorems 1 and 2, we see that a necessary and sufficient condition that there exists a hole Ω of $\sigma(N)$ such that every $S \in \mathscr{S}_p(N)$ has index -1 there, is that $\sigma(N)$ contains an outer hole.

e. The key to the proof of Theorem 2 will be that given any integer n (or ∞), we can write $\mu = \sum_{i=1}^{n} \mu_i$ where the μ_i's are pairwise singular and $\Omega \subset \operatorname{int} \widetilde{K}_i$, $i = 1, \ldots, n$, where $\widetilde{K}_i$ is the Sarason hull for μ_i. Intuitively, we can surround Ω as many times as we please by μ.

II. INDEX THEOREMS FOR $\mathscr{S}_p(N)$

We now begin the proof of Theorem 1 by developing some measure theoretic machinery. The Cauchy transform of a complex measure, ν, denoted by $\hat{\nu}$, is defined by

$$\hat{\nu}(z) = \int \frac{\mathrm{d}\nu(w)}{w - z}$$

for all z such that $\int |w - z|^{-1} \mathrm{d}|\nu|(w) < \infty$. ($|\nu|$ is the total variation measure of ν.) We refer the reader to [4] for an excellent account of the basic facts about the Cauchy transform of a measure.

A complex measure ν annihilates a continuous function f if $\int f \mathrm{d}\nu = 0$. We now prove our fundamental lemma on complex measures that annihilate the disc algebra, those continuous functions on $\bar{D}$ that are analytic in D. Recall that m denotes normalized Lebesgue measure on ∂D.

LEMMA 1. *Let $f \in L^1(m)$ and let ν be a complex measure supported by $\bar{D}$. Suppose $z f \mathrm{d}m + \nu$ annihilates the disc algebra. If Γ is an open subarc of ∂D such that* $(\operatorname{spt} \nu) \cap \Gamma = \emptyset$ *then for almost every θ with $\mathrm{e}^{\mathrm{i}\theta}$ in Γ,*

$$\lim_{r \to 1^-} (z f \mathrm{d}m + \nu)^\wedge \, (r\mathrm{e}^{\mathrm{i}\theta}) = f(\mathrm{e}^{\mathrm{i}\theta}).$$

Proof. For each $\lambda \in \mathbf{C} \setminus \bar{D}$ the function $(z - \lambda)^{-1}$ is in the disc algebra and thus is annihilated by $f \mathrm{d}z + \nu$. Hence $(f \mathrm{d}z + \nu)^\wedge$ vanishes identically outside $\bar{D}$. Let $F(\lambda) = (f \mathrm{d}z)^\wedge(\lambda)$. Then $F(\lambda) + \hat{\nu}(\lambda)$ vanishes outside $\bar{D}$ and in particular

$$\lim_{r \to 1^+} (F(r\mathrm{e}^{\mathrm{i}\theta})) + \hat{\nu}(r\mathrm{e}^{\mathrm{i}\theta})) = 0 \tag{1}$$

for each $e^{i\theta}$ in Γ. Since $\hat{\nu}$ is analytic off spt ν and hence in a neighborhood of Γ, it follows that

(2) $$\lim_{r\to 1} \hat{\nu}(re^{i\theta}) = \hat{\nu}(e^{i\theta})$$

for each $e^{i\theta} \in \Gamma$. Combining this last fact with (1), we obtain

(3) $$\lim_{r\to 1+} F(re^{i\theta}) = -\hat{\nu}(e^{i\theta}) \text{ for each } e^{i\theta} \in \Gamma.$$

We now employ an argument from Duren [10], p. 39. For $r \in (0,1)$

$$F(re^{i\theta}) - F\left(\frac{1}{r}e^{i\theta}\right) = \int \frac{zf\mathrm{d}m}{z - re^{i\theta}} - \int \frac{zf\mathrm{d}m}{z - r^{-1}e^{i\theta}} = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\, f(e^{it})\,\mathrm{d}t,$$

where $P(r,\theta) = (1 - r^2)/(1 - 2r\cos\theta + r^2)$ is the Poisson Kernel. Letting $r \to 1^-$, we obtain

(4) $$\lim_{r\to 1^-} \left(F(re^{i\theta}) - F\left(\frac{1}{r}e^{i\theta}\right)\right) = f(e^{i\theta})$$

for almost all θ. Applying (2), (3) and (4), we obtain

$$\lim_{r\to 1^-} (F(re^{i\theta}) + \hat{\nu}(re^{i\theta})) = f(e^{i\theta})$$

for almost all θ with $e^{i\theta}$ in Γ.

We now introduce a couple of more concepts from measure theory. Let W be a Dirichlet region, that is, a region for which the Dirichlet problem is solvable. For each complex measure ν with spt $\nu \subset \overline{W}$, there exists a unique complex measure, ν_s on ∂W, called the sweep of ν to ∂W, such that

$$\int u \mathrm{d}\nu_s = \int u \mathrm{d}\nu$$

for each function u that is continuous on $\overline{W}$ and harmonic in W. The existence and uniqueness of ν_s are guaranteed by the Riesz representation theorem, considering ν as a linear functional on the continuous functions on ∂W. Observe that if $\tau \leqslant \nu$, then $\tau_s \leqslant \nu_s$. (Here $\tau \leqslant \nu$ means $\tau(B) \leqslant \nu(B)$ for every Borel set B.) For a complex measure ν, the inequality $|\nu_s| \leqslant |\nu|_s$ holds. Also observe if $\nu \geqslant 0$ then $|\nu| = |\nu_s|$. Furthermore, sweeping a measure does not change its boundary values, i.e.,

$$\nu_s = (\nu_W|)_s + \nu|_{\partial W}.$$

Finally, if ν on $\overline{W}$ represents evaluation of the polynomials at a point in W, then so does ν_s.

A measure τ carried by D is a Carleson measure if there exists a constant $M > 0$ such that $\tau(R) \leqslant Mh$ for every set R of the form

$$R = \{re^{i\theta}: 1 - h \leqslant r < 1;\ \theta_0 \leqslant \theta \leqslant \theta_0 + h\}.$$

We shall refer to sets of the form of R as Carleson rectangles and to M as a Carleson constant. (See Duren [10], pp. 156—158.)

LEMMA 2. *If a measure* ν *with* spt $\nu \subset \bar{D}$ *represents evaluation at zero for the polynomials, then* $\nu|_D$ *is a Carleson measure and the Carleson constant can be chosen to be five.*

Proof. For each polynomial p, the function $|p|^2$ is subharmonic and hence

$$\int |p|^2 d\nu|_D \leqslant \int |p|^2\, d(\nu|_D)_s + \int |p|^2\, d\nu|_{\partial D} = \int |p|^2\, dm. \tag{5}$$

The equality follows from the uniqueness of representing measures on ∂D for evaluation at zero and the fact that $(\nu|_D)_s + \nu|_{\partial D}$ also represents evaluation at zero. It is well-known that (5) implies that $\nu|_D$ is a Carleson measure and the proof in [10], pp. 157—158 shows that the Carleson constant can be chosen to be five.

REMARK. The conclusion of the lemma also holds for a measure carried by D such that its sweep to ∂D is less than or equal to m.

A preprint version of this paper contained a proof of the following lemma. The authors wish to thank their colleague J. Ball for pointing out the fact that this lemma is a special case (by applying the reflection principle to the function f+g) of Lemma 6.6 in [19], p. 223.

LEMMA 3. *Let* Γ *be an open subarc of* ∂D. *Let* $f \in H^2(m)$ *and let* g *be analytic in a neighborhood of* Γ. *If* $\lim_{r \to 1^-} (f+g)(re^{i\theta})$ *is real for almost every* $e^{i\theta} \in \Gamma$, *then* $f+g$ *can be extended analytically across* Γ.

Let K be a compact subset of $\bar{D}$. Let U be a component of $D \setminus K$. Then $a \in \partial U \cap \cap \partial D$ is called a *strong boundary point of* U if for each $\alpha \in (0, \pi)$, there exists an isosceles triangle $T_{a\alpha}$ such that

(i) a is a vertex of $T_{a\alpha}$;

(ii) int $T_{a\alpha} \subset U$;

(iii) the interior angle of $T_{a\alpha}$ at a has measure α;

(iv) the radial line segment from zero to a bisects the interior angle at a.

Referring back to Remark c of the introduction, we note that almost every (m) point of C is a strong boundary point of int J.

LEMMA 4. *The set of strong boundary points of a component* U *of* $D \setminus K$ *is a Borel set.*

Proof. Let B_{mn} be the set of points a in $\partial U \cap \partial D$ for which there exists a triangle $T_{a,(\pi-1/m)}$ satisfying (i) through (iv) above such that the two sides meeting at a can be chosen to be of length n^{-1}. First observe that B_{mn} is compact. The conclusion then follows from the fact that the strong boundary equals $\bigcap_m \bigcup_n B_{mn}$.

Let N be the normal operator M_z on $L^2(\mu)$ for some measure μ. We assume $P^\infty(\mu) = H^\infty(D)$. Since $H^\infty(D)$ is antisymmetric, the measure $\mu|_{\partial D}$ is absolutely continuous with respect to m. Noting that μ is determined only up to mutual absolute continuity, we assume that there exists a measurable set $E \subset \partial D$ such that $\mu|_E = m|_E$ and $\mu(\partial D \setminus E) = 0$.

We are ready to prove Theorem 1. The reader should note that the statement of this theorem differs (in appearance) from that stated in the introduction. The difference is reconciled by Lemma 5, which shows that the concepts of a component of $D \setminus \operatorname{spt} \mu$ that is outer and one that has a strong boundary of positive m measure are equivalent. The reason for proceeding in this way will become apparent in the proof.

THEOREM 1. (The notation here is the same as that of the two preceding paragraphs.) *Let U be a component of $D \setminus \operatorname{spt} \mu$. If the strong boundary of U has positive Lebesgue measure, then* $\mathrm{i}(S - \lambda) = -1$ *for each $S \in \mathscr{S}_p(N)$ and each λ in U.*

Proof. We can assume $0 \in U$. [If not let $\lambda \in U$ and let $f(z) = (z - \lambda)(1 - \bar{\lambda} z)^{-1}$. Observing that f is analytic and one-to-one on a neighborhood of the unit disc and maps the disc onto the disc, we see that the index theorems for N and $\mathscr{S}(N)$ are identical with those for $f(N)$ and $\mathscr{S}_p(f(N))$. This requires some straightforward calculations. For example, strong boundary goes to strong boundary, $f(N)$ has a cyclic vector, etc., but all are easy computations.]

Our first step is the construction of some Jordan regions contained in U. Let B denote the strong boundary of U. For each $b \in B$ let T_b denote the boundary of the triangle given by the definition, where the angle α is $\pi - (\pi/10)$. For each $n \in Z^+$, let B_n be the set of those points b in B such that the length of each of the sides of T_b is at least n^{-1}. Since $B = \cup B_n$, the set B_n has positive measure if n is sufficiently large. Fixing such an n, choose a perfect subset A of B_n with $m(A) > 0$. By shortening the sides of each T_b, $b \in A$, we may assume that they are all congruent. Let β be the distance from the vertex b to the opposite side of T_b. Let Γ be the subarc of ∂D with endpoints in A such that $(\partial D \setminus \Gamma) \cap A = \emptyset$. We assume that Γ has length less than the minimum of $\pi/100$ and $\beta/10$. We further assume that Γ is so short that the corresponding sides of T_b's (under the natural congruence) do not intersect.

Let $G = \bigcup_{b \in A} \operatorname{int} T_b$. The set G is clearly open and connected. Looking at only the triangles corresponding to the endpoints of Γ, we see that

$$G \supset \left\{ r e^{i\theta} : e^{i\theta} \in \Gamma,\ 1 - \frac{\beta}{2} < r \leqslant 1 - \frac{\beta}{20} \right\}.$$

We shall be interested in the part of ∂G near Γ. Let $V_1 = \partial G \cap W$ where

$$W = \left\{re^{i\theta}: e^{i\theta} \in \Gamma,\ 1 - \frac{\beta}{20} < r \leqslant 1\right\}.$$

Observe that $V_1 \subset \bigcup_{b \in A} T_b$. This follows because the set A is closed and the T_b's are congruent.

Let R_b and S_b denote the sides of T_b meeting at b. Fix $b \in A$. Either b is an endpoint of a component of $\partial D \setminus A$ or there exists a sequence of points in A converging to b from both sides of b. In the latter case, $W \cap R_b \subset G \cup \{b\}$ and $W \cap S_b \subset G \cup \{b\}$. If b is an endpoint of Γ, then $W \cap R_b \subset V_1$ and $W \cap S_b \subset G \cup \{b\}$, or vice versa. If b is an endpoint of a component of $\Gamma \setminus A$, let a denote the other endpoint. Suppose R_a and S_b are the sides that meet and let p_{ab} denote the point of intersection. By the geometry $W \cap S_a \subset G \cup \{a\}$ and $W \cap R_b \subset G \cup \{b\}$. The line segments $[a, p_{ab}]$ and $[b, p_{ab}]$ are clearly contained in V_1 and the remaining points of $W \cap R_a$ and $W \cap S_b$ are contained in int T_b and int T_a, respectively. For each $e^{i\theta}$ in the shorter arc of ∂D joining a and b, it now follows that

$$V_1 \cap [0, e^{i\theta}] \subset [a, p_{ab}] \cup [b, p_{ab}].$$

(Here $[0, e^{i\theta}]$ denotes the straight line segment from 0 to $e^{i\theta}$.) Thus V_1 is the union of A and a countable collection of line segments whose lengths sum up to less than $2m(\Gamma \setminus A)$. Using this observation and the geometry, we see that V_1 is a rectifiable Jordan arc of length less than $2m(\Gamma)$.

We now construct an additional rectifiable Jordan arc joining the endpoints of Γ. Let M_i denote the i^{th} component of $\Gamma \setminus A$. Let N_i denote the union of the two line segments of V_1 that connect the endpoints of M_i. Let $P_i(Q_i)$ denote the union of the two line segments in G that connect the endpoints of M_i and that each make an angle of $\pi/10$ $(\pi/5)$ with the corresponding line segment of N_i. Let $V_2 = A \cup (\cup P_i)$. (The reason for constructing the Q_i's will be seen later.) By construction V_2 is a rectifiable Jordan arc of length less than $2m(\Gamma)$.

For λ on P_i, we shall be interested in estimating the distances to N_i and Q_i. By the geometry there exist constants δ_1 and δ_2, independent of i, such that for λ on P_i,

(6) $$d(\lambda, N_i) \geqslant \delta_1(1 - |\lambda|) \quad \text{and}$$

(7) $$d(\lambda, Q_i) \geqslant \delta_2(1 - |\lambda|).$$

We shall now construct a Jordan curve containing V_2. Let a and b denote the endpoints of Γ. Since U is open and connected and contains int $T_a \cup$ int T_b, there exists a polygonal Jordan path J_0 joining a and b such that

$$J_0 \setminus \{a, b\} \subset U \tag{8}$$

$$J_0 \cap V_2 = \{a, b\} \tag{9}$$

$$J_0 \cap Q_i = \emptyset \quad \text{for all } i \tag{10}$$

$$J_0 \cup V_2 \text{ is a Jordan curve} \tag{11}$$

$$0 \in \operatorname{int} J_0 \cup V_2. \tag{12}$$

Furthermore we may assume that the line segments of J_0 that contain a and b are contained in $[0, a) \cup [0, b]$. Let $J = J_0 \cup V_2$ and let $H = \operatorname{int} J$.

We shall be interested in approximating H by a simpler regions. Let

$$J_n = J_0 \cup A \cup (\bigcup_{i>n} M_i) \cup \left(\bigcup_{i=1}^{n} P_i\right)$$

and let H_n denote the interior of the rectifiable Jordan curve J_n. This finishes our topological constructions.

Fix $S \in \mathcal{S}_p(M_z)$, and denote the space on which S acts by $\mathcal{H}$. Hence $M_z \mathcal{H} \subset \mathcal{H}$. (Recall that we have normalized the component U that has a strong boundary of positive Lebesgue measure so that $0 \in U$.) We shall first establish that $\mathrm{i}(S) \geqslant -1$, by contradiction.

Suppose $\mathrm{i}(S) \leqslant -2$. Then $\dim \ker S^* \geqslant 2$. Thus there exist K_1 and K_2 in $\mathcal{H} \ominus S\mathcal{H}$ such that $\|K_1\| = \|K_2\| = 1$ and $K_1 \perp K_2$. It follows that

$$\int z^n K_1 \bar{K}_2 \mathrm{d}\mu = 0 \quad \text{for } n = 0, 1, 2, \dots$$

$$\int z^n \bar{K}_1 K_2 \mathrm{d}\mu = 0 \quad \text{for } n = 0, 1, 2, \dots$$

$$\int z^n |K_1|^2 \mathrm{d}\mu = 0 \quad \text{for } n = 1, 2, \dots$$

$$\int z^n |K_2|^2 \mathrm{d}\mu = 0 \quad \text{for } n = 1, 2, \dots .$$

With these equalities one sees that for $i = 1, 2$, the measure $|K_i|^2 \mathrm{d}\mu$ represents evaluation at zero for the polynomials. Thus, by Lemma 2, the measure $|K_i|^2 \mathrm{d}\mu|_D$ is a Carleson measure with Carleson constant five. Since

$$|K_1 \bar{K}_2| = |\bar{K}_1 K_2| \leqslant 1/2(|K_1|^2 + |K_2|^2),$$

the Carleson constant of five also works for $K_1\bar{K}_2 d\mu|_D$ and $\bar{K}_1 K_2 d\mu|_D$. Our next goal is to construct one bounded analytic function in H such that its product with each member of a natural collection of Cauchy transforms is bounded in H. Let ν be a complex measure supported on $\bar{D}$ such that $|\nu|(H) = 0$. Let $\{G_n\}$ be any decreasing sequence of open sets such that $\nu(G_n) < n^{-1}$ and each G_n contains some tail end of the open arcs M_i on the unit circle. Define ν_n to be the complex measure obtained by adding $\nu|_{\bar{D}\setminus(G_n\cup H_n)}$ and the sweep of $\nu|_{G_n\cup H_n}$ to ∂D. Note that $\|\nu_n - \nu\| \to 0$ as $n \to \infty$.

We now proceed to estimating Cauchy transforms on J. Let R_i be the region bounded by M_i and Q_i, and let $|M_i|$ equal the arc length of M_i. Note that R_i is contained in the Carleson rectangle that has M_i as one of its sides. For each i, sweep $(|K_1|^2 d\mu)_n|_{D\setminus R_i}$ to $\partial(D/R_i)$ and add on $(|K_1|^2 d\mu)_n|_{R_i\cup\partial D}$, calling the sum $(|K_1|^2 d\mu)_{R_{in}}$. This last measure represents evaluation at zero, so Lemma 2 applies to it. Thus

$$\int_{R_i\cup Q_i} (|K_1|^2 d\mu)_{R_{in}} \leqslant 5|M_i|. \tag{13}$$

The sweep of $(|K_1|^2 d\mu)_{R_{in}}$ to ∂D is m. Thus $dm - (|K_1|^2 d\mu)_{R_{in}}$, and hence $z dm - z(|K_1|^2 d\mu)_{R_{in}}$, has a total variation on ∂D of less than or equal to $5|M_i|$. For λ on P_i, the function $z(z-\lambda)^{-1}$ is analytic in $D\setminus R_i$ and hence

$$|[z(|K_1|d\mu)_n]^\wedge(\lambda)| = \left|\int_{\bar{D}} \frac{z}{z-\lambda} (|K_1|^2 d\mu)_n\right| = \left|\int_{\bar{D}} \frac{z}{z-\lambda} (|K_1|^2 d\mu)_{R_{in}}\right| =$$

$$= \left|\int \frac{z}{z-\lambda}\, dm - \int \frac{z}{z-\lambda}\, dm + \int_{\partial D} \frac{z}{z-\lambda} (|K_1|^2 d\mu)_{R_{in}} + \right.$$

$$\left. + \int_{Q_i} \frac{z}{z-\lambda} (|K_1|^2 d\mu)_{R_{in}} + \int_{R_i} \frac{z}{z-\lambda} (|K_1|^2 d\mu)_{R_{in}}\right|.$$

Using Cauchy's formula on the first term, our previous estimate on the norm of $z dm - z(|K_1|^2 d\mu)_{R_{in}}$ on ∂D for the second and third terms, and inequalities (6), (7) and (13) on the fourth and fifth terms, we have for λ on P_i the following estimate:

$$|[z(|K_1|^2 d\mu)_n]^\wedge(\lambda)| \leqslant 1 + \frac{5|M_i|}{1-|\lambda|} + \frac{5|M_i|}{\delta_2(1-|\lambda|)} + \frac{5|M_i|}{\delta_1(1-|\lambda|)}.$$

(Note: for the fifth term the only measure in R_i is supported between N_i and M_i.) Letting $\delta_3 = \min\{\delta_1, \delta_2\}$ and noting $|M_i| \geqslant 1 - |\lambda|$, we see the last sum is bounded by $16|M_i|[\delta_3(1-|\lambda|)]^{-1}$. By similar methods one obtains the same estimate for the Cauchy transforms of $z(|K_2|^2 d\mu)_n$, $(K_1\bar{K}_2 d\mu)_n$, and $(\bar{K}_1 K_2 d\mu)_n$ on P_i.

Let m_2 denote harmonic measure on J and define a function u on J by

$$u(\lambda) = \begin{cases} -\log \dfrac{|M_i|}{(1-|\lambda|)} & \lambda \in P_i \\ 0 & \lambda \in J \setminus (\cup P_i). \end{cases}$$

We shall show that $u \in L^1(m_2)$.

Let ℓ denote arc length measure on J and let P_{i1} and P_{i2} be the two line segments making up P_i. Since $m_2|_D$ is a Carleson measure with Carleson constant five, $m_2(\alpha) \leqslant 5\ell(\alpha)$ for each line segment α contained in P_{i1} with one endpoint on ∂D. Because $1 - |\lambda|$ is monotonically decreasing on P_{i1}, it now follows that

$$\int_{P_{i1}} |u| \mathrm{d}m_2 \leqslant 5 \int_{P_{i1}} |u| \, \mathrm{d}\ell = 5 \int_{P_{i1}} [\log|M_i| - \log(1 - |\lambda|)] \mathrm{d}\ell.$$

If $\sigma = \ell(P_{i1})$, then there exist β and γ with $|\beta| = |\gamma| = 1$, such that $r(t) = \beta + \gamma t$, $0 \leqslant t \leqslant \sigma$, is a parametrization of P_{i1}. Since P_{i1} makes an angle of $3\pi/20$ with ∂D, there exists a constant $\delta_4 > 1$ (independent of i) such that $t \leqslant \delta_4(1 - |r(t)|)$ for $0 \leqslant t \leqslant \sigma$. Using

$$0 < \log [1 - |r(t)|]^{-1} \leqslant \log \delta_4 t^{-1},$$

we obtain

$$\int_{P_{i1}} [\log|M_i| - \log(1 - |\lambda|)] \mathrm{d}\ell \leqslant \sigma \log |M_i| + \int_0^\sigma \log \frac{\delta_4}{t} \, \mathrm{d}t \leqslant$$

$$\leqslant \sigma \log|M_i| + |M_i| \log \delta_4 + \sigma - \sigma \log \sigma \leqslant |M_i| (1 + \log \delta_4).$$

For the last inequality, note that the inequality $x \geqslant 1 + \log x$ for $x \geqslant 1$ is used. A similar computation works for P_{i2}, so

$$\int_{P_i} |u| \mathrm{d}m_2 \leqslant 10(1 + \delta_4)|M_i|.$$

Thus

$$\int |u| \mathrm{d}m_2 = \sum \int_{P_i} |u| \mathrm{d}m_2 + \int_{J \setminus (\cup P_i)} |u| \mathrm{d}m_2 \leqslant 10(1 + \delta_4) \sum |M_i| \leqslant 10(1 + \delta_4) \mathrm{m}(\Gamma).$$

Hence $u \in L^1(m_2)$.

Let f be the Riemann map of D onto H. It is well known that since J is a Jordan curve, f has a homeomorphic extension to $\bar{D}$. Moreover, because J is also rectifiable, f is conformal at almost every point of ∂D ([10], p. 45).

The function $u \circ f \in L^1(m)$, [9], and hence has a harmonic extension to D. The last function is the real part of an analytic function g in D such that $g(0)$ is real. Let $h_1 = e^g \circ f^{-1}$. Then h_1 is a bounded analytic function in H and $|h_1(z)| = e^{u(z)}$ for almost all $z \in J$. (Here we use the results from [9] again.)

We shall now modify h_1 slightly to ensure uniform bounds on suitable Cauchy transforms near the endpoints a and b of the path J_0. Recall that J_0 is a polygonal path joining a and b, such that the line segments of J_0 that contain a and b are contained in $[0, a] \cup [0, b]$. There exists a constant $\delta_5 > 0$ such that for λ on the line segment of J_0 containing a, we have $d(\lambda, a) \leqslant \delta_5 d(\lambda, \operatorname{spt} \mu)$ and for λ on the line segment of J_0 containing b, we have $d(\lambda, b) \leqslant \delta_5 d(\lambda, \operatorname{spt} \mu)$.

Let $h = (z - a)(z - b)h_1$. For λ on the line segment of J_0 containing a, we now have

$$|h(\lambda)[z(|K_1|^2 d\mu)_n]^\wedge(\lambda)| \leqslant \frac{|\lambda - a|\,|\lambda - b|\,|h_1(\lambda)| \int (|K_1|^2 d\mu)_n}{d(\lambda, \operatorname{spt} \mu)} \leqslant 2\delta_5.$$

The same estimates hold for λ on the line segment of J_0 containing b. The sequence of function $\{[z(|K_1|^2 d\mu)_n]^\wedge\}$ is uniformly bounded on the remaining segments of J_0, since these segments comprise a compact subset of U and our measures are each of total variation norm less than or equal to one.

We now look at $h[z(|K_1|^2 d\mu)_n]^\wedge$ on $J \setminus J_0$ (which equals V_2). By the construction of h and the estimates of the Cauchy transforms indicated, we see that the sequence of functions $\{h[z(|K_1|^2 d\mu)_n]^\wedge\}$ is uniformly bounded on the P_i's.

We consider $[z(|K_1|^2 d\mu)_n]^\wedge$. Let f_n denote the Radon-Nikodym derivative of $(|K_1|^2 d\mu)_n|_{\partial D}$ with respect to Lebesgue measure. Since the sweep of $(|K_1|^2 d\mu)_n$ to ∂D is m, it follows that $\|f_n\|_\infty \leqslant 1$. In particular, $f_n \in L^2(m)$ and there exists a function in $H^2(m)$ whose Cauchy transform equals $(zf_n dm)^\wedge$ in D. By Lemma 1,

$$\lim_{r \to 1^-} [z(|K_1|^2 d\mu)_n]^\wedge (re^{i\theta}) = f_n(e^{i\theta}) \tag{14}$$

for almost every $e^{i\theta}$ in $A \cup (\bigcup_{i>n} M_i)$. Since f_n is real-valued we can apply Lemma 3 to obtain an extension of $[z(|K_1|^2 d\mu)_n]^\wedge|_D$ that is analytic across $A \cup (\bigcup_{i>n} M_i)$. Since $[z(|K_1|^2 d\mu)_n]^\wedge|_D$ is analytic across the P_i's, it follows that $[z(|K_1|^2 d\mu)_n]^\wedge|_H$ has an extension g_n that is analytic across ∂H. In particular, it is continuous on $\overline{H}$. Hence $g_n \circ f$ is in the disc algebra. (Remember $f\colon \overline{D} \to \overline{H}$ is the extended Riemann map.) Note that $g_n(e^{i\theta}) = f_n(e^{i\theta})$ for almost every $e^{i\theta} \in A$. Hence, by our previous estimates, $\{hg_n\}$ is a uniformly bounded sequence of analytic functions in H. Thus $\{(h \circ f)(g_n \circ f)\}$ is a uniformly bounded sequence of functions in H^∞. By construction of $(|K_1|^2 d\mu)_n$, it follows that $f_n|_A \to |K_1|^2|_A$ in $L^1(m|_A)$. By passing to a subsequence, if need be, we may assume the convergence is almost everywhere on A. By passing to a subsubsequence, we may also assume that $(h \circ f)(g_n \circ f)$ converges weak-star to a function G_1 in the Hardy space H^∞. Observe by our construction $G_1 \circ f^{-1}(e^{i\theta})$ equals $h(e^{i\theta})|K_1|^2(e^{i\theta})$ for almost every $e^{i\theta} \in A$.

Furthermore, for ζ in H we have

$$G_1 \circ f^{-1}(\zeta) = (z|K_1|^2 d\mu)^\wedge(\zeta)h(\zeta).$$

We construct a function G_2 in H^∞ in a similar manner corresponding to the measure $|K_2|^2 d\mu$.

We can (and do) construct $G_3 \in H^\infty$ from the measure $K_1\bar{K}_2 d\mu$ because of the following observations. Since $K_1\bar{K}_2 d\mu$ and $\bar{K}_1K_2 d\mu$ are annihilating measures of the disc algebra, it follows that $z\mathrm{Re}(K_1\bar{K}_2)d\mu$ and $z\mathrm{Im}(K_1\bar{K}_2)d\mu$ are annihilating measures. Secondly,

$$2|\mathrm{Re}\, K_1\bar{K}_2| \leqslant |K_1|^2 + |K_2|^2$$

and

$$2|\mathrm{Im}\, K_1\bar{K}_2| \leqslant |K_1|^2 + |K_2|^2$$

imply the total variations of all sweeps of $(\mathrm{Re}\, K_1\bar{K}_2)d\mu$ and $(\mathrm{Im}\, K_1\bar{K}_2)d\mu$ are less than or equal to m on ∂D. Constructing G's for the appropriate real and imaginary parts of $K_1\bar{K}_2$, we obtain a $G_3 \in H^\infty$ that satisfies

$$G_3 \circ f^{-1}(e^{i\theta}) = (K_1\bar{K}_2)(e^{i\theta})\, h(e^{i\theta})$$

for almost every $e^{i\theta} \in A$ and

$$G_3 \circ f^{-1}(\zeta) = (zK_1\bar{K}_2 d\mu)^\wedge(\zeta)\, h(\zeta)$$

for all $\zeta \in H$. We now construct a G_4 with the analogous properties for the measure $\bar{K}_1K_2 d\mu$.

Claim: $G_1G_2 = G_3G_4$. To see this, note that G_1G_2 and G_3G_4 are H^∞ functions that are equal almost everywhere on $f^{-1}(A)$, a set of positive Lebesgue measure. (Here we have used the rectifiability of ∂H and [10], p. 45.) Hence the functions must be equal almost everywhere on ∂D.

This yields the desired contradiction: For $G_1(f^{-1}(0)) = G_2(f^{-1}(0)) = h(0)$ and $G_3(f^{-1}(0)) = G_4(f^{-1}(0)) = 0$. Hence, if U is a component of $D \setminus \mathrm{spt}\, \mu$ and the strong boundary of U has positive Lebesgue measure, then $\mathrm{i}(S - \lambda) \geqslant -1$ for each $S \in \mathscr{S}_p(N)$ and each $\lambda \in U$.

We still need to show $\mathrm{i}(S - \lambda) \leqslant -1$ for $\lambda \in U$ (a component of $D \setminus \mathrm{spt}\, \mu$ with strong boundary of positive Lebesgue measure). We will demonstrate this by showing $\sigma(S) \supset U$. Before we do, we prove that Theorem 1, as stated in the preliminaries, is equivalent to the theorem under consideration.

LEMMA 5. *U is an outer component of $D\setminus\operatorname{spt}\mu$ if and only if the strong boundary of U has positive (Lebesgue) measure.*

Proof of Lemma 5. By definition of an outer component for $D\setminus\operatorname{spt}\mu$, there exists a $\lambda\in U$ and $f\in H^\infty$ such that $|f(\lambda)| > \|f\|_{\mu|_D}$. Let $\{f_n\}$ be a sequence of polynomials converging to f in $L^2(m)$. By passing to a subsequence, we may assume they converge almost everywhere (m).

Let τ_λ denote the harmonic measure on ∂U for evaluation at λ. By a simple modification of Lemma 2, we see that $\tau_\lambda|_D$ is a Carleson measure; therefore, $f_n \to f|_D$ in $L^2(\tau_\lambda|_D)$. The fact that the sweep of τ_λ to ∂D is $P_\lambda m$ (where $P_\lambda(e^{it}) = \operatorname{Re}[(1+\lambda e^{-it})(1-\lambda e^{-it})^{-1}]$) and the fact that this sweep leaves $\tau_\lambda|_{\partial D}$ fixed give rise to the conclusion that $\tau_\lambda|_{\partial D}$ is boundedly absolutely continuous with respect to $m|_{\partial D\cap\partial U}$. Hence $f_n\to f$ in $L^2(\tau_\lambda)$. Therefore $f\in H^2(\tau_\lambda)$, the $L^2(\tau_\lambda)$ closure of the polynomials, and hence

$$f(\lambda)=\int f\,\mathrm{d}\tau_\lambda.$$

We may assume f is not a constant (otherwise, $U = D$ and the result is clear) and therefore

$$\|f\|_{\tau_\lambda} > |f(\lambda)|.$$

Now

$$\|f\|_{\tau_{\lambda|D}} \leqslant \|f\|_{\mu|D}$$

because the carrier of $\tau_\lambda|_D$ is contained in the support of $\mu|_D$. Thus

$$\|f\|_{\tau_{\lambda|\partial D}} = \|f\|_{\tau_\lambda} > |f(\lambda)| > \|f\|_{\mu|D}.$$

It follows by these inequalities that there exists a set $E\subset\partial D\cap\partial U$ of positive τ_λ measure (hence of positive m measure) such that

$$|f(z)| > |f(\lambda)| \tag{15}$$

for all $z\in E$. We may assume that f has a nontangential limit at each point of E.

It follows from the above that we can find a perfect set $P\subset E$ of positive m measure, a positive integer n, and an angle $\alpha\in(0,\pi)$ such that for every $a\in P$, we have $T_{a\alpha}\cap\operatorname{spt}\mu\subset\{a\}$. (Here, $T_{a\alpha}$ is the isosceles triangle with a as a vertex, altitude of length $1/n$, such that the radial line segment from zero to a bisects the interior angle at a.) Hence, we can find a component W of $D\setminus\operatorname{spt}\mu$ such that B (= strong boundary of W) $\cap P$ has positive m measure. The first half of the proof will be finished if we show $W\cap U\neq\emptyset$.

Pick five distinct points a_i, $i = 1, 2, 3, 4, 5$, of B so close together so that

$$\bigcap_{1}^{5} T_{a_i, \alpha} \neq \emptyset.$$

The boundaries of these triangles and the boundary of the circle determine four regions, each of which has a vacuous intersection with the triangles. Because each of the points $a_i \in \partial U$, two of these regions must contain points of U. Since U is polygonally connected it follows that $U \cap W \neq \emptyset$. This finishes one-half of the lemma.

Now assume U is a component of $D \setminus \operatorname{spt} \mu$ with the property that its strong boundary has positive Lebesgue measure. We assume the topological construction found in the first half of the proof of Theorem 1 has been made. We modify this construction slightly for its application here (and for the other half of the proof o Theorem 1). By shortening Γ, we may assume that $\Gamma \cap A$ has two isolated points, the endpoints of Γ. (Now $A = \Gamma \cap A$, so A is no longer perfect.) Fix an i, and let a and b be the endpoints of M_i and p_{ab} as before. So

$$[a, p_{ab}] \cup [p_{ab}, b] = N_i.$$

Let C_i be the intersection of $\bar{D}$ and the circle passing through the points a, p_{ab} and b. Let

$$E = A \cup (\partial D \setminus \Gamma) \cup (\cup C_i).$$

Now E is a rectifiable Jordan curve and we define a function u on E via

$$u(z) = \begin{cases} 0 & z \in (\partial D \setminus \Gamma) \cup (\cup C_i) \\ 1 & z \in A \end{cases} .$$

(This function is well-defined off a countable subset of E). This function u has a harmonic extension to int E. Let v be the harmonic conjugate of u and consider the function $f = e^{u+iv}$. This function extends by the reflection principle to be analytic in D.

Let c and d be the endpoints of Γ. Let Y be the closed region bounded by $[(1-\beta/5)c, c]$, $[(1-\beta/5)d, d]$, $\{(1-\beta/5)e^{i\theta}\colon e^{i\theta} \in \Gamma\}$, and Γ. By the maximum principle, there exists $r < 1$ such that $u|_{\overline{D \setminus Y}} \leqslant r$. Since $u < 0$ on $(\operatorname{spt} \mu) \cap (\operatorname{int} Y)$, it now follows that $u|_{D \cap \operatorname{spt} \mu} \leqslant r$. Choose a point z_0 in D such that $u(z_0) > r$. Then

$$|f(z_0)| > \|f\|_{\mu|_D}.$$

REMARK. f belongs to H^∞, in fact $\|f\|_\infty = e$.

We now proceed to finish the proof of Theorem 1. It suffices to show $\sigma(S) \supset U$ for every $S \in \mathscr{S}_p(N)$. Suppose, to the contrary, there exists a pure subnormal S acting on $\mathscr{H}$ that has M_z on $L^2(\mu)$ as its minimal normal extension and $\sigma(S) \cap U = \emptyset$. (Here we are using the Bram-Halmos result about "hole filling", [3]). We assume that the topological construction in the last half of Lemma 5 has been made and that we have obtained the function f that showed U was an outer component. Since $f \in P^\infty(\mu) = H^\infty(D)$ and f is not constant, it follows from Corollary 6.4 of [8] that $f(S)$ is pure. Furthermore, by Theorem 6.1 of the reference cited, $f(N)$ is the minimal normal extension of $f(S)$.

We now apply the spectral mapping theorem ([8], Theorem 8.11) along with the properties of f to see that the spectrum of $f(S)$ is the union of a nonempty closed subset of the circle of radius e with a compact subset of the unit disc. Clearly this cannot be the spectrum of a pure subnormal operator, [7], hence $\sigma(S) \supset U$. This finishes the proof of Theorem 1.

REMARK. We now present a direct and shorter proof that $\sigma(S)$ contains every outer component of $D \setminus \operatorname{spt} \mu$. This proof depends upon the work of J. Chaumat [6].

Proof. Let $K = \sigma(S)$ and let $R^\infty(\mu, K)$ denote the weak-star closure in $L^\infty(\mu)$ of $R(K)$, the algebra of rational functions with poles off K. Observe that the purity of S implies that $R^\infty(\mu, K)$ has no L^∞ summand. Let $E(\mu, K)$ denote the set of those z in K for which there exists a measure $\nu_z << \mu$ with $\nu_z(\{z\}) = 0$ such that

$$g(z) = \int f \, d\nu_z$$

for each g in $R(K)$. Note that

$$E(\mu, K) \cap \partial D = \emptyset.$$

Let $\lambda_{E(\mu, K)}$ denote area measure on $E(\mu, K)$.

Now define a map

$$T_\mu \colon R^\infty(\mu, K) \to R^\infty(\lambda_{E(\mu, K)}, K)$$

by

$$(T_\mu h)(z) = \int h \, d\nu_z \quad \text{for } z \in E(\mu, K).$$

Since $R^\infty(\mu, K)$ has no L^∞ summand, a theorem of Chaumat ([6], Thm. 4, Chapt. IX) says that T_μ is an isometric isomorphism and a weak-star homeomorphism. (The homeomorphism property of the map T_μ follows easily from the Kreĭn-Smulian theorem.) The map T_μ sends a polynomial to itself, so $T_\mu f = f$ for each f in $P^\infty(\mu)$ by the weak-star continuity of T_μ. Thus

$$\|f\|_\mu = \|f\|_{E(\mu, K)}.$$

Since $E(\mu, K) \subset \sigma(S) \cap D$, it now follows from the definition of outer component that $\sigma(S)$ contains all of them.

Before we continue with our results on index theory we present an interesting consequence to the proof of Theorem 1. In order to do this, let us first summarize the key idea of this proof.

LEMMA 6. *Suppose* $P^\infty(\mu) = H^\infty$, $g \in L^1(\mu)$, $zg\mathrm{d}\mu$ *annihilates the disc algebra, and* $|g|\mathrm{d}\mu$ *is a Carleson measure. If* U *is an outer component of* $D\setminus\operatorname{spt}\mu$, *then for almost every strong boundary point* $e^{i\theta}$ *of* U,

$$\lim_{r\to 1^-} (zg\,\mathrm{d}\mu)^\wedge (r\,e^{i\theta}) = g(e^{i\theta})\frac{\mathrm{d}\mu}{dm}(e^{i\theta}).$$

Proof. Let A be a Borel subset of the strong boundary of U with $m(A)>0$. Then the proof of Theorem 1 shows that there exist a simply connected open set $V \subset U$ with rectifiable boundary and a nonzero function h in $H^\infty(V)$ such that

(i) The strong boundary B of V has positive m measure

(ii) $h(zg\mathrm{d}\mu)^\wedge$ is bounded in V

(iii) $\lim_{r\to 1^-} h(r\,e^{i\theta})(zg\mathrm{d}\mu)^\wedge(r\,e^{i\theta}) = h(e^{i\theta})g(e^{i\theta})\dfrac{\mathrm{d}\mu}{\mathrm{d}m}(e^{i\theta})$ for almost every $e^{i\theta}$ in B.

Since $\lim_{r\to 1^-} h(r\,e^{i\theta}) = h(e^{i\theta})$ for almost every $e^{i\theta}$ in B and $h(e^{i\theta})\neq 0$ for almost every $e^{i\theta}$ in B (see [10], p. 45), the conclusion of the lemma follows easily.

Let $H^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$. The space $H^2(\mu)$ has a bounded point evaluation at ζ if there exists a constant $C > 0$ such that

$$|p(\zeta)| \leqslant C\|p\|_2$$

for every polynomial p. In this case there exists K_ζ in $H^2(\mu)$ such that

$$p(\zeta) = \langle p, K_\zeta\rangle$$

for every polynomial p. This extends to a continuous linear functional on H^2 by letting

$$\tilde{f}(\zeta) = \langle f, K_\zeta\rangle$$

for each f in $H^2(\mu)$.

COROLLARY. *Suppose* $P^\infty(\mu) = H^\infty$ *and* $H^2(\mu)$ *has bounded point evaluations in an outer component* U *of* $D\setminus\operatorname{spt}\mu$. *If* $f \in H^2(\mu) \cap L^\infty(\mu)$, *then*

$$\lim_{r\to 1^-}\tilde{f}(r\,e^{i\theta}) = f(e^{i\theta})$$

for almost every $e^{i\theta}$ *in the strong boundary* B *of* U.

Proof. Let $f \in H^2(\mu) \cap L^\infty(\mu)$. Fix λ in U and let K_λ be the unique vector in $H^2(\mu)$ such that

$$p(\lambda) = \langle p, K_\lambda\rangle$$

for every polynomial p. It is easy to see that $fK_\lambda \in H^2(\mu)$ and that $(fK_\lambda)^\sim(\zeta) = \tilde{f}(\zeta)\tilde{K}_\lambda(\zeta)$ for each ζ in U. Let $\{p_n\}$, $\{q_n\}$ be sequences of polynomials converging to fK_λ and K_λ, respectively, in $L^2(\mu)$ norm. We compute:

$$(z(z-\lambda)p_n\bar{K}_\lambda d\mu)^\wedge(\zeta) = (z(z-\lambda)(p_n - p_n(\zeta))\bar{K}_\lambda d\mu)^\wedge(\zeta) + \\ + p_n(\zeta)(z(z-\lambda)\bar{K}_\lambda d\mu)^\wedge(\zeta) = p_n(\zeta)(z(z-\lambda)\bar{K}_\lambda d\mu)^\wedge(\zeta).$$

Similarly,

$$(z(z-\lambda)q_n\bar{K}_\lambda d\mu)^\wedge(\zeta) = q_n(\zeta)(z(z-\lambda)\bar{K}_\lambda d\mu)^\wedge(\zeta).$$

Noting that $(z(z-\lambda)\bar{K}_\lambda d\mu)^\wedge$ is analytic in U and nonzero at λ (if $\lambda \neq 0$), we see that

$$\frac{p_n(\zeta)}{q_n(\zeta)} = \frac{(z(z-\lambda)p_n\bar{K}_\lambda d\mu)^\wedge(\zeta)}{(z(z-\lambda)q_n\bar{K}_\lambda d\mu)^\wedge(\zeta)}$$

except for at most countably many ζ in U. Letting $n \to \infty$, we see that

$$\tilde{f}(\zeta) = \frac{\tilde{f}(\zeta)\tilde{K}_\lambda(\zeta)}{\tilde{K}_\lambda(\zeta)} = \frac{(z(z-\lambda)f|K_\lambda|^2 d\mu)^\wedge(\zeta)}{(z(z-\lambda)|K_\lambda|^2 d\mu)^\wedge(\zeta)} \tag{16}$$

except for at most countably many ζ in U.

Since $|K_\lambda|^2 d\mu$ represents a constant times evaluation at λ, it is a Carleson measure. Applying Lemma 6 to the numerator and denominator in (16), we obtain

$$\lim_{r\to 1^-} \tilde{f}(r\,e^{i\theta}) = f(e^{i\theta})$$

for almost every $e^{i\theta}$ in B such that $K_\lambda(e^{i\theta})\dfrac{d\mu}{dm}(e^{i\theta}) \neq 0$. Thus it now suffices to show that the latter function is nonzero almost everywhere on B.

Suppose not. Then by the proofs of Theorem 1 and Lemma 5, we can construct a function h in H^∞ such that

(i) $h(\lambda_0) = 1$ for some λ_0 in U

(ii) $\|h\|_{|K_\lambda|^2 d\mu} < 1$.

(iii) $(z(z-\lambda)|K_\lambda|^2 d\mu)^\wedge(\lambda_0) \neq 0$.

Then $h^n(\lambda_0) = [(z(z-\lambda)|K_\lambda|^2 d\mu)^\wedge(\lambda_0)]^{-1}\displaystyle\int z\frac{z-\lambda}{z-\lambda_0} h^n|K_\lambda|^2 d\mu$ for every $n \leqslant 1$, an obvious contradiction.

REMARK. In the event $\operatorname{spt}(\mu|_D)$ does not meet an arc of ∂D, it is easy to obtain boundary values for every function f in $H^2(\mu)$ on the arc by applying Lemma 1. $\left(\text{Just consider } \lim_{r\to 1^-}\dfrac{(z(z-\lambda)f\bar{K}_\lambda d\mu)^\wedge(r\,e^{i\theta})}{(z(z-\lambda)\bar{K}_\lambda)^\wedge(r\,e^{i\theta})}\cdot\right)$

The following lemma does most of the work for the proof of Theorem 2. The lemma establishes the fact that if λ is in a component of $D\setminus \operatorname{spt}\mu$ then "we can surround it as many times as we please".

LEMMA 7. *Let μ be a measure with support contained in $\bar{D}$ and $\mu(\partial D) = 0$. If $P^\infty(\mu) = H^\infty(D)$ then there exists a Borel partition $\{\Delta_1, \Delta_2\}$ of D such that $\mu = \mu|_{\Delta_1} + \mu|_{\Delta_2}$ and $P^\infty(\mu|_{\Delta_i}) = H^\infty(D)$ for $i = 1, 2$.*

Proof. For each positive integer greater than one, let

$$B(n) = D(n) = \{|z| \leqslant 1 - n^{-1}\}.$$

First, we claim that there exists an integer n_2 such that

$$\|f\|_{B(2)} \leqslant \|f\|_{\mu|D(n_2)} + \frac{1}{2}$$

for all f in H^∞ with $\|f\|_m \leqslant 1$. (Here, of course, $\|f\|_{B(2)} = \sup\{|f(z)|: z \in B(2)\}$.)

To see this, suppose the result is false. Then there exist sequences $\{f_n\}$ and $\{z_n\}$ contained in the unit ball of H^∞ and $B(2)$, respectively, such that

$$|f_n(z_n)| > \|f_n\|_{\mu|D(n)} + \frac{1}{2}$$

for all $n \geqslant 2$. Since the set $\{f_n\}$ is a normal family and $B(2)$ is a compact subset of D, we may assume that there exists a function f in the unit ball of H^∞ and a point z_0 in $B(2)$ such that

$$\|f_n - f\|_{B(2)} \to 0 \quad \text{and} \quad |z_n - z_0| \to 0.$$

A routine argument, based on Schwarz's lemma, now shows that

$$|f(z_0)| \geqslant \|f\|_\mu + \frac{1}{2},$$

a blatant contradiction.

Noting that $P^\infty(\mu_{D\setminus D(n_2)}) = P^\infty(\mu)$ by the maximum modulus principle, we can find n_3 so that

$$\|f\|_{B(3)} \leqslant \|f\|_{\mu|D(n_3)\setminus D(n_2)} + \frac{1}{3}$$

for all f in H^∞ with $\|f\|_m \leqslant 1$. Continuing this process, we obtain a sequence $\{n_j\}$ such that

$$\|f\|_{B(j)} \leqslant \|f\|_{\mu|D(n_{j+1})\setminus D(n_j)} + \frac{1}{j}$$

for all f in $H^\infty(D)$ with $\|f\|_m \leqslant 1$. Let $A_j = D(n_{j+1})\setminus D(n_j)$. Let $\Delta_1 = \bigcup_{j \text{ odd}} A_j$ and let $\Delta_2 = \bigcup_{j \text{ even}} A_j$.

It is clear now that for all f in the unit ball of H^∞,

$$\|f\|_m = \|f\|_{\mu|\Delta_1} = \|f\|_{\mu|\Delta_2}.$$

Using a routine argument involving normal families, we see that H^∞ is weak-star sequentially closed in $L^\infty(\mu|_{\Delta_i})$. Therefore, H^∞ is weak-star closed in $L^\infty(\mu|_{\Delta_i})$ for $i = 1,2$. Thus $P^\infty(\mu|_{\Delta_i}) = H^\infty$.

THEOREM 2. *Let $N = M_z$ on $L^2(\mu)$ and assume $P^\infty(\mu) = H^\infty(D)$. Assume further that for every $f \in P^\infty(\mu)$, we have*

$$\|f\|_\mu = \|f\|_{\mu|_D}.$$

Then if Ω is any component of $D\setminus$spt μ and $n \in \mathbf{Z}^+ \cup \{\infty\}$ there exists an $S \in \mathscr{S}_p(N)$ such that

$$\mathrm{i}(S - \lambda) = -n$$

for all $\lambda \in \Omega$.

Proof. We will prove the result for $n = 2$. (The proof for the case of any finite integer n is handled by induction. For $n = \infty$ the proof is handled by modifying the techniques of Lemma 7 and the following argument. We leave these details to the reader.)

The hypothesis that

$$\|f\| = \|f\|_{\mu|_D}$$

for every $f \in P^\infty(\mu)$ implies that

$$P^\infty(\mu|_D) = H^\infty(D).$$

Hence, by Lemma 7, we choose a measurable partition $\{\Delta_1, \Delta_2\}$ of D such that

$$\mu|_D = \mu|_{\Delta_1} + \mu|_{\Delta_2} \quad \text{and} \quad P^\infty(\mu|_{\Delta_i}) = H^\infty(D)$$

for $i = 1,2$. Let

$$\mu_1 = \mu|_{\Delta_1 \cup \partial D} \quad \text{and} \quad \mu_2 = \mu|_{\Delta_2}.$$

Then $\mu = \mu_1 + \mu_2$ with $\mu_1 \perp \mu_2$, and $P^\infty(\mu_i) = H^\infty(D)$ for $i = 1,2$.

Let $\lambda \in \Omega$. By using the techniques indicated in [2], Section 3, we can find measures β_1 and β_2 such that

17) β_i and μ_i are mutually absolutely continuous for $i = 1,2$, and

(18) $$\int p d\beta_i = p(\lambda) \quad \text{for all polynomials } p, \text{ for } i = 1,2.$$

Let $N_i = M_z$ on $L^2(\beta_i)$ and let $H^2(\beta_i)$ denote the closure of the polynomials in $L^2(\beta_i)$ for $i = 1,2$. Let $S_i = N_i|_{H^2(\beta_i)}$ for these i's.

First observe $S_i \in \mathscr{S}(N_i)$ by using the Stone-Weierstrass theorem. Now notice that (17) implies $\lambda \in \sigma(S_i)$ for $i = 1,2$. Hence, we have $\sigma(S_i) \supset \Omega$. Therefore, $\mathrm{i}(S_i - \lambda) = -1$ for $i = 1,2$ because each S_i has a cyclic vector (namely 1). Now observe $\bar{z} - \bar{\lambda} \perp H^2(\beta_i)$ and $|\bar{z} - \bar{\lambda}| > 0$ almost everywhere (β_i) for $i = 1,2$. Consequently, $S_i \in \mathscr{S}_p(N_i)$ by Proposition 3.10 of [2].

Hence $S_1 \oplus S_2 \in \mathscr{S}_p(N_1 \oplus N_2)$. Since $\mathrm{i}((S_1 \oplus S_2) - \lambda) = -2$ and $N_1 \oplus N_2$ is unitarily equivalent to N the theorem is established. (N is unitarily equivalent to $N_1 \oplus N_2$ because $\beta_1 \perp \beta_2$, and μ and $(\beta_1 + \beta_2)$ are mutually absolutely continuous).

REMARKS. The proof of Theorem 2 establishes the following fact: If Ω is an outer component of $D \setminus \operatorname{spt} \mu$ and $\mu = \mu_1 + \mu_2$ and $\mu_1 \perp \mu_2$ then

$$\Omega \cap \operatorname{int} \widetilde{K}_1 \cap \operatorname{int} \widetilde{K}_2 = \emptyset$$

where $\widetilde{K}_i$ is the set used by Sarason to describe $P^\infty(\mu_i)$ for $i = 1,2$. (Because if $\Omega \subset \operatorname{int} \widetilde{K}_i$ for $i=1,2$ then we would modify the above proof to produce $S \in \mathscr{S}_p(M_z)$ such that $\mathrm{i}(S - \lambda) = -2$ for $\lambda \in \Omega$, a contradiction of Theorem 1.)

The following result shows that the concept of surrounding a point λ in a hole (in relationship to index theory) must take into account the number of distinct times λ can be put into a Sarason hull.

PROPOSITION 2. *Let $\{\Gamma_n\}_{n=1}^q$ be a finite sequence of circles such that* $\operatorname{int} \Gamma_n \supset \operatorname{int} \Gamma_{n+1}$, *for* $1 \leqslant n \leqslant q - 1$. *Suppose μ is a measure such that* $\operatorname{spt} \mu = \bigcup_1^q \Gamma_n$ *and $P^\infty(\nu) = L^\infty(\nu)$ where $\nu = \mu|_{\operatorname{int} \Gamma_1}$. Let $N = M_z$ on $L^2(\mu)$. If $\lambda \in (\operatorname{int} \Gamma_1) \setminus \left(\bigcup_2^q \Gamma_n \right)$ then for all $S \in \mathscr{S}_p(N)$, it follows that*

$$\mathrm{i}(S - \lambda) = -1.$$

Proof. Since we are assuming $S \in \mathscr{S}_p(N)$, we have $P^\infty(\mu) = H^\infty(\operatorname{int} \Gamma_1)$. Therefore, without loss of generality, $\mu|_{\Gamma_1}$ is normalized Lebesgue measure on Γ_1. By a conformal map, we can assume $\lambda = 0$.

We first establish that $\sigma(S) = \overline{\operatorname{int} \Gamma_1}$. (Then $\mathrm{i}(S) \leqslant -1$). We proceed by contradiction. If this was not the case, there would exist a j such that $\sigma(S) \cap (\operatorname{int} \Gamma \setminus \overline{\operatorname{int} \Gamma_{j+1}}) = \emptyset$, but $\sigma(S) \supset \Gamma_j \cup \Gamma_{j+1}$. (Note: If $j = q$ we delete the terms above concerning Γ_{j+1}).

Let $f(z) = (z - \beta)^{-1}$ where $\beta \in (\operatorname{int} \Gamma_j) \setminus (\overline{\operatorname{int} \Gamma_{j+1}})$. By [12], $f(S)$ is a pure subnormal and using the standard functional calculus of an operator, we see that $\sigma(f(S)) \subset \overline{U}_1 \cup \overline{U}_2$ where $\overline{U}_1$ and $\overline{U}_2$ are disjoint closed discs. Furthermore, using [12] again, $f(N)$ is the minimal normal extension of $f(S)$. Hence $P^\infty(\mu \circ f^{-1})$ has no L^∞ summand, a contradiction that $P^\infty(\nu) = L^\infty(\nu)$.

Now suppose $\mathrm{i}(S) \leqslant -2$ and choose two orthogonal unit vectors K_1 and K_2 in $\mathscr{H} \ominus z\mathscr{H}$ where $\mathscr{H}$ is the space for S.

Let τ be any of the following measures: $z|K_i|^2\mathrm{d}\mu$ or $K_i\overline{K_j}\mathrm{d}\mu$ for $i = 1,2$ and $j = 1,2$. Then $\hat{\tau}$ is analytic and $|\hat{\tau}|^p$ has a harmonic majorant (for $p < 1$) in the regions $(\operatorname{int} \Gamma_{j-1})\setminus(\overline{\operatorname{int} \Gamma_j})$ for all $q \geqslant j \geqslant 1$ and also in int Γ_q, (see [10], pp. 39.). Consequently, $\hat{\tau}$ has nontangential boundary values almost everywhere (Lebesgue) from within and without on Γ_j ($j \geqslant 1$), (see [10], pp. 181—182).

These exterior and interior limits are equal wherever the Radon derivative of μ with respect to Lebesgue measure (on the Γ_j's) is zero, (see [10], pp. 39). But, by hypothesis, this Radon derivative is zero on a set of positive measure on each Γ_j, $j \geqslant 2$. By Lemma 1, we have

$$(z|K_1|^2\mathrm{d}\mu)^\wedge(z|K_2|^2\mathrm{d}\mu)^\wedge = z^2(K_1\overline{K_2}\mathrm{d}\mu)^\wedge(\overline{K_1}K_2\mathrm{d}\mu)^\wedge$$

on $(\operatorname{int} \Gamma_1)\setminus(\operatorname{int} \Gamma_2)^-$ since an H^p function is determined by its boundary values on a set of positive measure. Proceeding by induction, handling each hole in order, we see that

$$(z|K_1|^2\mathrm{d}\mu)^\wedge(z|K_2|^2\mathrm{d}\mu)^\wedge = z^2(K_1\overline{K_2}\mathrm{d}\mu)^\wedge(\overline{K_1}K_2\mathrm{d}\mu)^\wedge$$

on each hole. This is a contradiction because the left hand side of the equality is one at zero while the right hand side of the equality is zero at zero.

We conclude this section by asking a question. Note that Theorem 1 and Proposition 2 are answers to this question under more hypothesis.

QUESTION: Let $N = M_z$ on $L^2(\mu)$ and $S \in \mathscr{S}_p(N)$. Suppose $P^\infty(\mu) = H^\infty(D)$ and $\lambda \in D\setminus\operatorname{spt}\mu$. Suppose further that whenever $\mu = \mu_1 + \mu_2$ and $\mu_1 \perp \mu_2$, it follows that

$$\lambda \notin (\operatorname{int} \widetilde{K}_{\mu_1}) \cap (\operatorname{int} \widetilde{K}_{\mu_2}).$$

Then is $\mathrm{i}(S - \lambda) \geqslant -1$ for all $S \in \mathscr{S}_p(N)$?

REMARK. $\widetilde{K}_{\mu_i}$ are the sets used by Sarason, [17], to describe $P^\infty(\mu_i)$, for $i = 1,2$.

III. THE COLLECTION $\mathscr{S}_p(N)$

If N is a normal operator on a separable space, when is $\mathscr{S}_p(N)$ a nonempty collection? The answer is unknown. There are some partial results in the literature and we shall conclude this paper by showing how some of the ideas of the first part yield additional information concerning this problem.

Let μ be the scalar spectral measure for N. If N has a cyclic vector, then $\mathscr{S}_p(N)\neq\emptyset$ if and only if $P^\infty(\mu)$ has no L^∞ summand, (see [8], Prop. 9.6). For an arbitrary normal operator N, it is well-known that there exists a countable collection of measures $\{\mu_i\}_{i\in I}$ such that $\mu_1=\mu$ and $\mu_{i+1} << \mu_i$ for all $i \in I$ and N is unitarily equivalent to $\bigoplus_{i\in I} N_i$, where $N_i = M_z$ on $L^2(\mu_i)$. If $P^\infty(\mu_i)$ has no L^∞ summand for all $i \in I$ then $\mathscr{S}_p(M)\neq\emptyset$ by the proposition mentioned above.

We now focus our attention to the case where $N = N_1 \oplus N_2$ where $N_i = M_z$ on $L^2(\mu_i)$, $i=1, 2$, $\mu_2 << \mu_1$, and $P^\infty(\mu_1)$ has no L^∞ summand but $P^\infty(\mu_2) = L^\infty(\mu_2)$. (Therefore, N_2 is a reductive normal operator, i.e., every invariant subspace for N_2 is also invariant for N_2^*.) Even in this special case we cannot answer the aforementioned question. If N_2 is an operator on a one dimensional space, we can. Before we present the result we make the following observation:

OBSERVATION. *Let $\mathscr{H}$ be a Hilbert space, φ a nonzero densely defined linear functional on $\mathscr{H}$ with a closed graph. Then $\varphi \in \mathscr{H}^*$ (the dual of $\mathscr{H}$).*

Proof. It suffices to show φ is continuous on dom φ. Since φ is nonzero and has a closed graph, ker φ is not dense in dom φ. Hence by Theorem 1.18 of [15], φ is continuous on its domain. The following proposition ties together the concepts of surrounding a point not in the support of a measure and when $\mathscr{S}_p(N)$ is nontrivial.

PROPOSITION 3. *Let μ be a measure such that $P^\infty(\mu)$ is antisymmetric, $\lambda \notin \operatorname{spt}\mu$ and $\lambda \in \operatorname{int}\widetilde{K}$ (where $P^\infty(\mu) = H^\infty(\operatorname{int}\widetilde{K})$). Let δ_λ be any point mass measure at λ and*

$$N_1 = M_z \text{ on } L^2(\mu + \delta_\lambda),$$

$$N_2 = M_z \text{ on } L^2(\delta_\lambda),$$

$$N_3 = M_z \text{ on } L^2(\mu),$$

and

$$N = N_1 \oplus N_2 \textit{ on } L^2(\mu + \delta_\lambda) \oplus L^2(\delta_\lambda).$$

The following two conditions are equivalent:

(a) $\mathscr{S}_p(N) \neq \emptyset$.

(b) *There exists $T \in \mathscr{S}_p(N_3)$ such that* $\mathrm{i}(T-\lambda) \leqslant -2$.

Proof. (a) implies (b). Let $S \in \mathscr{S}_p(N)$ and let $\mathscr{H}$ be the space on which S is defined. Using the observation and the fact that $S \in \mathscr{S}_p(N)$, we see that there exists a closed subspace $\mathscr{H}_1$ contained in $L^2(\mu+\delta_\lambda)$ that is invariant for N_1, and a $\varphi \in \mathscr{H}_1^*$ such that

$$\mathscr{H} = \text{Graph}\,\varphi. \tag{1}$$

Moreover, since $\mathscr{H}$ is invariant for N, we have

$$\varphi N_1 x = N_2 \varphi x \tag{2}$$

for all $x \in \mathscr{H}_1$. Because φ is bounded, we get that $N_1|_{\mathscr{H}_1}$ is similar to $N|_{\mathscr{H}}$. Hence, by [18], $N_1|_{\mathscr{H}_1}$ is pure. Furthermore, $N_1|_{\mathscr{H}_1} \in \mathscr{S}(N_1)$; otherwise, S would not belong to $\mathscr{S}(N)$.

Since $N_1|_{\mathscr{H}_1}$ is pure, the function $g|_{\text{spt}\,\mu}$ determines $g(\lambda)$ for every $g \in \mathscr{H}_1$. As above, we see that there exists a closed subspace $\mathscr{H}_2$ contained in $L^2(\mu)$ that is invariant under N_3 and a $\psi \in \mathscr{H}_2^*$ such that

$$\mathscr{H}_1 = \text{Graph}\ \psi. \tag{3}$$

Arguing as before, we get that $N_3|_{\mathscr{H}_2} \in \mathscr{S}_p(N_3)$ and

$$\psi N_3 x = N_2 \psi x \tag{4}$$

for all $x \in \mathscr{H}_3$. Combining equations (3) and (1), we see there exists $\theta \in \mathscr{H}_2^*$ such that

$$\mathscr{H} = \text{Graph of}\ \psi \oplus \theta. \tag{5}$$

Claim: $\theta \neq \beta\psi$ for any $\beta \in \mathbf{C}$. To see this, suppose there exists a $\beta \in \mathbf{C}$ such that $\theta = \beta\psi$. Then (5) would imply the m.n.e. of $N|_{\mathscr{H}}$ equals $\{(f, r, \beta r)\colon f \in L^2(\mu)$ $r \in \mathbf{C}\}$, a contradiction.

Let $T = N_3|_{\mathscr{H}_2}$. To finish the proof, all we have left to show is that $\mathrm{i}(T - \lambda) \leqslant -2$. To do this, all we need to show is that there are two nonzero linearly independent vectors in $\mathscr{H}_3 \ominus [(N_3 - \lambda)\mathscr{H}_3]$.

Let K_1, K_2 be nonzero vectors in $\mathscr{H}_3$ such that $\psi(x) = \langle x, K_1\rangle$ and $\theta(x) = \langle x, K_2\rangle$ for all $x \in \mathscr{H}_3$. The vectors K_1 and K_2 are linearly independent because $\theta \neq \beta\psi$ for any $\beta \in \mathbf{C}$. Now for any $x \in \mathscr{H}_3$ we have

$$\langle (z - \lambda)x, K_1\rangle = \psi((N_3 - \lambda)x) = (N_2 - \lambda)\psi(x) = 0. \quad \text{(via (4))}$$

Hence $K_1 \perp (z - \lambda)\mathscr{H}_3$. A similar argument, using (2), establishes $K_2 \perp (z - \lambda)\mathscr{H}_3$.

To see that (b) implies (a), one needs only to reverse the above argument. That is, let $T \in \mathscr{S}_p(N_3)$, say T acts on $\mathscr{H}$, and $\mathrm{i}(T - \lambda) \leqslant -2$. Let K_1 and K_2 be orthogonal unit vectors in $\mathscr{H} \ominus [(N_3 - \lambda)\mathscr{H}]$. Let $\psi(x) = \langle x, K_1\rangle$ and $\theta(x) = \langle x, K_2\rangle$ for all $x \in \mathscr{H}$. One can directly verify (using the methods above) that $N|_{\mathscr{L}} \in \mathscr{S}_p(N)$ where $\mathscr{L}$ equals the graph of $\theta \oplus \psi$.

REMARKS: a. Let m_i denote Lebesgue measure on the circle $\{|z| = 1/i\}$ for $i = 1,2$. Let δ_0 denote the point mass measure at 0. Observe by the last proposition that $\mathscr{S}_p(N) = \emptyset$, where N is the direct sum of M_z on $L^2(m_1 + \delta_0)$ with M_z on $L^2(\delta_0)$, even through $P^\infty(m_1 + \delta_0)$ is antisymmetric. This shows that the assumption N has multiplicity one in [8], Proposition 9.6 is crucial. However, if we change the above example by replacing m_1 everywhere by $m_1 + m_2$, we see that $\mathscr{S}_p(N) \neq \emptyset$.

b. The proof of the above theorem shows that there exist two pure subnormal operators S_1 and S_2 that are similar but whose minimal normal extensions don't have the same multiplicity. (Consult [11], Problem 156). In fact, we could construct S_1 and S_2 such that the m.n.e. of S_1 has multiplicity one while the m.n.e. of S_2 has infinite multiplicity.

The following theorem gives a sufficient condition that $\mathscr{S}_p(N) \neq \emptyset$ in terms of the structure of $P^\infty(\mu)$ where $N = N_1 \oplus N_2$, N_2 is reductive and μ is the scalar spectral measure for N.

THEOREM 3. *Let μ be a measure such that $P^\infty(\mu) = H^\infty(G)$, i.e., $P^\infty(\mu)$ is antisymmetric and $G = \operatorname{int} \widetilde{K}$. Let β be a measure such that $\beta << \mu$ and $P^\infty(\beta) = L^\infty(\beta)$.*

Let $N = N_1 \oplus N_2$ where $N_1 = M_z$ on $L^2(\mu)$ and $N_2 = M_z$ on $L^2(\beta)$. If there exists a (nontrivial) Borel measurable partition $\{\Delta_1, \Delta_2\}$ of $\bar{G}$ satisfying

(a) *$P^\infty(\mu|_{\Delta_1} + \beta)$ is antisymmetric, and*

(b) *$P^\infty(\mu|_{\Delta_2})$ is antisymmetric with $\beta << \mu|_{\Delta_2}$,*

then $\mathscr{S}_p(N) \neq \emptyset$.

Proof. Since $P^\infty(\mu|_{\Delta_1} + \beta)$ is antisymmetric, we see that $\mathscr{S}_p(A) \neq \emptyset$, where $A = M_z$ on $L^2(\mu|_{\Delta_1} + \beta)$. Let $S \in \mathscr{S}_p(A)$, say S acts on $\mathscr{H}_1$. Observe that if $f, g \in \mathscr{H}_1$ and $f|_{\Delta_1} = g|_{\Delta_1}$, then $f|_{\operatorname{spt}\beta} = g|_{\operatorname{spt}\beta}$ because S is pure. Hence, $\mathscr{H}_1 = \text{Graph } B$, where B is a closed operator from $L^2(\mu|_{\Delta_1})$ to $L^2(\beta)$. Since $P^\infty(\beta) = L^\infty(\beta)$, it follows that B has dense range. (Recall $S \in \mathscr{S}(A)$).

Using the first part of hypothesis (b), we can choose $Q \in \mathscr{S}_p(C)$ where $C = M_z$ on $L^2(\mu|_{\Delta_2})$. Let $\mathscr{H}_2$ be the space on which Q acts. Let $\mathscr{K} = \mathscr{H}_2 \oplus \text{Graph } B$. It is easy to see that $\mathscr{K} \subset L^2(\mu) \oplus L^2(\beta)$, $\mathscr{K}$ is a closed invariant subspace for N and $N|_{\mathscr{K}}$ is pure.

Observe that the m.n.e. of $N|_{\mathscr{K}}$ clearly contains $L^2(\mu|_{\Delta_2})$. Since Δ_1 and Δ_2 are disjoint, there exists a sequence of continuous functions which converge boundedly pointwise almost everywhere (μ) to χ_{Δ_1}. Since $\beta \perp \mu|_{\Delta_1}$ and $S \in \mathscr{S}(A)$, we see that the m.n.e. of $N|_{\mathscr{K}} \supset L^2(\mu|_{\Delta_1})$. Hence the m.n.e. of $N|_{\mathscr{K}}$ is N because B has dense range in $L^2(\beta)$.

EXAMPLES. A. Let μ_1 denote planar Lebesgue measure on the open unit disc and β denote Lebesgue measure on the unit circle restricted to the set $\{\operatorname{Re} z > 0\}$. Let $\mu = \mu_1 + \beta$. Choose any infinite increasing sequence $\{a_i\}_{i=0}^\infty$ of nonnegative real numbers (with $a_0 = 0$) converging to 1, and set

$$\Delta_1 = \{z \in D\colon a_{2i} \leqslant |z| \leqslant a_{2i+1}, \quad i = 0, 1, \ldots\}$$

and

$$\Delta_2 = \{z\colon |z| = 1 \text{ and } \operatorname{Re} z > 0\} \cup \{z \in D\colon a_{2i-1} < |z| < a_{2i}, \quad i = 1, 2, \ldots\}.$$

Using the notation of the last theorem, we see that $\mathscr{S}_p(N) \neq \emptyset$.

B. The following gives an example of a normal operator N such that $\mathscr{S}_p(N) \neq \emptyset$ but N does not satisfy the hypothesis of either Theorem 3 or Theorem 4 (to follow). This example, intuitively speaking, shows that the answer to the question "when is $\mathscr{S}_p(N) \neq \emptyset$?" will depend on information other than $P^\infty(\mu)$ and how μ surrounds its holes.

Let G_1 be the open unit disc slit along the negative real axis from -1 to 0. Let $z^{1/2}$ denote the principal branch of $z^{1/2}$ and let G_2 denote the image of G_1 under $z^{1/2}$. Let ν denote normalized Lebesgue measure on ∂G_2 and let

$$\mu_1(E) = \nu(z^{1/2}(E))$$

for all Borel subsets $E \subset \{|z| = 1\}$ and

$$\beta(E) = \nu(z^{1/2}(E) \cap [0, \mathrm{i}])$$

for all Borel subsets $E \subset [-1, 0]$. Let $\mu = \mu_1 + \beta$. Let N be defined on $L^2(\mu) \oplus L^2(\beta)$ by

$$N(f \oplus g) = zf \oplus zg.$$

For convenience we shall write an element $f \oplus g \in L^2(\mu) \oplus L^2(\beta)$ as an ordered triple $(f|_{|z|=1}, f|_{[-1,0]}, g)$, suppressing the restriction signs when the context is clear. Let $\mathscr{H}$ be the closure of the linear manifold $\{(p+qz^{1/2}, p+qz^{1/2}, p-qz^{1/2})\colon p, q \text{ polynomials}\}$. We shall show that $N|_{\mathscr{H}} \in \mathscr{S}_p(N)$. Let $\mathscr{L}$ denote the closed reducing subspace for N such that $N|_{\mathscr{L}}$ is the minimal normal extension of $N|_{\mathscr{H}}$. By the Stone-Weierstrass theorem (gp, gp, gp) belongs to $\mathscr{L}$ for any continuous function g and any polynomial p. It follows easily that $(1, 0, 0) \in \mathscr{L}$ and, hence, $(f, 0, 0) \in \mathscr{L}$ for any $f \in L^2(\mu_1)$.

Using the Stone-Weierstrass theorem again we see that $(gp+fqz^{1/2}, gp+fqz^{1/2}, gp-fqz^{1/2})$ belongs to $\mathscr{L}$ for any continuous functions f and g and any polynomials p and q. Approximating $z^{1/2}$ by a sequence of continuous functions f_n and letting $p = q = g = 1$, we see that $(1+z, 1+z, 1-z)$ belongs to $\mathscr{L}$. Since (z, z, z) and $(1, 0, 0)$ belong to $\mathscr{L}$ so does $(0, z, 0)$. Clearly then $(0, \chi_{[-1,x]}z, 0) \in \mathscr{L}$ for all $x < 0$. Hence $(0, \chi_{[-1,x]}, 0) \in \mathscr{L}$ and therefore $(0, 1, 0) \in \mathscr{L}$. It is now obvious that $\mathscr{L} = L^2(\mu) \oplus L^2(\beta)$.

We still need to show $N|_{\mathscr{H}}$ is a pure operator. In this light define the map I from $(p+qz^{1/2}, p+qz^{1/2}, p-qz^{1/2})$ into $H^2(\nu)$ by

$$I(p+qz^{1/2}, p+qz^{1/2}, p-qz^{1/2})(w) = p(w^2) + q(w^2)w$$

for all $w \in \partial G_2$. We want to show I is an isometry. First note that for all Borel subsets $E \subset [0, \mathrm{i}]$ that $\nu(E) = \nu(-E)$. Hence

$$\|(p+qz^{1/2}, p+qz^{1/2}, p-qz^{1/2})\|^2 = \int_{\partial G_2 \cap \{|z|=1\}} |p(w^2) + q(w^2)w|^2 \mathrm{d}\nu +$$

$$+ \int_{[0,\mathrm{i}]} |p(w^2) + q(w^2)w|^2 \mathrm{d}\nu + \int_{[0,\mathrm{i}]} |p(w^2) - q(w^2)w|^2 \mathrm{d}\nu.$$

But the last integral equals $\int_{[-\mathrm{i},0]} |p(w^2) + q(w^2)w|^2 \mathrm{d}\nu$

so

$$\|(p+qz^{1/2}, p+qz^{1/2}, p-qz^{1/2})\|^2 = \|p+qw\|^2_{L^2(\nu)}$$

for all polynomials p and q. We extend I to an isometry of $\mathscr{H}$ onto $H^2(\nu)$. Clearly I intertwines multiplication by z on $\mathscr{H}$ and multiplication by w^2 on $H^2(\nu)$. Since multiplication by w on $H^2(\nu)$ is obviously pure, it follows from [8], Corollary 6.4, that multiplication by w^2 on $H^2(\nu)$ is pure. Since purity is preserved under unitary equivalence, $N|_{\mathscr{H}} \in \mathscr{S}_p(N)$.

If one combines the methods and ideas of Lemma 7 and the last theorem, one can verify the following:

THEOREM 4. *Let N be any normal operator with scalar spectral measure μ. If $P^\infty(\mu) = H^\infty(G)$ and $P^\infty(\mu|_G) = H^\infty(G)$ then $\mathscr{S}_p(N) \neq \emptyset$. (Here $G = \operatorname{int} \widetilde{K}$.)*

This research was supported in part by the National Science Foundation under Grant No. MCS77–00966.

REFERENCES

1. ABRAHAMSE, M. B.; DOUGLAS, R. G., A class of subnormal operators related to multiply connected domains, *Advances in Math.*, **19** (1976), 106–148.
2. BALL, J.; OLIN, R.; THOMSON, J., Weakly closed algebras of subnormal operators, *Illinois J. Math.*, **22** (1978), 315–326.
3. BRAM, J., Subnormal operators, *Duke Math. J.*, **22** (1955), 75–94.
4. BROWDER, A., *Introduction to Function Algebras*, W. A. Benjamin Inc., New York, 1969.
5. BROWN, S., Some invariant subspaces for subnormal operators, *Integral Equations Operator Theory*, **1** (1978), 310–333.
6. CHAUMAT, J., Adherence faible étoile d'algèbres de fractions rationnelles, *Publications Mathématiques d'Orsay*, **147**, pp. 75–40, U.E.R. Mathématique, Université Paris XI, Orsay, 1975.
7. CLANCEY, K.; PUTNAM, C. R., The local spectral behavior of completely subnormal operators, *Trans. Amer. Math. Soc.*, **163** (1972), 239–244.
8. CONWAY, J. B.; OLIN, R., A Functional Calculus for Subnormal Operators II, *Mem. Amer. Math. Soc.*, **184** (1977).
9. DAVIE, A., Dirichlet algebras of analytic functions, *J. Functional Analysis*, **6** (1970), 348–356.
10. DUREN, P., *Theory of H^p spaces*, Academic Press, New York, 1970.
11. HALMOS, P. R., *A Hilbert Space Problem Book*, D. Van Nostrand Co., Princeton, 1967.
12. OLIN, R., Functional relationships between a subnormal operator and its minimal normal extension, *Pacific J. Math.*, **63** (1976), 221–229.
13. OLIN, R.; THOMSON, J., The spectrum of a normal operator and the problem of filling in holes, *Indiana Univ. Math. J.*, **26** (1977), 541–544.
14. OLIN, R.; THOMSON, J., Algebras of subnormal operators, *J. Functional Analysis*, to appear.
15. RUDIN, W., *Functional Analysis*, McGraw-Hill Book Co., New York, 1973.
16. SARASON, D., The H^p Spaces of an Annulus, *Mem. Amer. Math. Soc.*, **56** (1965), 1–78.
17. SARASON, D., Weak-star density of polynomials, *J. Reine Agnew. Math.*, **252** (1972), 1–15.
18. STAMPFLI, J., Analytic extensions and spectral localization, *J. Math. Mech.*, **16** (1966), 287–296.
19. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on a Hilbert space*, American Elsevier, New York, 1970.

ROBERT F. OLIN
Mathematics Department
Virginia Polytechnic Institute
Blacksburg, VA 24061
U.S.A.

JAMES E. THOMSON
Mathematics Department
Virginia Polytechnic Institute
Blacksburg, VA 24061
U.S.A.

Received February 23, 1979; revised June 13, 1979.