J. OPERATOR THEORY
3(1980), 271–274

THE EXT GROUPS OF THE C^*-ALGEBRAS ASSOCIATED WITH IRRATIONAL ROTATIONS

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Let θ be an irrational number, and let A_θ denote the crossed product C^*-algebra [9] for the action of the group of integers, $\mathbf{Z}$, on the algebra of continuous functions on the circle by powers of the rotation of angle $2\pi\theta$. It is important for this note that A_θ is isomorphic to the C^*-algebra generated by any two unitaries U and V which satisfy $VU = \lambda UV$ where $\lambda = e^{2\pi i\theta}$. This can be seen by associating to such U and V the unitaries u and v in A corresponding respectively to rotation by $2\pi\theta$ and to the "identity" function which embeds the circle as the unit circle in the complex plane. The purpose of this note is to calculate the Ext groups [3], [1] of the A_θ and of the $n \times n$ matrix algebras over the A_θ. For related considerations see [14].

It is known that the A_θ are simple [7], [15]. Furthermore, they are strongly amenable, hence nuclear [17], so that their Ext semigroups are, in fact, groups [6], [1]. In addition, the A_θ are quasi-diagonal. This follows from results announced by Hadwin, and also from the stronger fact obtained in [13] that the A_θ can be embedded in AF algebras. This quasi-diagonality is important to us since it enables us to use the homotopy results from [12].

Let $L/K(H)$ denote the Calkin algebra for a separable Hilbert space H, and let $\mathrm{Ext}_s(A_\theta)$ and $\mathrm{Ext}_w(A_\theta)$ denote the strong and weak Ext groups of A_θ, whose elements consist of equivalence classes of unital $*$-monomorphisms (called extensions) of A_θ into $L/K(H)$. Observe that there is a bijection between extensions and the pairs (U, V) of unitaries in $L/K(H)$ satisfying $VU = \lambda UV$. Observe also that if ind denotes the index function on unitaries in $L/K(H)$, and if w is a unitary in A_θ, then $\mathrm{ind}(\tau(w))$ depends only on the class, $[\tau]$, in $\mathrm{Ext}_s(A_\theta)$ of the extension τ. It follows easily that the map φ from $\mathrm{Ext}_s(A_\theta)$ to $\mathbf{Z} \times \mathbf{Z}$ defined by

$$\varphi([\tau]) = (\mathrm{ind}(\tau(u)), \mathrm{ind}(\tau(v)))$$

is a well-defined group homomorphism.

THEOREM. *The map φ defined above is an isomorphism. Furthermore, weak equivalence classes are strong equivalence classes, so that* $\mathrm{Ext}_s(A_\theta)$ *and* $\mathrm{Ext}_w(A_\theta)$ *coincide. The* Ext-*group topology on* $\mathrm{Ext}_s(A_\theta)$ *and* $\mathrm{Ext}_w(A_\theta)$ *is the discrete topology.*

Let M_n denote the algebra of $n \times n$ matrices, so that $M_n \otimes A_\theta$ is the algebra of $n \times n$ matrices over A_θ. An easy application of [8], Proposition 2.2 or [11], Proposition 4 then yields:

COROLLARY. $\mathrm{Ext}_s(M_n \otimes A_\theta) = \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_n$ *where* $\mathbf{Z}_n$ *is the cyclic group of order* n, *while* $\mathrm{Ext}_w(M_n \otimes A_\theta) = \mathbf{Z} \times \mathbf{Z}$.

It follows that for fixed θ the $M_n \otimes A_\theta$ are non-isomorphic. This also follows from Theorem 3 of [16].

Proof of the Theorem. To see that φ is surjective, let H be a Hilbert space with orthonormal basis $\{e_n\}_{n \geqslant 0}$, let S be the unilateral shift on H defined by $Se_n = e_{n+1}$ for all n, and let M be the multiplication operator defined by $Me_n = \lambda^n e_n$ for all n. Then $MS = \lambda SM$, so that we can define an extension, τ, by setting $\tau(u) = \pi(S)$ and $\tau(v) = \pi(M)$, where π is the quotient map from the algebra of bounded operators to the Calkin algebra. It is clear that $\varphi([\tau]) = (1, 0)$. Similarly we can define an extension σ by $\sigma(u) = \pi(M^*)$ and $\sigma(v) = \pi(S)$, so that $\varphi([\sigma]) = (0, 1)$.

The hard part of the proof is to show that φ is injective. Since $\mathrm{Ext}_s(A_\theta)$ is a group, it suffices to show that if τ is an extension with $\mathrm{ind}(\tau(u)) = 0 = \mathrm{ind}(\tau(v))$ then τ is trivial. Let τ be such an extension. Since $\tau(v)\tau(u)\tau(v)^* = \lambda\tau(u)$, it follows that the spectrum of $\tau(u)$ is invariant under multiplication by λ, and so must be the entire unit circle since θ is irrational. Thus by the Weyl-von Neumann-Berg theorem [2] we can find a unitary, U, on H such that $\pi(U) = \tau(u)$ and U is a bilateral shift of infinite multiplicity, that is, U is of the form $B \otimes I_L$ for a decomposition $H = K \otimes L$ where L is infinite dimensional and K has a basis $\{f_n : n \in \mathbf{Z}\}$ such that $Bf_n = f_{n+1}$ for all n. Define W on H by $W = W' \otimes I_L$ where $W'f_n = \lambda^n f_n$ for all n, so that $WU = \lambda UW$. Let N' be any unitary on L whose spectrum is the entire circle, and let $N = I_K \otimes N'$, so that U and N commute and have as joint spectrum the torus $S^1 \times S^1$. Then U and NW determine a trivial extension, τ_0, of A_θ. By a theorem of Voiculescu [18], [1], $\tau \oplus \tau_0$ is then equivalent to τ, so that it suffices to show that the extension $\tau \oplus \tau_0$ is trivial. Let V be any unitary on H such that $\pi(V) = \tau(v)$, and note that $\tau \oplus \tau_0$ is determined by $\pi(U \oplus U)$ and $\pi(V \oplus NW)$.

Note now that the two unitaries $U \oplus U$ and $VW^* \oplus N$ $(=(V \oplus NW)(W^* \oplus \oplus W^*))$, clearly commute modulo compact operators on $H \oplus H$ and have as joint essential spectrum all of $S^1 \times S^1$. Thus they define an extension of $C(S^1 \times S^1)$, the algebra of continuous functions on $S^1 \times S^1$. But any extension of this algebra for which the two generating unitaries have index zero is trivial, as indicated in §2 of [5] (and also 2.4 of [3]). Thus we can find commuting unitaries U_0 and X_0 on $H \oplus H$ which are compact perturbations of $U \oplus U$ and $VW^* \oplus N$. It follows that $\pi(U \oplus U)$ and $\pi(X_0(W \oplus W))$ determine an extension equivalent to $\tau \oplus \tau_0$, and that it suffices for us to show that this new extension is trivial.

Since X_0 commutes with U_0, we can find a selfadjoint operator T on $H \oplus H$ such that $\exp(iT) = X_0$ and T commutes with U_0. Then $\exp(itT)$ will commute with

$U \oplus U$ modulo compact operators for every $t \in [0, 1]$. It follows that for every $t \in [0, 1]$ an extension σ_t of A_θ is determined by setting

$$\sigma_t(u) = \pi(U \oplus U)$$

$$\sigma_t(v) = \pi(\exp(itT))\pi(W \oplus W)\,.$$

Note that σ_1 is equivalent to $\tau \oplus \tau_0$, while σ_0 is trivial. It is easy to see that $t \to \sigma_t(a)$ is norm continuous for all $a \in A_0$, that is, σ_t depends continuously on t in the point-norm topology. Since A_θ is quasi-diagonal, it follows from the strong homotopy-invariance property proved in Proposition 5.7(ii) and Theorem 5.12 of [12] that σ_1 is equivalent to σ_0. Thus $\tau \oplus \tau_0$ is trivial as desired.

Note next that if τ is any extension of A_θ, then $\mathrm{ind}(\tau(u))$ and $\mathrm{ind}(\tau(v))$ are unchanged under conjugation by any unitary in the Calkin algebra. Since we have just shown that these indices determine the strong equivalence class of the extension, it follows that strong equivalence classes coincide with weak equivalence classes.

The Ext-group topology is defined, among other places, shortly before Lemma 1 of [10]. If $\{\tau_i\}$ is a net of extensions of A_θ which converges to the extension τ in that topology, then it follows immediately from the definition that eventually $\mathrm{ind}(\tau_i(u))$ and $\mathrm{ind}(\tau_i(v))$ must equal $\mathrm{ind}(\tau(u))$ and $\mathrm{ind}(\tau(v))$. From what we have seen above it follows that the topology is discrete. Q.E.D.

Proof of the Corollary. Because of the fact shown above that strong equivalence classes are weak equivalence classes, it follows imediately by [8], Proposition 2.2 or by [11], Proposition 4, that $\mathrm{Ext}_s(M_n \otimes A_\theta)$ is isomorphic to $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_n$. The isomorphism is obtained by sending an extension τ of $M_n \otimes A_\theta$ to

$$(\mathrm{ind}(\tau_0(u)),\ \mathrm{ind}(\tau_0(v)),\ [\tau_1])$$

where τ_0 is the restriction of τ to $I \otimes A_\theta$, τ_1 is the restriction of τ to $M_n \otimes I$ and where we use the isomorphism of $\mathrm{Ext}_s(M_n)$ with $\mathbf{Z}_n$ and the fact that $\mathrm{Ext}(A_\theta) \simeq \simeq \mathbf{Z} \times \mathbf{Z}$ is torsion free.

The second part of the Corollary follows by the general fact (see for instance [11], Proposition 4) that given an arbitrary separable unital C^*-algebra A, $\mathrm{Ext}_w(M_n \otimes A)$ is naturaly isomorphic with $\mathrm{Ext}_w(A)$. Anyway the preceding results and the fact that $\mathrm{Ext}_w(M_n) = \{0\}$ give an explicit isomorphism of $\mathrm{Ext}_w(M_n \otimes A_\theta)$ with $\mathbf{Z} \times \mathbf{Z}$. This isomorphism sends the extension τ to

$$(\mathrm{ind}(\tau_0(u)),\ \mathrm{ind}(\tau_0(v))).$$

Q.E.D.

The second author's contribution was made while he was visiting the Mathematics Institute of the University of Copenhagen. He would like to thank the members to of the Institute, especially Dorte Oleson and Gert Pedersen, for their very warm hospitality. His research was partially supported by National Sciences Foundation grant MCS 77-13070.

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Received January 10, 1980.