J. OPERATOR THEORY
4(1980), 3–23

© Copyright by INCREST, 1980

SIMPLEXES OF STATES OF C^*-ALGEBRAS

C. J. K. BATTY

1. INTRODUCTION

The state space $S(A)$ of a unital C^*-algebra A becomes a compact convex set when equipped with the weak* topology, and this fact has been the motivation for much of the general theory of compact convex subsets of locally convex spaces [1]. The outstanding problem of which compact convex sets are affinely homeomorphic to state spaces of C^*-algebras has recently been solved by Alfsen and Shultz [3], but for Choquet simplexes the solution has long been known — a simplex is the state space of a C^*-algebra if and only if it is a Bauer simplex (i.e. its extreme boundary is compact), in which case the C^*-algebra is commutative. However other simplexes may occur as closed faces of state spaces. Among the properties of state spaces observed by Alfsen and Shultz and inherited by their faces is the "3-ball property", namely that the face generated by two pure states of a C^*-algebra is either their convex hull or is affinely isomorphic to a 3-dimensional Euclidean ball. In the reverse direction, it is immediately apparent that any simplex has the "1-ball property", i.e. the convex hull of any two extreme points is a face.

In many physical applications, a restricted class of states of the C^*-algebra is of special interest, and attempts to decompose these into extremal states involve determining whether the class forms a simplex. For example, if A is the C^*-algebra of observables of a quantum system, the symmetries of the system are represented by a group G of $*$-automorphisms of A, and the space $S_G(A)$ of G-invariant states merits special attention. Various notions of "asymptotic abelianness" were introduced, which ensure that $S_G(A)$ forms a simplex [10], [11], [12], [13], [18], [24], [25], [31]. Lanford and Ruelle [19], [25], [26] considered G-invariant states φ with associated covariant representations $(\mathscr{H}_\varphi, \pi_\varphi, u_\varphi)$, which are G-abelian in the sense that $\pi_\varphi(A)''$ restricts to an abelian von Neumann algebra on the space $\mathscr{H}_\varphi^G$ of u_φ-invariant vectors in $\mathscr{H}_\varphi$. They showed that any G-abelian state is represented by a unique maximal measure on $S_G(A)$, and that $\mathscr{H}_\varphi^G$ is one-dimensional whenever φ is ergodic and G-abelian. The one-dimensionality of $\mathscr{H}_\varphi^G$ is known to be equivalent to the "weak cluster" property, namely

$$\inf\{|\varphi(a'b) - \varphi(a)\varphi(b)|\} = 0$$

for all a and b in A, where the infimum is taken over all a' in the convex hull of the G-orbit of a. Subsequently Dang-Ngoc and Ledrappier [7], [8] obtained global converses of the results in [19]. Specifically they showed that if $S_G(A)$ forms a simplex, or if A is separable and every ergodic state is weakly clustering, then every G-invariant state is G-abelian.

For quantum dynamical systems, the group G is a strongly continuous one--parameter group $\{\alpha(t) : t \in \mathbf{R}\}$ representing time evolution, and the ground states of the system [4], [29], [30] form a class S_0 of special significance, not only physically but also in terms of the geometry of $S(A)$, since S_0 is a closed face of $S(A)$ contained in $S_G(A)$. The results and methods of [7], [8], [19] are immediately applicable in this setting, so it can be deduced that S_0 forms a simplex if and only if every ground state is G-abelian, or equivalently (at least if A is separable) every extremal ground state is weakly clustering.

This paper originated out of an attempt to obtain an abstract approach which would determine which faces F of $S_G(A)$ are simplexes, and an earlier version was written from this point of view. This contained theorems which immediately gave the results of [7], [8], [19] on putting $F = S_G(A)$, but also included the case when G is trivial and F is a face of $S(A)$. The referee then showed that $S_G(A)$ is canonically affinely homeomorphic to a face of the state space of the C^*-crossed product $G \times A$. Therefore the results about faces of $S_G(A)$ could be deduced from the special case when G is trivial. Thus the primary objects of study in the version presented here are closed faces F of $S(A)$.

In § 2, we introduce the notion of F-abelian states akin to that of G-abelian invariant states. Theorem 2.5 presents the main characterisations of faces which are simplexes, valid in all separable and some non-separable cases. Some of thesi are expressed in terms of F-abelianness or multiplicity conditions on the representations $(\mathscr{H}_\varphi, \pi_\varphi)$. However it is also shown that in order for F to be a simplex, it is sufficient that F should have the 1-ball property.

In § 3, we extend a result of Alfsen and Shultz [3] by showing that the convex hull of any set of pairwise inequivalent pure states is a face of $S(A)$, and we consider some circumstances under which its closure is also a face. Methods are given to construct simplicial faces with non-compact extreme boundary. In § 4, the results of § 2 and § 3 are converted into theorems about $S_G(A)$ by means of the crossed product, and in the final section they are applied to ground states. In this latter setting, no separability conditions are needed, and one of the equivalent criteria for S_0 to be a simplex is that every extremal ground state is a physical ground state in the terminology of [29], [30]. A method will be given to construct uniformly continuous one-parameter automorphism groups on simple C^*-algebras whose ground states form simplexes of any given finite dimension. Since such systems have non-unique physical ground states, this answers a question of Sakai [29].

I am deeply indebted to the referee for several perceptive comments on the original version of this paper, and in particular for providing Theorem 4.2. I am also extremely grateful to O. Bratteli for providing background information on the subjects discussed in this paper, especially for making available a preprint of [6] and drawing attention to [4] and [8].

2. ABELIAN FACES

Throughout the paper, A will be a unital C^*-algebra with state space $S(A)$, which will be considered to have the weak* topology in which it is a compact convex subset of A^*. Alfsen and Shultz [3, Corollary 3.4] have shown that $S(A)$ (and hence all its faces) has the "3-ball property", i.e. the face of $S(A)$ generated by any two distinct pure states is either their convex hull or is affinely homeomorphic to a 3--dimensional Euclidean ball. Moreover the first possibility occurs precisely when the given states are inequivalent (i.e. their associated irreducible representations are not unitarily equivalent). In Proposition 3.1 we shall extend one half of this result to larger sets of inequivalent pure states.

The result of Alfsen and Shultz shows that compact convex sets without the 3-ball property can never be affinely homeomorphic to faces of $S(A)$. In this section we shall give algebraic and geometrical characterisations of closed faces of $S(A)$ which are Choquet simplexes. We shall say that a convex set has the *1-ball property* if the convex hull of any two of its extreme points is a face. Any simplex has the 1-ball property, but there are elementary examples (even in $\mathbf{R}^5$) of compact convex sets with the 1-ball property which are not simplexes. We shall however see in Theorem 2.5 that a closed face of $S(A)$ with the 1-ball property is a simplex, provided certain separability conditions are satisfied. Thus there is a further class of metrisable compact convex sets which can never be affinely homeomorphic to a closed face of $S(A)$, at least if A is separable.

For φ in $S(A)$, let $(\mathscr{H}_\varphi, \pi_\varphi, \xi_\varphi)$ be the associated Hilbert space, representation and cyclic vector, and for η in $\mathscr{H}_\varphi$ let ω_φ^η be the functional in A^* defined by

$$\omega_\varphi^\eta(a) = \langle \pi_\varphi(a)\,\eta\,,\,\eta \rangle\,.$$

It is well-known [9, Proposition 2.5.1] that there is a linear order-isomorphism Θ_φ of the self-adjoint part of $\pi_\varphi(A)'$ onto the space J_φ of functionals ψ in A^* satisfying $-\lambda\varphi \leqslant \psi \leqslant \lambda\varphi$ for some scalar $\lambda \geqslant 0$, given by

$$\Theta_\varphi(x)(a) = \langle \pi_\varphi x \xi_\varphi,\ \xi_\varphi \rangle.$$

Thus $J_\varphi \cap S(A)$ is the smallest face of $S(A)$ containing φ.

Now consider a closed face F of $S(A)$, and let

$$\widetilde{F} = \{\lambda\varphi : \lambda \geqslant 0,\ \varphi \in F\},$$

so that $\widetilde{F}$ is a weak* closed order-ideal in the positive cone of A^*. For φ in F, let

$$\mathscr{H}_\varphi^F = \{\eta \in \mathscr{H}_\varphi : \omega_\varphi^\eta \in \widetilde{F}\}.$$

For η and η' in $\mathscr{H}_\varphi^F$, $\omega_\varphi^{\eta+\eta'}$ is dominated by $2(\omega_\varphi^\eta + \omega_\varphi^{\eta'})$, and therefore belongs to $\widetilde{F}$. Hence $\mathscr{H}_\varphi^F$ is a closed linear subspace of $\mathscr{H}_\varphi$ containing ξ_φ, whose dimension will be called the *F-multiplicity* of φ. The orthogonal projection of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\varphi^F$ will be denoted by p_φ^F.

LEMMA 2.1. *For φ in F, p_φ^F belongs to $\pi_\varphi(A)''$.*

Proof. For x in $\pi_\varphi(A)'$ and η in $\mathscr{H}_\varphi^F$, $\omega_\varphi^{x\eta}$ is dominated by $\|x\|^2\omega_\varphi^\eta$ and therefore belongs to $\mathscr{H}_\varphi^F$. Thus $\mathscr{H}_\varphi^F$ is $\pi_\varphi(A)'$-invariant, so $p_\varphi^F \in \pi_\varphi(A)''$.

It follows from Lemma 2.1 that for φ in F, $p_\varphi^F \pi_\varphi(A)'' p_\varphi^F$ is a von Neumann algebra. We shall say that φ is *F-abelian* if $p_\varphi^F \pi_\varphi(A)'' p_\varphi^F$ is abelian, and that F is *abelian* if every state in F is F-abelian.

PROPOSITION 2.2. *The set of F-abelian states is a σ-convex face of F.*

Proof. Suppose that ψ is a state dominated by $\sum_{n=1}^{\infty} \lambda_n \varphi_n$ where $\lambda_n \geqslant 0$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ and φ_n is F-abelian. Then π_ψ is unitarily equivalent to a subrepresentation of $\bigoplus_n \pi_{\varphi_n}$. By Lemma 2.1, $\mathscr{H}_\psi^F$ is a subspace of $\bigoplus_n \mathscr{H}_{\varphi_n}^F$, and $p_\psi^F \pi_\psi(A)'' p_\psi^F$ is a subalgebra of a restriction of $\bigoplus_n p_{\varphi_n}^F \pi_{\varphi_n}(A)'' p_{\varphi_n}^F$, which is abelian.

LEMMA 2.3. *For φ in F, the following are equivalent:*

(i) *φ has F-multiplicity* 1.

(ii) *φ is pure and F-abelian.*

(iii) *φ is pure and F contains no other state equivalent to φ.*

Proof. Suppose φ is of F-multiplicity 1. Lemma 2.1 shows that for x in $\pi_\varphi(A)'$ $x\xi_\varphi = \lambda\xi_\varphi$ for some scalar λ. Since ξ_φ is separating for $\pi_\varphi(A)'$, $x = \lambda 1_{\mathscr{H}_\varphi}$, so $\pi_\varphi(A)'$ is one-dimensional, and φ is pure.

Now suppose φ is pure. Then $p_\varphi^F \pi_\varphi(A)'' p_\varphi^F$ contains all bounded linear operators on $\mathscr{H}_\varphi^F$, and the equivalence of (i) and (ii) follows immediately. Also, for any unit vector η in $\mathscr{H}_\varphi^F$ other than a scalar multiple of ξ_φ, there is an operator a in A with $\|\pi_\varphi(a)\eta\| < 1/2$ and $\|\pi_\varphi(a)\xi_\varphi\| > 1/2$. Hence ω_φ^η is distinct from φ. The equivalence of (i) and (iii) follows.

In Lemma 2.4 and the proof of Theorem 2.5, we shall consider a Radon probability measure μ on $S(A)$ representing a state φ in the sense that

$$\int \psi(a)\,\mathrm{d}\mu(\psi) = \varphi(a) \qquad (a \in A).$$

Here μ is to be regarded only as a Baire measure, and the phrase "μ-a.e." is to be interpreted as "except on a μ-null Baire set". Similarly the statement that a function f defined μ-a.e. is μ-integrable indicates that f is Baire measurable. For the general theory of representing measures and Choquet simplexes, the reader is referred to [1].

LEMMA 2.4. *Let μ be a Baire probability measure on $S(A)$ representing a state φ of A, η be a vector in $\mathscr{H}_\varphi$ and a_n be a sequence in A such that $\sum_n \|\pi_\varphi(a_n)\xi_\varphi - \eta\| < \infty$. Then $\pi_\psi(a_n)\xi_\psi$ converges μ-a.e. to a vector η_ψ in $\mathscr{H}_\psi$. For b in A, $\psi \to \omega_\psi^{\eta_\psi}(b)$ is μ-integrable and*

$$\int \omega_\psi^{\eta_\psi}(b)\,\mathrm{d}\mu(\psi) = \omega_\varphi^\eta(b).$$

If a_n' is another sequence in A such that $\sum_n \|\pi_\varphi(a_n')\xi_\varphi - \eta\| < \infty$, then $\pi_\psi(a_n')\xi_\psi$ converges to η_ψ μ-a.e.. In particular, if $\eta = \pi_\varphi(b)\,\xi_\varphi$ for some b in A, then $\eta_\psi = \pi_\psi(b)\xi_\psi$ μ-a.e..

Proof. Let f_n be the non-negative continuous function on $S(A)$ defined by

$$f_n(\psi) = \|\pi_\psi(a_n - a_{n+1})\,\xi_\psi\| = [\psi((a_n - a_{n+1})^*(a_n - a_{n+1}))]^{\frac{1}{2}}.$$

Then

$$\int f_n(\psi)^2\,\mathrm{d}\mu(\psi) = \int \psi((a_n - a_{n+1})^*(a_n - a_{n+1}))\,\mathrm{d}\mu(\psi) =$$
$$= \varphi((a_n - a_{n+1})^*(a_n - a_{n+1})) =$$
$$= \|\pi_\varphi(a_n - a_{n+1})\xi_\varphi\|^2.$$

Hence $\sum f_n$ is an absolutely convergent series in $L^2(\mu)$, and therefore in $L^1(\mu)$, so $\sum f_n(\psi) < \infty$ μ-a.e.. For any such ψ, $\pi_\psi(a_n)\xi_\psi$ converges to some limit η_ψ in $\mathscr{H}_\psi$.

Let $g_n(\psi) = \|\pi_\psi(ba_n)\xi_\psi\|$. Then $|g_n - g_{n+1}| \leqslant \|b\| f_n$, so g_n is a Cauchy sequence in $L^2(\mu)$ converging pointwise to $\omega_\psi^{\eta_\psi}(b^*b)^{\frac{1}{2}}$. Thus this function is square-integrable, and

$$\int \omega_\psi^{\eta_\psi}(b^*b)\,\mathrm{d}\mu(\psi) = \lim_n \int g_n(\psi)^2\,\mathrm{d}\mu(\psi) =$$
$$= \lim_n \varphi(a_n^* b^* b a_n) = \omega_\varphi^\eta(b^*b).$$

Since A is spanned by its positive part, it follows that

$$\int \omega_{\psi}^{\eta_{\psi}}(b)\, \mathrm{d}\mu(\psi) = \omega_{\varphi}^{\eta}(b) \qquad (b \in A).$$

The same argument as in the first part of the proof shows that $\|\pi_{\psi}(a_n - a_n')\xi_{\psi}\| \to 0$ μ-a.e., so $\pi_{\psi}(a_n')\xi_{\psi} \to \eta_{\psi}$ μ-a.e.. The last statement follows on taking $a_n' = b$.

We are now ready to state the main theorem of this section.

THEOREM 2.5. *Let F be a closed face of $S(A)$, and consider the following statements:*

(i) *F is abelian.*

(ii) *For each φ in F, $\pi_{\varphi}(A)'$ is abelian.*

(iii) *F is a Choquet simplex.*

(iv) *Any factorial state in F is pure.*

(v) *No two distinct pure states in F are equivalent.*

(vi) *F has the 1-ball property.*

(vii) *Every pure state in F has F-multiplicity* 1.

The following implications are valid:

(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii).

If F is a G_δ-set in $S(A)$ (in particular if A is separable), then conditions (i) *to* (vii) *are equivalent.*

Proof. (i) $\Rightarrow$ (ii). Since $\pi_{\varphi}(A)' = (\pi_{\varphi}(A) \cup \{p_{\varphi}^F\})'$ by Lemma 2.1, this follows immediately from [26, Theorem 1.1].

(ii) $\Leftrightarrow$ (iii). It is known that φ is represented by a unique maximal measure on $S(A)$ (and hence on F) if and only if $\pi_{\phi}(A)'$ is abelian [6, Theorem 4.2.4].

(iii) $\Rightarrow$ (iv). If $\pi_{\varphi}(A)'$ is abelian and a factor, then it is one-dimensional.

(iv) $\Rightarrow$ (v). If F contains distinct equivalent pure states φ and ψ, then $\frac{1}{2}(\varphi + \psi)$ is a factorial state in F which is not pure [6, Proposition 2.4.26].

(v) $\Leftrightarrow$ (vi). This follows immediately from [3, Corollary 3.4].

(vi) $\Leftrightarrow$ (vii). This follows immediately from Lemma 2.3.

Now suppose that φ is a state in F, a and a' are operators in the unit ball of A, and η is a vector in $\mathscr{H}_{\varphi}^F$. Let μ be any Baire probability measure on F representing φ which is pseudo-carried by the pure states in F, in the sense that any Baire subset of F containing no pure states is μ-null. For example μ might by any maximal measure on F [1, Corollary I.4.12] or the measure associated with a maximal abelian von

Neumann subalgebra of $\pi_\varphi(A)'$ [26; 28, § 3.1]. Let b_n be a sequence in A such that $\|\pi_\varphi(b_n)\xi_\varphi - \eta\| < 2^{-n-1}$. By Lemma 2.1, there are operators a_n in A such that

$$\|\pi_\varphi(a_n a b_n)\,\xi_\varphi - p_\varphi^F \pi_\varphi(a)\eta\| < 2^{-n},$$

$$\|\pi_\varphi(a_n a' b_n)\,\xi_\varphi - p_\varphi^F\,\pi_\varphi(a')\,\eta\| < 2^{-n},$$

$$\|\pi_\varphi(a_n^*)\,\xi_\varphi - \xi_\varphi\| < 2^{-n}.$$

By Lemma 2.4, there is a Baire subset E of F with $\mu(E) = 1$ such that for ψ in E, $\pi_\psi(b_n)\xi_\psi$, $\pi_\psi(a_n a b_n)\xi_\psi$ and $\pi_\psi(a_n a' b_n)\xi_\psi$ converge to some limits η_ψ, ζ_ψ and ζ_ψ' respectively in $\mathscr{H}_\psi$ and $\pi_\psi(a_n^*)\xi_\psi$ converges to ξ_ψ.

Next suppose that F is a G_δ in $S(A)$, and for an integer n, let

$$Q_n = \{\rho \in (A^*)^+ : \|\rho\| \leqslant n\}\,.$$

Then $\widetilde{F} \cap Q_n$ is a closed face and a G_δ-subset of Q_n. Let θ be the mapping of E into $(A^*)^+$ defined by $\theta(\psi) = \omega_\psi^{\eta_\psi}$. Since θ is the pointwise weak* limit of the sequence of (weak*) continuous mappings $\psi \to \psi(b_m^* \cdot b_m)$, $\theta^{-1}(Q_n)$ is a Baire subset of F. and θ is a Baire measurable function of $\theta^{-1}(Q_n)$ into Q_n. Hence $\theta^{-1}(\widetilde{F})$ is a Baire, subset of F. By Lemma 2.4, for any b in A,

$$\int_{\theta^{-1}(Q_n)} \theta(\psi)\,(b)\,\mathrm{d}\mu(\psi) \leqslant \int_E \omega_\psi^{\eta_\psi}\,(b)\mathrm{d}\mu(\psi) = \omega_\varphi^\eta(b).$$

Thus the barycentre of the restriction ν_n to Q_n of the image of μ under θ is dominated by ω_φ^η and therefore belongs to the closed face $\widetilde{F} \cap Q_n$ of Q_n. Since ν_n is a Baire measure, it is carried by the Baire set $\widetilde{F} \cap Q_n$. Hence μ is carried by $\theta^{-1}(\widetilde{F})$, so $\eta_\psi \in \mathscr{H}_\psi^F$ μ-a.e. .

Applying similar arguments to ζ_ψ and ζ_ψ', we can find a Baire subset E_1 of F, carrying μ and contained in E, such that for ψ in E_1, η_ψ, ζ_ψ and ζ_ψ' all belong to $\mathscr{H}_\psi^F$. In particular if ψ in E_1 has F-multiplicity 1, then

$$\eta_\psi = \langle \eta_\psi\,,\ \xi_\psi \rangle\, \xi_\psi\,,$$

$$\zeta_\psi = \langle \zeta_\psi\,, \xi_\psi \rangle \xi_\psi = \lim \psi(a_n a b_n)\,\xi_\psi\,,$$

$$p_\psi^F \pi_\psi(a)\,\eta_\psi = \langle \pi_\psi(a)\eta_\psi\,,\ \xi_\psi \rangle\, \xi_\psi = \lim \psi(a b_n)\,\xi_\psi\,.$$

But

$$|\psi(a b_n - a_n a b_n)| \leqslant \|\pi_\psi(a_n^*)\,\xi_\psi - \xi_\psi\|\,\|a\|\,\|\pi_\psi(b_n)\,\xi_\psi\| \to 0 \qquad \text{as } n \to \infty.$$

Thus

$$\zeta_\psi = p_\psi^F \pi_\psi(a)\,\eta_\psi = \psi(a)\,\langle \eta_\psi\,,\ \xi_\psi \rangle\, \xi_\psi\,,$$

and similarly

$$\zeta'_\psi = \psi(a')\,\langle \eta_\psi,\ \xi_\psi\rangle\,\xi_\psi.$$

Hence

$$\langle \pi_\psi(a')\zeta_\psi, \eta_\psi\rangle = \psi(a)\psi(a')|\langle \eta_\psi, \xi_\psi\rangle|^2 = \langle \pi_\psi(a)\,\zeta'_\psi, \eta_\psi\rangle.$$

Finally suppose also that condition (vii) is valid. Then

$$\{\psi \in E_1 : \langle \pi_\psi(a')\,\zeta_\psi, \eta_\psi\rangle \neq \langle \pi_\psi(a)\,\zeta'_\psi, \eta_\psi\rangle\}$$

is a Baire subset of F containing no pure states, and is therefore μ-null. By Lemma 2.4 and standard polarisation identities,

$$\langle \pi_\varphi(a')\,p^F_\varphi \pi_\varphi(a)\eta, \eta\rangle = \int_F \langle \pi_\psi(a')\,\zeta_\psi, \eta_\psi\rangle\, \mathrm{d}\mu(\psi) =$$

$$= \int_F \langle \pi_\psi(a)\,\zeta'_\psi,\ \eta_\psi\rangle\, \mathrm{d}\mu(\psi) = \langle \pi_\varphi(a)\,p^F_\varphi \pi_\varphi(a')\eta, \eta\rangle.$$

Thus φ is F-abelian.

When A is separable, Theorem 2.5 can alternatively be proved by means of a direct integral decomposition with respect to the measure associated with a maximal abelian von Neumann subalgebra of $\pi_\varphi(A)'$ (cf. [7]). It seems likely that the conditions (i) to (vii) are equivalent for any closed face, even if A is non-separable.

3. INEQUIVALENT PURE STATES

Any Bauer simplex (Choquet simplex whose extreme boundary is closed) is affinely homeomorphic to the state space of the commutative C^*-algebra of continuous functions on its extreme boundary, and conversely the state space of a C^*-algebra A is a Choquet simplex only if A is commutative, in which case $S(A)$ is a Bauer simplex. In this section, we shall consider the general problem of finding faces of $S(A)$ which are simplexes. In view of condition (v) of Theorem 2.5, we must consider sets of inequivalent pure states of A. The following proposition extends one half of [3, Corollary 3.4].

PROPOSITION 3.1. *Let P be a set of pairwise inequivalent pure states of A. Then the convex hull* co$P$, *and the σ-convex hull σ-co$P$, of P are faces of $S(A)$. If $\sum_{n=1}^{\infty} \lambda_n \varphi_n = 0$ or some real numbers λ_n with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ and distinct states φ_n in P, then $\lambda_n = 0$ for all n.*

Proof. Any state ψ in σ-co$P$ has the form

$$\psi = \sum_n \lambda_n \varphi_n$$

where λ_n is a finite or infinite sequence of positive real numbers with sum 1 and φ_n is a sequence of distinct states in P. Let $(\mathscr{H}, \pi) = \bigoplus_n (\mathscr{H}_{\varphi_n}, \pi_{\varphi_n})$. Since the representations π_{φ_n} are disjoint and irreducible, $\pi(A)' = \bigoplus_n \mathbf{C} \cdot 1_n$, where 1_n is the identity operator on $\mathscr{H}_{\varphi_n}$ [9, Proposition 5.2.4]. Let $\xi = \bigoplus_n \lambda_n^{\frac{1}{2}} \xi_{\varphi_n} \in \mathscr{H}$. Then the cyclic subspace $[\pi(A)\xi]$ associated with ξ satisfies

$$[\pi(A)\xi] = [\pi(A)''\xi] = \bigoplus_n \mathscr{H}_{\varphi_n} = \mathscr{H} ,$$

so ξ is cyclic for $\pi(A)$. Moreover $\langle \pi(a)\xi, \xi \rangle = \psi(a)$ $(a \in A)$, so $(\mathscr{H}_\psi, \pi_\psi, \xi_\psi)$ is unitarily equivalent to $(\mathscr{H}, \pi, \xi)$. Thus $\pi_\psi(A)'$ may be identified with $\bigoplus_n \mathbf{C} \cdot 1_n$ in such a way that

$$\Theta_\psi(\bigoplus_n \lambda_n' 1_n)(a) = \sum_n \lambda_n' \langle \pi_{\varphi_n}(a) \lambda_n^{\frac{1}{2}} \xi_{\varphi_n}, \lambda_n^{\frac{1}{2}} \xi_{\varphi_n} \rangle =$$

$$= \sum_n \lambda_n \lambda_n' \varphi_n(a).$$

Furthermore $\Theta_\psi(\bigoplus_n \lambda_n' 1_n)$ is a state if and only if $\lambda_n' \geqslant 0$ and $\sum_n \lambda_n \lambda_n' = 1$, in which case $\Theta_\psi(\bigoplus_n \lambda_n' 1_n) \in \sigma$-co$P$. Thus the face generated by any state in σ-co$P$ is contained in σ-co$P$, so σ-co$P$ is a face of $S(A)$. The same proof shows that co$P$ is a face.

Now suppose that $\sum_n \lambda_n \varphi_n = 0$ for some absolutely summable sequence λ_n and distinct φ_n in P. Let $P_1 = \{\varphi_n : \lambda_n > 0\}$ and $P_2 = \{\varphi_n : \lambda_n < 0\}$. Since

$$\sum_{\lambda_n > 0} \lambda_n \varphi_n = \sum_{\lambda_n < 0} (-\lambda_n) \varphi_n ,$$

P_1 is contained in the face generated by σ-coP_2. By the first part of the proof, σ-coP_2 is itself a face of $S(A)$, so P_1 is contained in σ-coP_2. Since P_1 consists of pure states, P_1 is contained in P_2. But P_1 is disjoint from P_2, so P_1 and similarly P_2 are empty.

Proposition 3.1 shows that one can construct n-dimensional simplexes as faces of $S(A)$ simply by taking the convex hulls of sets of $(n+1)$ pairwise inequivalent pure states. One might hope to construct infinite-dimensional Choquet simplexes by taking the (weak*) closed convex hulls $\overline{\text{co}}P$ of certain infinite sets P of pairwise inequivalent pure states. Effros [14] has shown that the norm-closure of a face of $S(A)$ is a face, but the weak* closure may not be. In Theorem 3.2 below, we obtain necessary and sufficient conditions for $\mathrm{co}P$ to be a face when P has a unique limit point, and the subsequent corollaries show that this special case is sufficient to include many examples both where $\overline{\mathrm{co}}P$ is not a face, and where it is a simplicial face with non-compact extreme boundary.

Let Ω be a compact Hausdorff space, $C(\Omega)$ be the commutative C^*-algebra of all continuous complex-valued functions on Ω, and $K(\Omega)$ be the state space $S(C(\Omega))$ of $C(\Omega)$ identified with the space of Radon probability measures on Ω. For ω in Ω and μ in $K(\Omega)$, let

$$C_{\mu\omega}(\Omega) = \left\{ f \in C(\Omega) : f(\omega) = \int f \, d\mu \right\}$$

and $K_{\mu\omega}(\Omega)$ be the state space of the order-unit space $C_{\mu\omega}(\Omega)$. If μ is the point mass ε_ω at ω, then $K_{\mu\omega}(\Omega)$ is the Bauer simplex $K(\Omega)$. Otherwise $K_{\mu\omega}(\Omega)$ is a Choquet simplex with extreme boundary $\{\theta(\varepsilon_{\omega'}) : \omega' \in \Omega,\ \omega' \neq \omega\}$ where θ is the restriction map from $C(\Omega)^*$ into $C_{\mu\omega}(\Omega)^*$. A typical point of $K_{\mu\omega}(\Omega)$ is uniquely of the form $\theta(\nu)$ where ν is a measure in $K(\Omega)$ for which ω is not an atom, and the unique maximal representing measure of $\theta(\nu)$ is the image of ν under the mapping $\omega' \to \theta(\varepsilon_{\omega'})$ (cf. [1, Proposition II.7.17]).

THEOREM 3.2. *Let P be a set of pairwise inequivalent pure states of A with a unique limit point φ_0. The weak* closed convex hull $\overline{\mathrm{co}}P$ of P coincides with the σ-convex hull of the weak* closure $\bar{P} = P \cup \{\varphi_0\}$ of P. Furthermore:*

(i) *If $\varphi_0 = \sum_{n=1}^{\infty} \lambda_n \varphi_n$ for some φ_n in P and $\lambda_n \geqslant 0$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, then $\overline{\mathrm{co}}P$ is a face of $S(A)$ affinely homeomorphic to $K_{\mu\varphi_0}(\bar{P})$ where $\mu = \sum_{n=1}^{\infty} \lambda_n \varepsilon_{\varphi_n}$.*

(ii) *If φ_0 is a pure state equivalent to no state in P, then $\overline{\mathrm{co}}P$ is a face of $S(A)$ affinely homeomorphic to $K(\bar{P})$.*

(iii) *If neither* (i) *nor* (ii) *applies, then $\overline{\mathrm{co}}P$ is not a face of $S(A)$.*

Proof. By Proposition 3.1 any state ψ in σ-co$\bar{P}$ can be written uniquely as $\sum_{\varphi \in \bar{P}} \lambda_\psi(\varphi)\varphi$ where λ_ψ is an absolutely summable function of $\bar{P}$ into $[0, 1]$ with $\sum_{\varphi \in \bar{P}} \lambda_\psi(\varphi) = 1$. Let ψ_γ be a net in σ-co$\bar{P}$ which is weak* convergent to a state ψ in $S(A)$. Passing to a subnet we may assume that λ_{ψ_γ} converges pointwise on P to a summable function λ with $\sum_{\varphi \in P} \lambda(\varphi) = \varkappa \leqslant 1$. Let $\psi' = \sum_{\varphi \in P} \lambda(\varphi)\varphi + (1 - \varkappa)\varphi_0 \in \sigma\text{-co}\bar{P}$. For any convex circled, weak* neighbourhood U of φ_0 in A^*,

$$\psi_\gamma - \psi' = \sum_{\bar{P}} \lambda_{\psi_\gamma}(\varphi)\varphi - \sum_{\bar{P}} \lambda(\varphi)\varphi - \sum_{\bar{P}} (\lambda_{\psi_\gamma}(\varphi) - \lambda(\varphi))\, \varphi_0 =$$

$$= \sum_{P \setminus U} (\lambda_{\psi_\gamma}(\varphi) - \lambda(\varphi))(\varphi - \varphi_0) + \sum_{\bar{P} \cap U} (\lambda_{\psi_\gamma}(\varphi) - \lambda(\varphi))(\varphi - \varphi_0).$$

The second sum lies in $2(U-\varphi_0)$ for each γ, while the first is (norm) convergent to 0 since $P\setminus U$ is finite. Thus ψ_γ is weak* convergent to ψ', so $\psi=\psi'$. This shows that $\bar{\mathrm{co}}P=\sigma\text{-co}\bar{P}$.

If the condition (i) is satisfied, it follows immediately that $\bar{\mathrm{co}}\,P=\sigma\text{-co}P$, which is a face of $S(A)$ by Proposition 3.1. Given f in $C_{\mu\varphi_0}(\bar{P})$, there is an affine function a on $\bar{\mathrm{co}}P$ given by

$$a(\psi)=\sum_P \lambda_\psi(\varphi)f(\varphi).$$

For φ in P, $a(\varphi)=f(\varphi)$, and

$$a(\varphi_0)=\sum_n \lambda_n f(\varphi_n)=\int f\,\mathrm{d}\mu=f(\varphi_0).$$

If ψ_γ, ψ and U are as in the first part of the proof, then

$$a(\psi_\gamma)-a(\psi)=\sum_P \lambda_{\psi_\gamma}(\varphi)f(\varphi)-\sum_P \lambda(\varphi)f(\varphi)-\sum_P(\lambda_{\psi_\gamma}(\varphi)-\lambda(\varphi))f(\varphi_0)=$$

$$=\sum_{P\setminus U}(\lambda_{\psi_\gamma}(\varphi)-\lambda(\varphi))(f(\varphi)-f(\varphi_0))+$$

$$+\sum_{P\cap U}(\lambda_{\psi_\gamma}(\varphi)-\lambda(\varphi))(f(\varphi)-f(\varphi_0)).$$

Since f is continuous, U may be chosen to ensure that this expression is small for all large γ. Thus a is continuous. It follows that the restriction mapping is an isometry of the space of continuous affine functions on $\bar{\mathrm{co}}P$ onto $C_{\mu\varphi_0}(\bar{P})$, and hence that $\bar{\mathrm{co}}P$ is affinely homeomorphic to $K_{\mu\varphi_0}(P)$.

If the condition of (ii) is satisfied, then we can replace P by $\bar{P}$ and apply part (i) with $\mu=\varepsilon_{\varphi_0}$. Thus $\bar{\mathrm{co}}P=\mathrm{co}\bar{P}$ is a face affinely homeomorphic to $K_{\mu\varphi_0}(\bar{P})=K(\bar{P})$.

Now suppose that $\bar{\mathrm{co}}P$ is a face, but neither the assumption of (i) nor (ii) is satisfied. If $\varphi_0=1/2\,(\psi+\psi')$ for some ψ and ψ' in $S(A)$, then ψ and ψ' belong to $\bar{\mathrm{co}}P=\sigma\text{-co}\bar{P}$, so

$$\psi=\sum_{n=0}^{\infty}\lambda_n\varphi_n \qquad \psi'=\sum_{n=0}^{\infty}\lambda_n'\varphi_n$$

for some φ_n in $P\,(n>0)$ and non-negative scalars λ_n and $\lambda_n'\,(n\geqslant 0)$ with $\sum_{n=0}^{\infty}\lambda_n=\sum_{n=0}^{\infty}\lambda_n'=1$. Then

$$\left(1-1/2\,\lambda_0-1/2\,\lambda_0'\right)\varphi_0=\sum_{n=1}^{\infty}1/2\,(\lambda_n+\lambda_n')\,\varphi_n.$$

Since (i) does not apply, $\lambda_0=\lambda_0'=1$, so $\psi=\psi'=\varphi_0$. Thus φ_0 is a pure state. Since (ii) does not apply, φ_0 is equivalent to some φ_0' in P. Since (i) does not apply, φ_0

is distinct from φ_0'. The set $P_1 = P\setminus\{\varphi_0'\}$ has unique limit point φ_0 and satisfies the condition of part (ii). By the first part of the proof, $\overline{\overline{\text{co}}}P_1 = \sigma\text{-co}\overline{P_1}$ which is a face of $S(A)$, where $\overline{P_1}= P_1 \cup \{\varphi_0\}$. Since φ_0' is pure and does not belong to $\overline{P_1}$, it does not belong to $\sigma\text{-co}\overline{P_1}$. Since $\overline{\overline{\text{co}}}P$ is the convex hull of the face $\sigma\text{-co}\overline{P_1}$ and the pure state φ_0', it is easy to see that the convex hull of φ_0 and φ_0' is a face of $\text{co}P$ and hence of $S(A)$. It follows from [3, Corollary 3.4] that φ_0 and φ_0' are inequivalent, which contradicts the choice of φ_0'.

The referee has pointed out that if the weak* closure of P is countable and contained in the norm-closed convex hull $\overline{\overline{\text{co}}}P$ of P, then $\overline{\overline{\text{co}}}P$ coincides with $\overline{\text{co}}P$, and is therefore a face of $S(A)$ (by Proposition 3.1 and [14, Corollary 4.7]).

The author is also indebted to the referee for observing that the following two corollaries of Theorem 3.2 are valid for the given class of C^*-algebras rather than just UHF-algebras.

COROLLARY 3.3. *Suppose A contains a maximal ideal M such that A/M is separable and infinite-dimensional. Then there is a sequence of pairwise inequivalent pure states of A whose (weak*) closed convex hull is not a face of $S(A)$.*

Proof. Passing to the quotient A/M whose state space may be identified with a closed split face of $S(A)$, we may assume that A is simple, separable and infinite-dimensional. Then any one equivalence class of pure states is dense in $S(A)$ ([15], Theorem 2; [9], Lemme 11.2.1), and there are infinitely many such equivalence classes. (Otherwise A is of type I [16], and therefore has a composition series consisting of C^*-algebras with continuous trace [9, Théorème 4.5.5]. Being simple, A is then of continuous trace, and being unital, A has finite-dimensional representations, which are necessarily faithful.) Since $S(A)$ is metrisable, given any two distinct states ψ_1 and ψ_2 of A and a in A with $\psi_1(a) < \psi_2(a)$, it is possible to find inductively a sequence of pairwise inequivalent pure states φ_n which converge to $1/2\,(\psi_1 + \psi_2)$ but satisfy $\varphi_n(a) < 1/2(\psi_1(a) + \psi_2(a))$. It follows immediately from Theorem 3.2(iii) that $\overline{\text{co}}\,\{\varphi_n\}$ is not a face of $S(A)$.

COROLLARY 3.4. *Suppose A contains a maximal ideal M such that A/M is separable and infinite-dimensional, and let μ be any probability measure on the one-point compactification $\overline{\mathbf{N}}$ of $\mathbf{N}$. Then there is a face of $S(A)$ affinely homeomorphic to $K_{\mu\infty}(\mathbf{N})$.*

Proof. Suppose $\mu = \sum_{n\in\overline{\mathbf{N}}} \mu_n\varepsilon_n$ where $\mu_n \geqslant 0$, $\sum_{n\in\overline{\mathbf{N}}} \mu_n = 1$. If $\mu_\infty = 1$, then as in Corollary 3.3, we can find inequivalent pure states φ_n ($n\in \overline{\mathbf{N}}$) such that φ_n converges to φ_∞ as $n \to \infty$. Otherwise, again as in Corollary 3.3, we can find inequivalent pure states φ_n $(n \in \mathbf{N})$ such that

$$\left(\sum_{j=1}^{n} \mu_j\right) \varphi_{n+1} - \sum_{j=1}^{n} \mu_j \varphi_j \to 0 ,$$

so φ_n converges to $\varphi_\infty = (1-\mu)^{-1} \sum_{j=1}^{\infty} \mu_j \varphi_j$. In both cases, the mapping $n \to \varphi_n$ $(n \in \bar{\mathbf{N}})$ is a homeomorphism of $\bar{\mathbf{N}}$ onto $\bar{P}$, where $P = \{\varphi_n : n \in \mathbf{N}\}$, and by Theorem 3.2, $\overline{\text{co}}\, P$ is a face affinely homeomorphic to $K_{\mu\infty}(\bar{\mathbf{N}})$.

It is tempting to conjecture that every (metrisable) Choquet simplex is affinely homeomorphic to a face of the state space of some (separable) C^*-algebra. There is a metrisable simplex whose extreme boundary is dense [22], and this simplex is unique and contains every other metrisable simplex as a face (up to affine homeomorphism) [20]. Thus the separable version of the conjecture is equivalent to the existence of a C^*-algebra A and a closed face F of $S(A)$ whose extreme points are inequivalent pure states and are dense in F (by Theorem 2.5).

The referee has further conjectured that if A is any order-unit space such that A^{**} is isomorphic (as an order-unit space) to the self-adjoint part of a von Neumann algebra, then the state space of A is affinely homeomorphic to a face of the state space of a C^*-algebra.

4. G-ABELIAN C^*-ALGEBRAS

Now suppose that there is a group G acting on the C^*-algebra A via a homomorphism α of G into the group of $*$-automorphisms of A. The set $S_G(A)$ of G-invariant states of A is a compact convex subset of $S(A)$, and its extreme points are known as *ergodic* states. For φ in $S_G(A)$, there is a unitary representation u_φ of G on $\mathscr{H}_\varphi$ uniquely determined by the condition $u_\varphi(g)\,\xi_\varphi = \xi_\varphi$ and the covariance relation

$$u_\varphi(g)\, \pi_\varphi(a) u_\varphi(g^{-1}) = \pi_\varphi(\alpha(g)(a)) .$$

Let $\mathscr{H}_\varphi^G$ be the set of all u_φ-invariant vectors in $\mathscr{H}_\varphi$, and p_φ^G be the projection of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\varphi^G$. Then $p_\varphi^G \pi_\varphi(A)'' p_\varphi^G$ is a von Neumann algebra [13, Corollary 2], and φ is said to be *G-abelian* if $p_\varphi^G \pi_\varphi(A)'' p_\varphi^G$ is abelian. The C^*-algebra A is said to be *G-abelian* if every G-invariant state is G-abelian.

LEMMA 4.1. *For φ in $S_G(A)$, $\mathscr{H}_\varphi^G$ is the largest closed linear subspace $\mathscr{K}$ of $\mathscr{H}_\varphi$ containing ξ_φ such that ω_φ^η is G-invariant for all η in $\mathscr{K}$.*

Proof. It is clear that $\mathscr{H}_\varphi^G$ is a closed linear subspace containing ξ_φ, and that ω_φ^η is G-invariant for η in $\mathscr{H}_\varphi^G$. Conversely if $\mathscr{K}$ is such a subspace, then for a in A,

g in G, η in $\mathscr{H}$ and scalar λ,

$$\langle \pi_\varphi(a)\, u_\varphi(g)\,(\eta + \lambda\xi_\varphi),\ u_\varphi(g)(\eta + \lambda\xi_\varphi)\rangle = \omega_\varphi^{\eta+\lambda\xi_\varphi}(\alpha(g^{-1})(a)) =$$

$$= \omega_\varphi^{\eta+\lambda\xi_\varphi}(a) = \langle \pi_\varphi(a)(\eta + \lambda\xi_\varphi), \eta + \lambda\xi_\varphi\rangle\,.$$

Comparing coefficients of $\bar\lambda$ gives

$$\langle \pi_\varphi(a)\, u_\varphi(g)\eta,\ \xi_\varphi\rangle = \langle \pi_\varphi(a)\eta, \xi_\varphi\rangle\,.$$

Since ξ_φ is cyclic, it follows that η is u_φ-invariant.

It is known [6, Theorem 4.3.22] that $\mathscr{H}_\varphi^G$ is one-dimensional if and only if φ is *weakly clustering* in the sense that

$$\inf\{|\varphi(a'b) - \varphi(a)\varphi(b)| : a' \in \mathrm{co}\alpha(G)(a)\} = 0$$

for any a and b in A.

As in [32] or [21, § 7.6] we can form the crossed product of the discrete system (A, G, α). This is a C^*-algebra $G \times A$ generated by $\sigma(A)$ and $\theta(G)$, where σ is a faithful *-homomorphism of A into $G \times A$ and θ is a unitary representation of G in $G \times A$ related by the covariance formula

$$\theta(g)\,\sigma(a)\theta(g^{-1}) = \sigma(\alpha(g)\,(a)).$$

There is a bijective correspondence between covariant representations of (A, G, α) and non-degenerate representations of $G \times A$, given by $(\mathscr{H}, \pi, u) \to (\mathscr{H}, \pi \times u)$ where

$$(\pi \times u)(\sigma(a)) = \pi(a),$$

$$(\pi \times u)(\theta(g)) = u(g)\,.$$

There is also a bijective correspondence between the state space of $G \times A$ and the set of all bounded functions Φ from G into A^* such that $\Phi(e)$ is a state (where e is the identity of G) and Φ is positive-definite in the sense that

$$\sum_{i,j=1}^{n} \Phi(g_i^{-1}g_j)\,(\alpha(g_i^{-1})\,(a_i^* a_j)) \geqslant 0$$

for any a_i in A and g_i in G $(1 \leqslant i \leqslant n)$. This correspondence is given by $\rho \to \Phi_\rho$ where

$$\Phi_\rho(g)(a) = \rho(\sigma(a)\theta(g))$$

and is a homeomorphism for the weak* topology on $S(G \times A)$ and the topology of pointwise weak* convergence on the space of functions from G into A^*.

The following theorem is due to the referee.

THEOREM 4.2. *Let $F = F_G(A)$ be the set of states ρ of $G \times A$ satisfying:*

$$\rho(\theta(g_1)\sigma(a)\theta(g_2)) = \rho(a) \quad (g_1, g_2 \in G, a \in A).$$

Then F is a closed face of $S(G \times A)$, the mapping $\rho \to \Phi_\rho(e)$ is an affine homeomorphism of F onto $S_G(A)$, and for ρ in F, $(\mathscr{H}_\rho, \mathscr{H}_\rho^F, \pi_\rho, \xi_\rho)$ is unitarily equivalent to $(\mathscr{H}_\varphi, \mathscr{H}_\varphi^G, \pi_\varphi \times u_\varphi, \xi_\varphi)$, where $\varphi = \Phi_\rho(e)$.

Proof. For ρ in F, g and h in G and a in A,

$$\Phi_\rho(g)(\alpha(h)(a)) = \rho(\theta(h)\,\sigma(a)\theta(h^{-1}g)) = \rho(a) = \Phi_\rho(e)\,(a).$$

Thus $\Phi_\rho(g) = \Phi_\rho(e) \in S_G(A)$, so $\rho \to \Phi_\rho(e)$ is an affine homeomorphism of F into $S_G(A)$. Given a state φ in $S_G(A)$, the function $\Phi(g) = \varphi$ is easily seen to be positive--definite, so $\Phi = \Phi_\rho$ for some ρ in $S(G \times A)$. Now

$$\rho(\theta(g_1)\sigma(a)\theta(g_2)) = \Phi(g_1^{-1}g_2)(\alpha(g_1)(a)) = \varphi(a) = \rho(\sigma(a)),$$

so ρ belongs to F, and $\Phi_\rho(e) = \varphi$.

The vector ξ_φ is clearly cyclic for $(\pi_\varphi \times u_\varphi)(G \times A) \supset \pi_\varphi(A)$, and

$$\langle(\pi_\varphi \times u_\varphi)(\sigma(a)\theta(g))\,\xi_\varphi, \xi_\varphi\rangle = \langle\pi_\varphi(a)u_\varphi(g)\xi_\varphi, \xi_\varphi\rangle =$$
$$= \varphi(a) = \rho(\sigma(a)\theta(g)).$$

Hence $(\mathscr{H}_\rho, \pi_\rho, \xi_\rho)$ is unitarily equivalent to $(\mathscr{H}_\varphi, \pi_\varphi \times u_\varphi, \xi_\varphi)$.

For a unit vector η in $\mathscr{H}_\varphi$,

$$\langle(\pi_\varphi \times u_\varphi)\,(\theta(g_1)\sigma(a)\theta(g_2))\eta, \eta\rangle = \langle u_\varphi(g_1)\,\pi_\varphi(a)\,u_\varphi(g_2)\eta, \eta\rangle.$$

Thus the vector state determined by η in $\mathscr{H}_\varphi^G$ belongs to F. Taking $\eta = x\xi_\varphi$ for some x in $\pi_\rho(G \times A)'$, it follows that F is a face of $S(G \times A)$. Conversely if the vector state determined by η in $\mathscr{H}_\varphi$ belongs to F, then on taking $g_2 = g_1^{-1}$ it follows that ω_φ^η is G-invariant. Finally it follows from Lemma 4.1 that the unitary equivalence takes $\mathscr{H}_\rho^F$ onto $\mathscr{H}_\varphi^G$.

Two G-invariant states φ and ψ of A are said to be *covariantly equivalent* if $(\mathscr{H}_\varphi, \pi_\varphi, u_\varphi)$ and $(\mathscr{H}_\psi, \pi_\psi, u_\psi)$ are unitarily equivalent. It is clear from Theorem 4.2 that states in $S_G(A)$ are covariantly equivalent if and only if the corresponding states in $F_G(A)$ are equivalent.

COROLLARY 4.3. *The convex and σ-convex hulls of any set of covariantly inequivalent ergodic states are faces of $S_G(A)$. The face generated by two covariantly* *equivalent ergodic states is affinely homeomorphic to a 3-dimensional Euclidean ball. In particular, $S_G(A)$ has the 3-ball property.*

Proof. This follows immediately from Theorem 4.2, Proposition 3.1 and [3, Corollary 3.4].

COROLLARY 4.4. *Consider the following conditions on (A, G, α):*

(i) *A is G-abelian.*

(ii) *For each φ in $S_G(A)$, $\pi_\varphi(A)' \cap u_\phi(G)'$ is abelian.*

(iii) *$S_G(A)$ is a Choquet simplex.*

(iv) *Any state φ in $S_G(A)$ for which $\pi_\varphi(A)' \cap u_\varphi(G)'$ is a factor is ergodic.*

(v) *No two distinct ergodic states are covariantly equivalent.*

(vi) *$S_G(A)$ has the 1-ball property.*

(vii) *Every ergodic state is weakly clustering.*

The following implications are valid:

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Leftrightarrow \text{(vi)} \Leftrightarrow \text{(vii)}.$$

If A is separable, then all seven conditions are equivalent.

Proof. In the general case, the implication (iii) $\Rightarrow$ (i) is proved in [8, Theorem 1] and [6, Corollary 4.3.11], and the remaining implications follow immediately when Theorem 2.5 is applied to the face $F_G(A)$ of $S(G \times A)$ considered in Theorem 4.2.

If A is separable, then $\alpha(G)$ is also separable in the strong operator topology. Furthermore replacing G by a countable dense subgroup of $\alpha(G)$ (and α by the identity representation) does not affect the validity of any of the conditions (i) to (vii). Thus we may assume that G is countable, so that $G \times A$ is separable. Thus the equivalence of all the conditions again follows from Theorem 2.5.

For the most part, Corollary 4.4 is not original, although the equivalence of (vi) with (v) and (vii), and the equivalence of (iv) in separable cases, appear to be new. Various parts of the rest of Corollary 4.4 may be found in [6], [7], [8], [19], [25], [28].

It seems likely that the conditions of Corollary 4.4 are all equivalent in general. In fact if A is non-separable but G is σ-compact this can be proved by amending the proof of (vii) $\Rightarrow$ (i) in Theorem 2.5. One now takes μ to be a maximal measure on $F = F_G(A)$, and one can show that for any fixed g in G,

$$u_{\Phi_\psi(e)}(g)\,\eta_\psi = \eta_\psi \quad \mu\text{-a.e.}(\psi)$$

(cf. the proof of Theorem 5.1 below). Since Haar measure on G is σ-finite, an application of Fubini's theorem shows that $\eta_\psi \in \mathscr{H}_\psi^F$ μ-a.e.. The proof is now completed as in Theorem 2.5.

If F_0 is a closed face of $S_G(A)$, then $F_1 = \{\rho \in F_G(A) : \Phi_\rho(e) \in F_0\}$ is a closed face of $S(G \times A)$. Identifying $\mathscr{H}_\rho$ with $\mathscr{H}_\varphi$ as in Theorem 4.2, it is easily seen that

$$\mathscr{H}_\rho^{F_1} = \{\eta \in \mathscr{H}_\varphi^G : \omega_\varphi^\eta \in \widetilde{F}_0\}.$$

Now Theorem 2.5 can be applied to F_1 to obtain seven conditions on F_0 and the covariant representations associated with states in F_0, all of which are equivalent, at least if A is separable.

5. GROUND STATES

Now suppose that $G = \mathbf{R}$, and that $\{\alpha(t) : t \in \mathbf{R}\}$ is a strongly continuous one-parameter group of *-automorphisms of A, whose generator δ is a *-derivation of A defined on a dense *-subalgebra $\mathscr{D}(\delta)$ of A. A state φ of A is said to be a *ground state* if $-i\varphi(a^*\delta(a)) \geqslant 0$ for all a in $\mathscr{D}(\delta)$. The compact convex subset of $S(A)$ consisting of all ground states will be denoted by $S_0(A, \alpha)$ or simply S_0.

The above definition of ground states is the one adopted in [6], [29], [30], and various equivalent ones are known [6], [21], [23]. For example, φ is a ground state if and only if φ is $\mathbf{R}$-invariant and the strongly continuous unitary group u_φ is generated by ih_φ where h_φ is a positive (unbounded) operator on $\mathscr{H}_\varphi$, in which case h_φ is affiliated with $\pi_\varphi(A)''$ [6, Proposition 5.3.19]. Now for φ in S_0, η in $\mathscr{H}_\varphi^{\mathbf{R}}$ and a in $\mathscr{D}(\delta)$, $\pi_\varphi(a)\eta$ belongs to $\mathscr{D}(h_\varphi)$ and $ih_\varphi\pi_\varphi(a)\eta = \pi_\varphi(\delta(a))\eta$. Hence

$$-i\omega_\varphi^\eta(a^*\delta(a)) = \langle h_\varphi\pi_\varphi(a)\eta, \pi_\varphi(a)\eta\rangle \geqslant 0 .$$

In particular for x in $\pi_\varphi(A)' \subset u_\varphi(\mathbf{R})'$, $x\xi_\varphi$ belongs to $\mathscr{H}_\varphi^{\mathbf{R}}$ and $\Theta_\varphi(x^*x) = \omega_\varphi^{x\xi_\varphi} \in \widetilde{S}_0$. Thus S_0 is a face of $S(A)$, and the above argument shows that $\mathscr{H}_\varphi^{S_0}$ contains $\mathscr{H}_\varphi^{\mathbf{R}}$. On the other hand, Lemma 4.1 shows that $\mathscr{H}_\varphi^{\mathbf{R}}$ contains $\mathscr{H}_\varphi^{S_0}$.

THEOREM 5.1. *Let $\{\alpha(t) : t \in \mathbf{R}\}$ be a strongly continuous one-parameter group of *-automorphisms of A, and S_0 be the set of all ground states of A. The following are equivalent:*

(i) *S_0 is abelian.*

(i)′ *Every ground state is $\mathbf{R}$-abelian.*

(ii) *For each φ in S_0, $\pi_\varphi(A)'$ is abelian.*

(iii) *S_0 is a Choquet simplex.*

(iv) *Every factorial ground state is a pure state.*

(v) *No two distinct pure ground states are equivalent.*

(vi) *S_0 has the 1-ball property.*

(vii) *Every pure ground state has S_0-multiplicity 1.*

(vii)′ *Every pure ground state is weakly clustering.*

Proof. The equivalence of (i) with (i)′ and of (vii) with (vii)′ follow from the fact that $\mathscr{H}_\varphi^{\mathbf{R}} = \mathscr{H}_\varphi^{S_0}$, together with [6, Theorem 4.3.22]. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) follow from Theorem 2.5. To prove

(vii) $\Rightarrow$ (i) it suffices to observe that the corresponding proof in Theorem 2.5 applies verbatim except for the section showing that $\eta_\psi \in \mathscr{H}_\psi^{S_0}$ μ-a.e. . However we now have, for any rational t,

$$\|\pi_\varphi(\alpha(t)(b_n))\,\xi_\varphi - \eta\| = \|u_\varphi(t)\,(\pi_\varphi(b_n)\,\xi_\varphi - \eta)\| < 2^{-n-1}.$$

By Lemma 2.4, $\pi_\psi(\alpha(t)(b_n))\xi_\psi = u_\psi(t)\pi_\psi(b_n)\xi_\psi$ converges μ-a.e. to η_ψ. Thus $u_\psi(t)\eta_\psi = \eta_\psi$ μ-a.e.. Since u_ψ is strongly continuous, it follows that $\eta_\psi \in \mathscr{H}_\psi^{\mathbf{R}} = \mathscr{H}_\psi^{S_0}$ μ-a.e..

Parts of Theorem 5.1 can be found in [4]. The implication (i) $\Rightarrow$ (v) was first proved by Haag [17].

Condition (iv) of Theorem 5.1 can be formally rewritten as:

(iv)′ Any ground state φ with $u_\varphi(\mathbf{R})' \cap \pi_\varphi(A)' \cap \pi_\varphi(A)''$ one-dimensional is ergodic.

Dang-Ngoc [7, Theorem 3] considered systems in which any G-invariant state φ with $u_\varphi(G)' \cap \pi_\varphi(A)' \cap \pi_\varphi(A)''$ one-dimensional is ergodic, and showed that this is equivalent (at least in separable cases) to other properties variously known as "quasi-largeness" [8] or "G-centrality" [6] of (A, G, α), which are known to be strictly stronger than G-abelianness [13]. Thus Theorem 5.1 shows that the ground states do not distinguish between $\mathbf{R}$-abelian and $\mathbf{R}$-central systems in the way that the invariant states do.

Sakai has called a ground state of S_0-multiplicity 1 a *physical ground state*, and raised the problem of determining when every pure ground state is a physical ground state [29, Problem 9] and [30, Problem 6.4]. Although Theorem 5.1 can be regarded as giving an answer to this, it would be of more significance to find criteria which depend only on knowledge of α (and therefore of the generator δ) rather than the whole structure of S_0.

Sakai [29, Problem 10] has also asked whether any physical ground state is necessarily unique when A is simple. Corollary 5.3 below shows that this is very far from being the case.

A one-parameter dynamical system $(A, \mathbf{R}, \alpha)$ is said to be *inner* if there is a self-adjoint operator b in A such that $\alpha(t)(a) = e^{itb}ae^{-itb}$ $(a \in A)$, in which case b *implements* α. If A is simple, any uniformly continuous one-parameter system is inner [27]. A system $(A, \mathbf{R}, \alpha)$ is *approximately inner* if there is a net of uniformly continuous systems $(A, \mathbf{R}, \alpha_\gamma)$ such that

$$\|\alpha_\gamma(t)(a) - \alpha(t)(a)\| \to 0 \quad (a \in A,\ t \in \mathbf{R}).$$

The approximating systems can always be taken to be inner, and if A is separable, the net can be assumed to be a sequence [21, Proposition 8.12.7].

THEOREM 5.2. *Suppose A is separable and F is a closed face of $S(A)$. Then F is of the form $S_0(A, \alpha)$ for some approximately inner system $(A, \mathbf{R}, \alpha)$ if and only* *f F intersects every non-empty closed split face of $S(A)$. In this case, α can be taken to be inner.*

Proof. Any non-empty closed split face F_1 of $S(A)$ is the annihilator $I^\perp$ of some closed two-sided ideal I of A [2, Proposition 7.1]. Now I is invariant under any uniformly continuous one-parameter system, and hence under any approximately inner system $(A, \mathbf{R}, \alpha)$, so there is an induced approximately inner system $(A/I, \mathbf{R}, \alpha_I)$ given by

$$\alpha_I(t)(\pi_I(a)) = \pi_I(\alpha(t)(a)) \qquad (a \in A, t \in \mathbf{R})$$

where π_I is the quotient map of A onto A/I. By [23, Theorem 2.3] this induced system possesses a ground state φ_I, and it is easy to see that $\varphi_I \circ \pi_I$ belongs to $F_1 \cap S_0(A, \alpha)$.

Now suppose F is a closed face intersecting every non-empty closed split face of $S(A)$. There is a closed left ideal L of A such that $F = L^\perp$ [14, Theorem 4.9], so F is semi-exposed. Since A is separable, F is exposed [1, Proposition II.5.16], so there is a positive operator b in A such that $F = \{\varphi \in S(A) : \varphi(b) = 0\}$ (b is a strictly positive element of L). For any proper closed two-sided ideal I of A, $I^\perp$ is a closed split face which therefore intersects F. Hence $\pi_I(b)$ is not invertible so it follows from [5, Corollary 2.4] and the Kreĭn-Milman theorem that F is the set of all ground states for the inner system implemented by b.

COROLLARY 5.3. *Suppose A is simple, separable and infinite-dimensional. Then any closed face of $S(A)$ is the set of all ground states for some inner system $(A, \mathbf{R}, \alpha)$. For any positive integer n, there is an inner system on A whose ground states form an $(n-1)$-dimensional simplex with exactly n physical ground states. For any probability measure μ on $\bar{\mathbf{N}}$, there is an inner system on A whose ground states form a simplex affinely homeomorphic to $K_{\mu\infty}(\mathbf{N})$ with infinitely many physical ground states*

Proof. This follows immediately from Theorems 5.1 and 5.2, Proposition 3.1 and Corollary 3.4.

It may be noted that the separability of A was not used in the first half of the proof of Theorem 5.2.

In view of Corollary 5.3, Theorems 2.5 and 5.1 are logically equivalent for separable simple C^*-algebras.

REFERENCES

1. ALFSEN, E. M., *Compact convex sets and boundary integrals*, Springer, Berlin, 1971.
2. ALFSEN, E. M.; ANDERSEN, T. B., Facial structure of compact convex sets, *Proc. London Math. Soc.*, **18** (1968), 385–404.
3. ALFSEN, E. M.; SHULTZ, F. W., State spaces of C^*-algebras, to appear.

4. ARAKI, H., On the algebra of all local observables, *Progr. Theoret. Phys.*, **32** (1964), 844–854.

5. BATTY, C. J. K., Ground states of uniformly continuous dynamical systems, *Quart. J. Math. Oxford Ser.*, **31** (1980), 37–47.

6. BRATTELI, O.; ROBINSON. D. W., *Operator algebras and quantum statistical mechanics*, Springer, Berlin, 1979.

7. DANG-NGOC, N., On the integral representation of states on a C^*-algebra, *Comm. Math. Phys.*, **40** (1975), 223–233.

8. DANG-NGOC, N.; LEDRAPPIER, F., Sur les systèmes dynamiques simpliciaux, *C.R. Acad. Sci. Paris Sér. A*, **277** (1973), 777–779.

9. DIXMIER, J., *Les C^*-algèbres et leurs représentations*, 2nd ed., Gauthier-Villars, Paris, 1969.

10. DOPLICHER, S.; KADISON, R. V.; KASTLER, D.; ROBINSON, D. W., Asymptotically abelian systems, *Comm. Math, Phys.*, **6** (1967), 110–120.

11. DOPLICHER, S.; KASTLER, D., Ergodic states in a non-commutative ergodic theory, *Comm Math. Phys.*, **7** (1968), 1–20.

12. DOPLICHER, S.; KASTLER, D.; ROBINSON, D. W., Covariance algebras in field theory and statistical mechanics, *Comm. Math. Phys.*, **3** (1966), 1–28.

13. DOPLICHER, S.; KASTLER, D.; STØRMER, E., Invariant states and asymptotic abelianness *J. Functional Analysis*, **3** (1969), 419–434.

14. EFFROS, E. G., Order ideals in a C^*-algebra and its dual, *Duke Math. J.*, **30** (1963), 391–412.

15. GLIMM, J., A Stone-Weierstrass theorem for C^*-algebras, *Ann. of Math.*, **72** (1960), 216–244.

16. GLIMM, J., Type I C^*-algebras, *Ann. of Math.*, **73** (1961), 572–612.

17. HAAG, R., On quantum field theories, *Danske Vid. Selsk. Mat.-Fys. Medd.*, **29**:12 (1955).

18. KASTLER, D.; ROBINSON, D. W., Invariant states in statistical mechanics, *Comm. Math. Phys.*, **3** (1966), 151–180.

19. LANFORD, O.; RUELLE, D., Integral representations of invariant states on B^*-algebras, *J. Mathematical Phys.*, **8** (1967), 1460–1463.

20. LINDENSTRAUSS, J.; OLSEN, G.; STERNFELD, Y., The Poulsen simplex, *Ann. Inst. Fourier (Grenoble)*, **28** (1978), 91–114.

21. PEDERSEN, G. K., *C^*-algebras and their automorphism groups*, Academic Press, New-York, 1979.

22. POULSEN, E. T., A simplex with dense extreme points, *Ann. Inst. Fourier (Grenoble)*, **11** (1961), 83–87.

23. POWERS, R. T.; SAKAI, S., Existence of ground states and KMS states for approximately inner dynamics, *Comm. Math. Phys.*, **39** (1975), 273–288.

24. RUELLE, D., States of physical systems, *Comm. Math. Phys.*, **3** (1966), 133–150.

25. RUELLE, D., *Statistical mechanics*, Benjamin, New York, 1969.

26. RUELLE, D., Integral representations of states on a C^*-algebra, *J. Functional Analysis*, **6** (1970), 116–151.

27. SAKAI, S., Derivations of simple C^*-algebras, *J. Functional Analysis*, **2** (1968), 202–206.

28. SAKAI, S., *C^*-algebras and W^*-algebras*, Springer, Berlin, 1971.

29. SAKAI, S., Recent developments in the theory of unbounded derivations in C^*-algebras, in "*C^*-algebras and applications to physics*", Lecture Notes in Mathematics 650, pp. 85–122, Springer, Berlin, 1978.

30. SAKAI, S., The theory of unbounded derivations in C^*-algebras, Newcastle University Lecture Notes, preprint (1977).

31. STØRMER, E., Large groups of automorphisms of C^*-algebras, *Comm. Math. Phys.*, **5** (1967), 1–22.

32. ZELLER-MEIER, G., Produits croisés d'une C^*-algèbre par un groupe d'automorphismes, *J. Math. Pures Appl.*, **47** (1968), 101–239.

C. J. K. BATTY
Department of Mathematics
University of Edinburgh
King's Buildings
Edinburgh EH9 3JZ
Scotland

Received December 29, 1978; revised October 19, 1979.

Added in proof. Since completing this paper, the author has proved that the seven conditions of Theorem 2.5 (and hence those of Corollary 4.4) are equivalent without the restriction that F be a G_δ or A be separable. Furthermore the conjecture of § 3 that every metrisable simplex is a face of a state space is valid. Proofs of these results will be published elsewhere.