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ON THE SPECTRAL PICTURE OF AN OPERATOR

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1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. The concept of the spectral picture of an operator, introduced by Pearcy in [8], has proved to be useful in various ways. For example, one of the main theorems of the Brown-Douglas-Fillmore theory [4] can be stated in terms of spectral pictures thus: Two essentially normal operators in $\mathscr{L}(\mathscr{H})$ are compalent if and only if they have the same spectral picture. (See [8] for definitions.) Also the Romanian characterization of quasitriangular operators [1] can be formulated concisely in terms of spectral pictures: An operator in $\mathscr{L}(\mathscr{H})$ is quasitriangular if and only if its spectral picture contains no negative number.

The purpose of this paper is to make a contribution toward our understanding of the notion of spectral picture. In Section 2 we give a concise exposition of the unpublished result of John Conway to the effect that all spectral pictures are possible. In Section 3 we discuss the behavior of the concept of spectral picture with respect to the relation of quasisimilarity and give a definitive result in an important case. The remainder of the paper (Sections 4 and 5) may be considered as an attempt to study the "continuity" properties of the spectral picture. To make this question precise, a metric is introduced in Section 4 on the set of all spectral pictures. The main result of Section 5 (Theorem 5.6) can be paraphrased by saying that the map sending an operator to its spectral picture is pathologically discontinuous at every operator in $\mathscr{L}(\mathscr{H})$. As a by-product of this negative result we show that the norm-closure of the set of all n-cyclic operators has empty interior.

We conclude this section by some terminology. For our purposes it will be convenient to use a slightly more formal definition of spectral picture than the one initially introduced in [8]. An *abstract spectral picture* is a pair $\mathscr{P} = (Y, \gamma)$ where Y is a nonempty compact subset of the complex plane and γ is a continuous function $\hat{Y} \setminus Y \to \tilde{\mathbf{Z}} = \mathbf{Z} \cup \{-\infty, \infty\}$ where $\hat{Y}$ denotes the complement of the unbounded component of Y. (Throughout this paper $\tilde{\mathbf{Z}}$ is equipped with the discrete topology.)

Equivalently one can say that γ is an integer valued function constant on the holes (that is the bounded components of the complement) of Y. The set of spectral pictures will be denoted by $\mathscr{SP}$. As usual, for an operator T in $\mathscr{L}(\mathscr{H})$, $\sigma_e(T)$, $\sigma_{le}(T)$ and $\sigma_{re}(T)$ denote respectively its essential, left essential and right essential spectra, that is respectively the spectrum, left spectrum and right spectrum of the image $\pi(T)$, of T in the Calkin algebra (i.e. the quotient of the algebra $\mathscr{L}(\mathscr{H})$ by the ideal of compact operators $\mathscr{K}(\mathscr{H})$). It will also be convenient to set

$$\sigma_{lre}(T) = \sigma_{le}(T) \cap \sigma_{re}(T).$$

Equivalently

$$\sigma_{lre}(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \mathscr{S}\mathscr{F}(\mathscr{H})\}.$$

Here $\mathscr{S}\mathscr{F}(\mathscr{H})$ denotes the set of semi-Fredholm operators (recall that an operator T in $\mathscr{L}(\mathscr{H})$ is semi-Fredholm if it has closed range and either $\ker T$ or $\ker T^*$ is finite dimensional). For an operator T in $\mathscr{S}\mathscr{F}(\mathscr{H})$ its index is the element, $\mathrm{i}(T)$, of $\tilde{\mathbf{Z}}$ defined by $\mathrm{i}(T) = \dim \ker T - \dim \ker T^*$. The set of Fredholm operators (that is, the semi-Fredholm operators with finite index) will be denoted by $\mathscr{F}(\mathscr{H})$ (for details see [8]). We define the spectral picture, $SP(T)$, of an operator T in $\mathscr{L}(\mathscr{H})$ by

$$SP(T) = (\sigma_{lre}(T), \mathrm{i}_T)$$

where, for the appropriate values of λ, $\mathrm{i}_T(\lambda) = \mathrm{i}(T - \lambda)$. (Observe that with our definition a hole in $\sigma_{lre}(T)$ is a hole or a pseudohole in $\sigma_e(T)$ (in the sense of [8]) depending on whether its index is finite or not.) Finally we will often talk about an operator T without mentioning the space on which it acts; it is to be understood then, that this space is always a separable, infinite dimensional, complex Hilbert space which will be implicitly identified with $\mathscr{H}$ whenever such an identification seems convenient.

2. UNIVERSALITY OF THE SPECTRAL PICTURE

In this section we prove that the map $SP : \mathscr{L}(\mathscr{H}) \to \mathscr{SP}$ is onto. Our proof is constructive and uses the operator of multiplication by z, V_Ω, on $R^2(\Omega)$ as a "building block". (Recall that, if Ω is a bounded open set in $\mathbf{C}$, $R^2(\Omega)$ denotes the closure of the set of rational functions with poles off Ω^- in the Hilbert space $L^2(\Omega, m)$ where m denotes the Lebesgue measure.) The properties of V_Ω that are relevant to our construction are summarized in the following proposition which has been extensively used in the literature. (See for example [3] and [4].)

PROPOSITION 2.1. *Let Ω be a bounded open set in the complex plane and let V_Ω be the operator of multiplication by z on $R^2(\Omega)$. Then we have*

(1) $\sigma(V_\Omega) = \Omega^-$; *moreover* $\|(V_\Omega - \lambda)^{-1}\| \leqslant 1/\mathrm{dist}(\lambda, \Omega)$ *for* $\lambda \notin \Omega^-$,

(2) *for each* $\lambda \in \Omega$, $(V_\Omega - \lambda)$ *is a Fredholm operator and* $\mathrm{i}(V_\Omega - \lambda) = -1$ (*consequently*, $\sigma_e(V_\Omega) \subset \partial\Omega$).

PROPOSITION 2.2. *Let* Ω *be a bounded open set and let* n *belong to* $\tilde{\mathbf{Z}}$. *Then there exists an operator* A^n_Ω *such that:*

(1) $\sigma(A^n_\Omega) \subset \Omega^-$ *and, for* $\lambda \notin \Omega^-$, $\|(A^n_\Omega - \lambda)^{-1}\| \leqslant 1/\mathrm{dist}(\lambda, \Omega^-)$,

(2) *for each* λ *in* Ω $(A^n_\Omega - \lambda)$ *is semi-Fredholm of index* n.

Proof. For $n > 0$ let A^n_Ω be the direct sum of n copies of V_Ω and for $n < 0$ let $A^n_\Omega = (A^{-n}_{\Omega^*})^*$ where $\Omega^* = \{\lambda : \bar{\lambda} \in \Omega\}$. Verifications of (1) and (2) are straightforward. Finally we take A^0_Ω to be a diagonal operator such that $\sigma(A^0_\Omega) = \sigma_e(A^0_\Omega) = \partial\Omega$. ▨

We are now ready to prove that the map SP is onto. The author is grateful to Professor J. Conway for allowing him to present a proof of his unpublished result.

THEOREM 2.3. (J. Conway). *Given an abstract spectral picture* $\mathscr{P}$ *there exists an operator* T *in* $\mathscr{L}(\mathscr{H})$ *such that* $SP(T) = \mathscr{P}$.

Proof. Let $\mathscr{P} = (Y, \gamma)$ be the given spectral picture. For each n in $\tilde{\mathbf{Z}}$ we define Ω_n to be the union of the bounded components B of $\mathbf{C} \setminus Y$ such that $\gamma(B) = n$ and denote by A_n the operator $A^n_{\Omega_n}$ (we define A_n only when $\Omega_n \neq \emptyset$). Let

$$T = N \oplus \oplus \Big(\sum_{n \in \tilde{\mathbf{Z}},\ \Omega_n \neq \emptyset} \oplus A_n \Big)$$

where N is anormal operator such that $\sigma(N) = \sigma_e(N) = Y$ (if all the Ω_n's are empty, we just set $T = N$). We now discuss the semi-Fredholm status of $T - \lambda$ as λ runs over the complex plane. We distinguish three cases.

Case 1: $\lambda \in Y$. Since $\pi(N)$ is normal, the left and right essential spectra of N are equal to $\sigma_e(N) = Y$. Thus, $(N - \lambda)$ is not semi-Fredholm and, consequently, $(T - \lambda)$ is not semi-Fredholm either.

Case 2: $\lambda \notin \hat{Y} = Y \cup (\bigcup_{n \in \tilde{\mathbf{Z}}} \Omega_n)$. Then there exists $d > 0$ such that $\mathrm{dist}(\lambda, \Omega_n^-) \geqslant d$ for all n. Therefore, the sequence $\{(A_n - \lambda)^{-1}\}_{n \in \tilde{\mathbf{Z}},\ \Omega_n \neq \emptyset}$ is norm-bounded and $(T - \lambda)$ is invertible.

Case 3: $\lambda \in \hat{Y} \setminus Y = \bigcup_{n \in \tilde{\mathbf{Z}}} \Omega_n$. Let k in $\tilde{\mathbf{Z}}$ such that $\lambda \in \Omega_k$. Then there exists a common positive lower bound for $\mathrm{dist}(\lambda, \Omega_n^-)$, $n \neq k$, and, therefore, $\sum_{n \neq k} \oplus (A_n - \lambda)$ is invertible. Since $(N - \lambda)$ is invertible and $(A_k - \lambda)$ is semi-Fredholm of index k (cf. Proposition 2.2), the operator $(T - \lambda)$ is semi-Fredholm of index k.

It follows from the above considerations that $Y = \{\lambda \in \mathbf{C} : (T - \lambda) \notin \mathscr{S}\mathscr{F}\}$ and that, for any λ in $\hat{Y} \setminus Y$, $\mathrm{i}(T - \lambda) = \gamma(\lambda)$. Thus, $SP(T) = \mathscr{P}$, as desired. ▨

3. SPECTRAL PICTURES AND QUASISIMILARITY

Recall that an operator $A : \mathscr{H} \to \mathscr{H}$ is a *quasiaffinity* if A has trivial kernel and dense range. An operator T in $\mathscr{L}(\mathscr{H})$ is said to be a *quasiaffine transform* of an operator T' in $\mathscr{L}(\mathscr{H})$ (notation: $T \prec T'$) if there exists a quasiaffinity $A : \mathscr{H} \to \mathscr{H}$ such that $AT = T'A$. If both relations, $T \prec T'$ and $T' \prec T$, hold, the operators T and T' are said to be quasisimilar. In [15], it was shown that two quasisimilar operators have overlapping essential spectra. This result suggests the following question: What conditions on the spectral pictures $\mathscr{P} = (Y, \gamma)$ and $\mathscr{P}' = (Y', \gamma')$ will insure the existence of two quasisimilar operators T and T' such that $SP(T) = \mathscr{P}$ and $SP(T') = \mathscr{P}'$? The purpose of this section is to answer completely this question when Y and Y' are connected sets. Our result is as follows. (In particular, it shows that there exist quasisimilar operators with "very different" spectral pictures.)

THEOREM 3.1. *Let $\mathscr{P} = (Y, \gamma)$ and $\mathscr{P}' = (Y', \gamma')$ be two spectral pictures such that Y and Y' are connected sets. Then there exist two quasisimilar operators T and T' in $\mathscr{L}(\mathscr{H})$ such that $SP(T) = \mathscr{P}$ and $SP(T') = \mathscr{P}'$ if and only if the following conditions are satisfied:*

(1) $Y \cap Y' \neq \emptyset$.

(2) *The functions γ and γ' coincide on the intersection of their domains (that is, on $(\hat{Y} \setminus Y) \cap (\hat{Y}' \setminus Y')$).*

(3) $\{\lambda \in (\hat{Y} \setminus Y) : \gamma(\lambda) \neq 0\} \subset \hat{Y}'$ and $\{\lambda \in (\hat{Y}' \setminus Y') : \gamma'(\lambda) \neq 0\} \subset \hat{Y}$.

The proof of Theorem 3.1 will be broken into several propositions. One of these (Theorem 3.6) says that, if T and T' are quasisimilar operators, then $\sigma_{\mathrm{lre}}(T) \cap \cap \sigma_{\mathrm{lre}}(T') \neq \emptyset$, thus improving the above result of [5]. We begin with an elementary fact on quasiaffine transforms.

LEMMA 3.2. *If $T \prec T'$ and λ is an eigenvalue of T with multiplicity n, then λ is an eigenvalue of T' with multiplicity $m \geqslant n$.*

Proof. Let A be a quasiaffinity such that $AT = T'A$. Then we have the equality $A(T - \lambda) = (T' - \lambda)A$, from which we get the inclusion, $A(\ker(T - \lambda)) \subset \ker(T' - - \lambda)$. Since A is one-to-one, the result follows. ▨

LEMMA 3.3. *Let T and T' be quasisimilar operators. Then, for any λ such that both $(T - \lambda)$ and $(T' - \lambda)$ are semi-Fredholm operators, we have $\mathrm{i}(T - \lambda) = = \mathrm{i}(T' - \lambda)$.*

Proof. The result follows at once from the definition of the index and from Lemma 3.2 via the relations, $T \prec T'$, $T' \prec T$, $T'^* \prec T^*$ and $T^* \prec T'^*$. ▨

LEMMA 3.4. *Let T and T' be quasisimilar operators. Then any λ such that $(T - \lambda)$ is semi-Fredholm of nonzero index belongs to $(\sigma_{\mathrm{lre}}(T'))^\wedge$.*

Proof. If the operator $(T' - \lambda)$ is not semi-Fredholm, then λ belongs to $\sigma_{lre}(T')$ and, hence, to $(\sigma_{lre}(T'))^{\wedge}$. If $(T' - \lambda)$ is semi-Fredholm, the result follows from Lemma 3.3. ▨

The proof of Lemma 2.9 of [5] can be easily adapted to obtain the following result.

LEMMA 3.5. *If the operators T and T' are quasisimilar, and if there is a hole H_0 in $\sigma_{lre}(T)$ contained in $\sigma(T)$, then $\sigma_{lre}(T) \cap \sigma_{lre}(T') \neq \emptyset$.*

THEOREM 3.6. *If T and T' are quasisimilar operators in $\mathcal{L}(\mathcal{H})$, then $\sigma_{lre}(T) \cap$ $\cap\ \sigma_{lre}(T') \neq \emptyset$.*

Proof. If neither $\sigma_{lre}(T)$ nor $\sigma_{lre}(T')$ have holes of infinite index, then $\sigma_{lre}(T) =$ $= \sigma_e(T)$ and $\sigma_{lre}(T') = \sigma_e(T')$, and the conclusion is just the result of [5] recalled at the beginning of the present section. Otherwise, one of the sets $\sigma_{lre}(T)$ or $\sigma_{lre}(T')$, say $\sigma_{lre}(T)$, has a hole of infinite index and such a hole is contained in $\sigma(T)$. In that case, the conclusion follows from Lemma 3.5. ▨

We conclude our sequence of intermediate results with two specialized versions of Theorem 3.1. The first one is essentially Theorem 3.11 of [5] and corresponds to the case where Y' is a singleton. (For any subset E of $\mathbf{C}$, 0_E denotes the zero function on E.)

THEOREM 3.7. *Let Y be a subset of the complex plane and let λ be a complex number. Then there exists an operator T in $\mathcal{L}(\mathcal{H})$ with spectrum Y, quasisimilar to an operator Q whose spectrum is the singleton $\{\lambda\}$ if and only if Y is compact, connected, and contains λ. Moreover, any such operator T satisfies $SP(T) = (Y, 0_{(\hat{Y} \setminus Y)})$.*

Proof. By translation we may assume $\lambda = 0$. Then the first part of the theorem is Theorem 3.11 of [5]. To prove the last assertion observe that if there is any complex number z in $\sigma(T) \setminus \sigma_{lre}(T)$ it must be an eigenvalue of either T or T^* and, consequently, of Q or Q^*. Therefore, such a number z has to be 0, in contradiction with the fact that 0 belongs to $\sigma_{lre}(T)$. Thus, $\sigma_{lre}(T) = \sigma(T)$ and this equality implies $i(T - z) = 0$ for any z not in $\sigma_{lre}(T)$. ▨

COROLLARY. 3.8. *Theorem 3.1 is true in the case where γ and γ' are the zero functions (on $\hat{Y} \setminus Y$ and $\hat{Y}' \setminus Y'$, respectively).*

Proof. The "only if" part follows from Theorem 3.6 which corresponds to condition (1). (The other two conditions are trivially satisfied in this case.)

Conversely, let Y and Y' be nondisjoint compact connected subsets of $\mathbf{C}$. Let λ be an element of $Y \cap Y'$. By Theorem 3.7 there exist quasisimilar operators Q and W (respectively, Q' and W') such that $\sigma(Q) = \sigma(Q') = \{\lambda\}$, $SP(W) =$ $= (Y, 0_{(\hat{Y} \setminus Y)})$, and $SP(W') = (Y', 0_{(\hat{Y}' \setminus Y')})$. It is easy to check that the operators

$T = Q' \oplus W$ and $T' = Q \oplus W'$ are quasisimilar and satisfy $SP(T) = SP(W) = (Y, 0_{\hat{Y}\setminus Y})$ and $SP(T') = SP(W') = (Y', 0_{\hat{Y}'\setminus Y'})$. ▨

Proof of Theorem 3.1. We first prove the "only if" part. Let T and T' be quasisimilar operators with spectral pictures (Y, γ) and (Y', γ'), respectively. Since $Y = \sigma_{\mathrm{lre}}(T)$ and $Y' = \sigma_{\mathrm{lre}}(T')$, Theorem 3.6 yields $Y \cap Y' \neq \emptyset$, that is, Condition (1). Similarly, Condition (2) follows from Lemma 3.3 and Condition (3) from Lemma 3.4. (Note that we have not used the fact that Y and Y' are connected.)

Conversely, to prove the "if" part, let $\mathcal{P} = (Y, \gamma)$ and $\mathcal{P}' = (Y', \gamma')$ be spectral pictures such that Y and Y' are connected sets and such that Conditions (1), (2), and (3) are satisfied. Let $\Omega = \{\lambda \in \hat{Y}\setminus Y : \gamma(\lambda) \neq 0\}$ and $\Omega' = \{\lambda \in \hat{Y}'\setminus Y' : \gamma'(\lambda) \neq 0\}$. (We assume that $\Omega \cup \Omega' \neq \emptyset$, otherwise the proof is over by Corollary 3.8.) Note that, by virtue of Condition (3), the open set $\Omega \cup \Omega'$ is contained in $\hat{Y} \cap \hat{Y}'$ and so is its boundary, Γ. We want to show that, in fact, Γ is contained in $Y \cap Y'$. Suppose not; then there exists λ in Γ which does not belong to, say, Y. From $\Gamma \subset \hat{Y}$ it follows that λ is in some hole H of Y; moreover, we have $H \cap \Omega = \emptyset$ (otherwise H is contained in Ω and λ is an interior point of $\Omega \cup \Omega'$). But then, the set $H \cap \Omega'$ must be nonempty and we have a contradiction with Condition (2) (γ vanishes on H but there are points in H where γ' is nonzero). Thus, $\Gamma \subset Y \cap Y'$. Condition (2) guarantees the existence of a (continuous) function $\gamma_1 : \hat{\Gamma}\setminus\Gamma \to \tilde{\mathbf{Z}}$, extending γ and γ' on $\Omega \cup \Omega'$ and vanishing elsewhere. Let R be an operator such that $SP(R) = (\Gamma, \gamma_1)$ and let W and W' be quasisimilar operators such that $SP(W) = (Y, 0_{(\hat{Y}\setminus Y)})$ and $SP(W') = (Y', 0_{(\hat{Y}'\setminus Y')})$. (The existence of such operators W and W' is insured by Corollary 3.8.) It is easy to check that the operators $T = R \oplus W$ and $T' = R \oplus W$ are quasisimilar. We now compute $SP(T)$ by inspection of the semi-Fredholm status of $T - \lambda$, distinguishing three cases. First, if λ belongs to Y then $(T - \lambda)$ is not semi-Fredholm (because $W - \lambda$ is not). Next, if λ belongs to Ω, then $(W - \lambda)$ is semi-Fredholm of index 0 (in fact, invertible) and $(R - \lambda)$ is semi-Fredholm of index $\gamma_1(\lambda) = \gamma(\lambda)$. Thus, in that case, $(T - \lambda)$ is semi-Fredholm of index $\gamma(\lambda)$. Finally, if λ does not belong to $\Omega \cup Y$, then, since $\gamma_1(\lambda)$ is either equal to 0 or undefined, $(R - \lambda)$ is semi-Fredholm of index 0; the operator $(W - \lambda)$ being invertible, we have that $(T - \lambda)$ is semi-Fredholm of index $0 = \gamma(\lambda)$. Therefore, $SP(T) = (Y, \gamma)$ and similarly $SP(T') = (Y', \gamma')$. ▨

4. A METRIC ON THE SET OF SPECTRAL PICTURES

The metric we introduce on $\mathcal{SP}$ is based on the Hausdorff metric, d_H, on the set of nonempty, compact subsets of the complex plane. (For details on the Hausdorff metric, the reader is referred to [12].) For a subset $\mathcal{M}$ of $\tilde{\mathbf{Z}}$, let $\mathcal{SP}_{\mathcal{M}}$ be the subset of $\mathcal{SP}$ consisting of the spectral pictures (Y, γ) such that $\gamma^{-1}(n)$ is nonempty if and only if n belongs to $\mathcal{M}$. We first define a metric on $\mathcal{SP}_{\mathcal{M}}$.

PROPOSITION 4.1. *Let $\mathcal{M}$ be a subset of $\tilde{\mathbf{Z}}$. Then the map $\mathrm{d}_{\mathcal{M}} : \mathcal{SP}_{\mathcal{M}} \times \mathcal{SP}_{\mathcal{M}} \to \mathbf{R}$ defined by*

$$\mathrm{d}_{\mathcal{M}}(\mathcal{P}, \mathcal{P}') = \mathrm{d}_H(Y, Y') + \sum_{n \in \mathcal{M} \setminus \mathbf{Z}} \mathrm{d}_H(\Omega_n^-, \Omega_n'^-) +$$

$$+ \sum_{n \in \mathcal{M} \cap \mathbf{Z}} 2^{-|n|} \mathrm{d}_H(\Omega_n^-, \Omega_n'^-)$$

(where $\mathcal{P} = (Y, \gamma)$, $\mathcal{P}' = (Y', \gamma')$, $\Omega_n = \gamma^{-1}(n)$ and $\Omega_n' = \gamma'^{-1}(n)$) is a metric on $\mathcal{SP}_{\mathcal{M}}$.

Proof. Since the sets Ω_n are all contained in the complement of the unbounded component of $\mathbf{C} \setminus Y$ (and similarly for the Ω_n''s), the numbers $\mathrm{d}_H(\Omega_n^-, \Omega_n'^-)$ are bounded independently of n. Thus, $\mathrm{d}_{\mathcal{M}}$ is well-defined. The symmetry and the triangular inequality (for $\mathrm{d}_{\mathcal{M}}$) follow at once from the corresponding properties of the Hausdorff metric. Suppose now that $\mathrm{d}_{\mathcal{M}}(\mathcal{P}, \mathcal{P}') = 0$. This implies the equality $\mathrm{d}_H(Y, Y') = 0$ and, hence, that $Y = Y'$. As a consequence any hole of Y is a hole of Y'. It remains to show that γ and γ' are equal. Suppose there exists a hole H of Y on which the value of γ is n and the value of γ' is $m \neq n$. Then $H \cap \Omega_n'^- = \emptyset$ and $\Omega_n^- \neq \Omega_n'^-$. This inequality implies that $\mathrm{d}_H(\Omega_n^-, \Omega_n'^-)$ and, hence, $\mathrm{d}_{\mathcal{M}}(\mathcal{P}, \mathcal{P}')$ are positive. Consequently, there is no such hole as H and $\gamma = \gamma'$. ▨

THEOREM 4.2. *The map $\mathrm{d} : \mathcal{SP} \times \mathcal{SP} \to \mathbf{R}$ defined by*

$$\mathrm{d}(\mathcal{P}, \mathcal{P}') = \begin{cases} \inf(\mathrm{d}_{\mathcal{M}}(\mathcal{P}, \mathcal{P}'), 1) & \text{if } \mathcal{P} \text{ and } \mathcal{P}' \text{ belong to the same } \mathcal{SP}_{\mathcal{M}} \\ 1 & \text{if not} \end{cases}$$

is a metric on $\mathcal{SP}$.

Proof. Since $\mathcal{SP}$ is the disjoint union of the $\mathcal{SP}_{\mathcal{M}}$ when $\mathcal{M}$ runs over the set $2^{\tilde{\mathbf{Z}}}$, the map d is well-defined. The verification of the axioms of a metric is straightforward. ▨

It is easy to check that the topology induced on $\mathcal{SP}$ by this metric is such that a set $\mathcal{O}$ in $\mathcal{SP}$ is open if and only if $\mathcal{O} \cap \mathcal{SP}_{\mathcal{M}}$ is open for each $\mathcal{M} \subset \tilde{\mathbf{Z}}$. Moreover, a spectral picture $\mathcal{P}'$ sufficiently close to the spectral picture $\mathcal{P}$ must belong to the same $\mathcal{SP}_{\mathcal{M}}$ and, therefore, the range of their index functions must be the same. Thus, the metric we have defined provides an intuitively "reasonable" topology on $\mathcal{SP}$. The next section will show, however, that this topology is somewhat too rigid to expect nice continuity properties of the map: $T \to SP(T)$. Nevertheless, we hope that this example will encourage further studies to find a more suitable topology on $\mathcal{SP}$.

5. CONTINUITY OF THE SPECTRAL PICTURE

In this section we will show that given any operator T in $\mathscr{L}(\mathscr{H})$ and any $\varepsilon > 0$, there exist operators T' in $\mathscr{L}(\mathscr{H})$ such that $\|T' - T\| < \varepsilon$ and $SP(T')$ is arbitrarily different from $SP(T)$ in terms of number of holes and values of indices. In view of the earlier remarks on the topology induced on $\mathscr{S}\mathscr{P}$ by the metric defined in Section 4 this clearly implies that the map $T \to SP(T)$ is discontinuous at any operator in $\mathscr{L}(\mathscr{H})$. In this section a "hole in $SP(T)$" is to be understood as a hole (of finite index) in $\sigma_e(T)$ (though our method can be used to affect the number of pseudoholes, it does not enable us to change the value of their indices and, therefore, we are not going to consider them).

We begin by recalling a theorem which has proved to be very useful in the recent developments in operator theory, especially in approximation problems (c.f. [3], [4]). To state it in a form convenient for our purposes we introduce the following notation. If A is any operator $\mathscr{L}(\mathscr{H})$, then we denote by $\tilde{A}$ the direct sum of countably many copies of A acting on $\mathscr{H}_\infty$, the direct sum of countably many copies of $\mathscr{H}$. We identify $\mathscr{H}_\infty$ with $\mathscr{H}$ via a unitary transformation (the same one throughout this section) and thus view $\tilde{A}$ as an element of $\mathscr{L}(\mathscr{H})$.

THEOREM 5.1. *Let T belong to $\mathscr{L}(\mathscr{H})$ and let ε be any positive number. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence in $\sigma_{1e}(T)$ and let D be a diagonal normal operator with entries $\{\lambda_n\}$. Then there exist a unitary operator $U : \mathscr{H} \oplus \mathscr{H} \to \mathscr{H}$, a compact operator K on $\mathscr{H}$ of norm less than ε, and an operator T' in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ such that $T = UT'U^* + K$ and T' has the form*

$$T' = \begin{pmatrix} D & A \\ 0 & S \end{pmatrix}.$$

The following lemma contains a few facts on the Fredholm theory for operators in 2×2 upper triangular form that will be needed later on. The author is indebted to Vern Paulsen for a simplification in the proof.

LEMMA 5.2. *Let A, R, C be operators in $\mathscr{L}(\mathscr{H})$ and let T be the operator in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ defined by*

$$T = \begin{pmatrix} A & R \\ 0 & C \end{pmatrix}.$$

Then we have:

(1) *If A is a Fredholm operator of index n then T is a semi-Fredholm operator of index m if and only if C is a semi-Fredholm operator of index $m - n$.*

(2) *If A is a normal operator then $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(C)$ and, for any $\lambda \notin \sigma_e(T)$, $\mathrm{i}(T - \lambda) = \mathrm{i}(C - \lambda)$.*

Proof (1). Let S be an operator such that $\pi(S)$ is the inverse of $\pi(A)$ in the Calkin algebra. We can write

$$\begin{pmatrix} A & R \\ 0 & C \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} 1 & SR \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$$

where K is a compact operator. Since $\begin{pmatrix} 1 & SR \\ 0 & 1 \end{pmatrix}$ is invertible, the operator T is semi-Fredholm of index m if and only if so is the operator $A \oplus C$; that is, if and only if the operator C is semi-Fredholm of index $m - n$.

(2) The inclusion $\sigma_e(T) \subset \sigma_e(A) \cup \sigma_e(C)$ follows easily from (1). It is always true that $\sigma_{le}(A) \subset \sigma_e(T)$. Here, since A is normal, $\sigma_e(A) = \sigma_{le}(A)$ and therefore, $\sigma_e(A) \subset \sigma_e(T)$. Let now λ belong to $\sigma_e(C) \setminus \sigma_e(A)$. Then (again because A is normal) the operator $(A - \lambda)$ is Fredholm of index 0 and it now follows from (1) that $(T - \lambda)$ cannot be a Fredholm operator. This proves the inclusion $\sigma_e(C) \subset \sigma_e(T)$ and, consequently, $\sigma_e(A) \cup \sigma_e(C) \subset \sigma_e(T)$. The equality of the indices follows from (1) (via the fact that, for any $\lambda \notin \sigma_e(T)$, $(A - \lambda)$ is Fredholm of index 0). ▨

We now establish our basic approximation theorem.

THEOREM 5.3. *Let T belong to $\mathcal{L}(\mathcal{H})$ and let F be a closed non-empty subset of* $\mathbf{C}$ *such that, for each μ in F,* $\mathrm{d}(\mu, \sigma_{le}(T)) < \varepsilon$. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a dense sequence in F and let D be a diagonal normal operator with entries $\{\mu_n\}_{n=1}^{\infty}$. Then there exists an operator T' such that $\|T' - T\| < \varepsilon$ and T' is unitarily equivalent to an operator of the form*

$$\begin{pmatrix} \tilde{D} & A \\ 0 & S \end{pmatrix}.$$

Moreover, we can assume that $\sigma_e(T') = \sigma_e(T) \cup F$ and that, for any $\lambda \notin \sigma_e(T')$, $\mathrm{i}(T - \lambda) = \mathrm{i}(T' - \lambda)$.

Proof. Since F is compact, there exists a number $\alpha < \varepsilon$ such that $\mathrm{d}(\mu, \sigma_{le}(T)) < \alpha$, for all μ in F. We now choose a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $\sigma_{le}(T)$ such that $|\mu_n - \lambda_n| < \alpha$, for all n, and we let D_1 be the diagonal normal operator with entries $\{\lambda_n\}_{n=1}^{\infty}$ (in the orthonormal basis in which the entries of D are $\{\mu_n\}_{n=1}^{\infty}$). By Theorem 5.1 there exist a unitary operator $U : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$, a compact operator K on $\mathcal{H}$ of norm less than $\varepsilon - \alpha$, and an operator T_1 in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ of the form

$$T_1 = \begin{pmatrix} \tilde{D}_1 & A \\ 0 & S \end{pmatrix}$$

such that $T = UT_1U^* + K$. Let T_2 be the operator on $\mathcal{H} \oplus \mathcal{H}$ obtained by replacing $\tilde{D}_1$ by $\tilde{D}$ in the matricial form of T_1; then we have $\|T_2 - T_1\| \leqslant \|D - D_1\| < \alpha$.

Setting $T' = UT_2U^*$ we have

$$\|T - T'\| \leqslant \|U\| \|(T_1 - T_2)\| \|U^*\| + \|K\| < \alpha + \varepsilon - \alpha = \varepsilon.$$

We now prove the assertion on the essential spectra and the indices. Note first, that, since T_1 is unitarily equivalent to a compact perturbation of T, the operators T_1 and T have the same spectral picture. Therefore, by (3) Lemma 5.2, we obtain $\sigma_e(T) = \sigma_e(\tilde{D}_1) \cup \sigma_e(S)$ and, for any $\lambda \notin \sigma_e(T)$, $i(T - \lambda) = i(S - \lambda)$. Since $\tilde{D}_1$ is unitarily equivalent to $\tilde{D}_1 \oplus \tilde{D}_1$, we can replace S by an operator of the form

$$\begin{pmatrix} \tilde{D}_1 & * \\ 0 & S \end{pmatrix}.$$

In other words, we can assume (again by Lemma 5.2) that $\sigma_e(D) \subset \sigma_e(S)$ and, therefore, that $\sigma_e(S) = \sigma_e(T)$. Applying Lemma 5.2 once more, we obtain $\sigma_e(T') = \sigma_e(\tilde{D}) \cup \sigma_e(S) = F \cup \sigma_e(T)$ (it is easy to see that $\sigma_e(\tilde{D}) = F$) and, for any $\lambda \notin \sigma_e(T')$, $i(T' - \lambda) = i(S - \lambda) = i(T - \lambda)$. ▨

Since the boundary of the essential spectrum of an operator T always contains the left essential spectrum of T, the above theorem shows that the number of holes in the spectral picture of T can be arbitrarily modified by arbitrarily small perturbations: one can introduce countably many Jordan curves, as close as desired to $\partial\sigma_e(T)$ in the unbounded component of $(\mathbf{C}\setminus\sigma_e(T))$ or in any existing hole of $\sigma_e(T)$. However, Theorem 3.1 does not enable us to affect the values of the index function: wherever they are both defined, the indices $i(T - \lambda)$ and $i(T' - \lambda)$ coincide. Our next step will be to show that the indices can also be modified in an arbitrary fashion by small perturbations. This will be done, first, for a unitary operator whose spectrum is the whole unit circle and, then, extended to an arbitrary operator by means of Theorem 5.3. To that end we establish some auxiliary results on bilateral shifts. Let ρ and τ be nonnegative real numbers and let $B_{\rho,\tau}$ be the weighted bilateral shift of multiplicity one defined in the canonical basis $\{e_n\}_{n\in\mathbf{Z}}$ of $\ell_2(\mathbf{Z})$ by $B_{\rho,\tau}(e_n) = \rho e_{n+1}$ for $n \geqslant 0$ and $B_{\rho,\tau}(e_n) = \tau e_{n+1}$ for $n < 0$. The unweighted bilateral shift $B_{1,1}$ is denoted simply by B.

LEMMA 5.4. *Suppose $\tau < \rho$. Then we have*

(1) $\sigma(T) = \{z : \tau \leqslant |z| \leqslant \rho\}$.

(2) $\sigma_e(T) = \rho\Gamma \cup \tau\Gamma$ *(where Γ is the unit circle) and the annulus $\sigma(T)\setminus\sigma_e(T)$ is a hole of index* -1.

(3) *For any $\lambda \notin \sigma(T)$,* $\|(B_{\rho,\tau} - \lambda)^{-1}\| \leqslant 1/\mathrm{dist}(\lambda, \sigma(T))$.

When $\rho < \tau$ the above results remain valid switching ρ and τ and replacing "index -1" by "index 1".

Proof (1). This is a consequence of the general result on the spectrum of a shift (see [10]).

(2) In the decomposition $\ell_2(\mathbf{Z}) = \bigvee_{i=0}^{\infty} \{e_i\} \oplus \bigvee_{i=1}^{\infty} \{e_{-i}\}$ the operator $B_{\rho,\tau}$ has the matricial form

$$\begin{pmatrix} \rho V & R \\ 0 & \tau V^* \end{pmatrix}$$

where V is the (unweighted) forward unilateral shift and R is a rank-one operator. Thus the spectral picture of $B_{\rho,\tau}$ is equal to the spectral picture of the operator $C = \rho V \oplus \tau V^*$. Elementary considerations now show that the latter is as follows. We have $\sigma_{\mathrm{lre}}(C) = \sigma_{\mathrm{e}}(C) = \rho\Gamma \cup \tau\Gamma$, $\{\lambda : \tau < |\lambda| < \rho\}$ is a hole of index -1, and $\{\lambda : |\lambda| < \tau\}$ is a hole of index 0.

(3) Easy computations show that $B_{\rho,\tau}$ is hyponormal and the result now follows from [11].

The proof of these results when $\tau > \rho$ is entirely similar. ▨

REMARK. We will say that an operator T has the property $(\mathscr{R})$ if its resolvent satisfies the norm-inequality $\|(T - \lambda)^{-1}\| \leqslant 1/\mathrm{dist}(\lambda, \sigma(T))$, for any $\lambda \notin \sigma(T)$. Then (3) of Lemma 5.4 says that the shifts $B_{\rho,\tau}$ have the property $(\mathscr{R})$. We observe for further reference that property $(\mathscr{R})$ is preserved under unitary equivalence and under direct sums, as well as under translations and multiplications by nonzero scalars. The interested reader can find more details about operators having the property $(\mathscr{R})$ in [9].

PROPOSITION 5.5. *Let U be a unitary operator whose spectrum is the whole unit circle Γ and let $\{\alpha_n\}_{n\in J\subset \mathbf{N}}$ be a sequence (finite or not) of disjoint open annulii centered at the origin. Suppose associated to each α_n an integer $\gamma(n)$. Let $\{z : \alpha < |z| < \beta\}$ be the smallest annulus containing all the annuli α_n. Then, for any $\mu > \mathrm{Max}(|1 - \alpha|, |1-\beta|)$, there exists an operator W such that $\|U - W\| < \mu$ and such that for each n, α_n is a hole of index $\gamma(n)$ for $SP(W)$.*

Proof. Suppose $\alpha_n = \{z : \tau_n < |z| < \rho_n\}$. We define the operator B_n to be the direct sum of $-\gamma(n)$ copies of B_{ρ_n,τ_n} if $\gamma(n) < 0$, γ_n copies of B_{τ_n,ρ_n} if $\gamma(n) > 0$ and finally $\rho_n B \oplus \tau_n B$ if $\gamma(n) = 0$. Let $\hat{B}$ be the direct sum of $\sum_{n\in J} |\gamma(n)|$ copies of B and let W_1 be the direct sum of the operators B_n. We clearly have $\|\hat{B} - W_1\| \leqslant \delta = \mathrm{Max}(|1 - \alpha, |1 - \beta|)$. Since $\hat{B}$ and U are normal (in fact unitary) operators with the same essential spectrum, it follows from Corollary 2.13 of [8] that there exist a unitary operator A and a compact operator K of norm less than $\mu - \delta$ such that

$$U = A\hat{B}A^* + K.$$

Let $W = AW_1A^*$; then,

$$\|W - U\| \leqslant \|A\| \, \|(W_1 - \hat{B})\| \, \|A^*\| + \|K\| < \delta + \mu - \delta = \mu.$$

Since $SP(W) = SP(W_1)$, it remains to show that each a_n is a hole of index $\gamma(n)$ for $SP(W_1)$. Let $a_n = \{z : \tau_n < z < \rho_n\}$ and let λ in a_n. For any $k \neq n$ belonging to J, $(B_k - \lambda)$ is invertible and by the remark following Lemma 5.4 (together with the fact that $a_k \cap a_n = \emptyset$) its inverse satisfies $\|(B_k - \lambda)^{-1}\| \leqslant 1/\min\{\rho_n - |\lambda|, \tau_n - |\lambda|\}$. Therefore, the operator $\sum_{k \in J \setminus \{n\}} \oplus (B_k - \lambda)$ is invertible and $(W_1 - \lambda)$, which is its direct sum with $B_n - \lambda$, has the same Fredholm status as $B_n - \lambda$, that is, is Fredholm of index $\gamma(n)$. The proof is complete. ▨

REMARK. It is clear that slightly different versions of Proposition 5.5 can be formulated. For instance, one can "thicken" (i.e., introduce in $\sigma_e(W)$) a closed annulus a^- disjoint from $\bigcup_{n \in J} a_n$ but contained in $\{z : \alpha < |z| < \beta\}$ by adding to the direct sum defining W_1, a direct sum $\sum_{n \in \mathbf{N}} \oplus \tau_n B$ such that $\bigcup_{n \in \mathbf{N}} \tau_n \Gamma$ is dense in a^-.

We are now ready to establish the main result of this section.

THEOREM 5.6. *Let T be any operator in $\mathscr{L}(\mathscr{H})$ and let ε be any positive number. Let $\{\Delta_n\}_{n \in \mathbf{N}}$ be a sequence of pairwise disjoint closed discs in $\mathbf{C} \setminus \sigma_e(T)$ of center t_n and radius r_n such that, for all n, $r_n \leqslant \operatorname{dist}(r_n, \sigma_e(T)) < \varepsilon/3$. Finally, for each n, let $\{a_{n,j}\}_{j \in J_n \subset \mathbf{N}}$ be a sequence (finite or not) of disjoint open annulii of center t_n contained in Δ_n, together with a map, $\gamma_n : J_n \to \mathbf{Z}$. Then there exists an operator T' such that $\|T' - T\| < \varepsilon$ and such that each $a_{n,j}$ is a hole of index $(\gamma_n(j) + \mathrm{i}(T - r_n))$ for $SP(T')$.*

Proof. Let $\{\mu_n\}_{n \in \mathbf{N}}$ be a countable dense subset of $\bigcup_{n \in \mathbf{N}} \partial\Delta_n$ and let D be a diagonal operator with entries $\{\mu_n\}$. Since $\mathrm{d}(\mu_n, \sigma_e(T)) < 2\varepsilon/3$, by Theorem 5.3 there exist a unitary operator R and an operator T_1 of the form

$$T_1 = \begin{pmatrix} \tilde{D} & * \\ 0 & S \end{pmatrix}$$

such that $\|T - R\,T_1R^*\| < 2\varepsilon/3$. The operator $\tilde{D}$ can be written $\tilde{D} = \sum_{n \in \mathbf{N}} \oplus N_n$ where each N_n is a diagonal normal operator such that $\sigma_e(N_n) = \partial\Delta_n$. We can write

$N_n = t_n + r_n U_n$ where U_n is a unitary operator satisfying $\sigma(U_n) = \Gamma$ and we also write $\mathscr{a}_{n,j} = t_n + r_n \mathscr{b}_{n,j}$ where, for each n, the $\mathscr{b}_{n,j}$'s are annulii concentrical to and contained in the unit disc. For each n we now apply Proposition 5.5 to the operator U_n and the annuli $\{\mathscr{b}_{n,j}\}_{j \in J_n}$ together with a number μ_n such that $\mu_n r_n < \varepsilon/3$. We obtain, for each n, an operator W_n such that $\|U_n - W_n\| < \mu_n$ and, for any j in J_n, $\mathscr{b}_{n,j}$ is a hole of index $\gamma_n(j)$ in $SP(W_n)$. Let $V_n = t_n + r_n W_n$; then $\|V_n - N_n\| < \varepsilon/3$ and each $\mathscr{a}_{n,j}$ is a hole of index $\gamma_n(j)$ in $SP(V_n)$. Let $\hat{V}$ be the operator obtained by replacing each N_n by V_n in the decomposition of $\tilde{D}$. Clearly, $\|\hat{V} - \tilde{D}\| < \varepsilon/3$ and it follows from the remark following Lemma 5.4 together with an argument similar to the one used in the proof of Proposition 5.5 that each $\mathscr{a}_{n,j} (n \in \mathbf{N}, j \in J_n)$ is a hole of index $\gamma_n(j)$ in $SP(\hat{V})$. Finally let T_2 be the operator obtained by replacing $\tilde{D}$ by $\hat{V}$ in the matricial form of the operator T_1 and let $T' = RT_2R^*$. Then,

$$\|T - T'\| \leqslant \|T - RT_1R^*\| + \|R(T_1 - T_2)R^*\| < 2\varepsilon/3 + \varepsilon/3 = \varepsilon,$$

and the desired equalities for indices are obtained via Lemma 5.2. ▨

We conclude this section by an application of Theorem 5.6 to a question involving cyclic operators. For any positive integer n we denote by $C_n(\mathscr{H})$ (or C_n) the set of n-cyclic operators in $\mathscr{L}(\mathscr{H})$. (Recall that an operator T in $\mathscr{L}(\mathscr{H})$ is said to be n-cyclic if there exists a subset E of $\mathscr{H}$ of cardinal n such that the smallest invariant subspace for T that contains E is $\mathscr{H}$.) In [7] Herrero showed that the norm-closure of C_n can be characterized simply in terms of spectral pictures: An operator T in $\mathscr{L}(\mathscr{H})$ belongs to C_n^- if and only if there are no holes of index less than $-n$ in $SP(T)$ and all the holes of index $-n$ are simply connected. In [6] it was shown that the set of cyclic operators has empty interior. Using Theorem 5.6 we can improve this result.

COROLLARY 5.7. *The normclosure, C_n^-, of the set of n-cyclic operators has empty interior. In fact $\bigcup_{n>0} C_n^-$ has empty interior.*

Proof. It is enough to prove the second assertion. Let T belong to some C_k^- and let $\varepsilon > 0$. Let $\{\Delta_n\}_{n \in \mathbf{N}}$ a sequence of discs as in Theorem 5.6 but in the unbounded component of $\mathbf{C} \setminus \sigma_e(T)$ and let i_n be a sequence of integers decreasing to $-\infty$. By Theorem 5.6, there is an operator T' in $\mathscr{L}(\mathscr{H})$ such that $\|T' - T\| < \varepsilon$ and each $\mathring{\Delta}_n$ is a hole of index i_n in $SP(T')$. By Herrero's result the operator T' cannot belong to $\cup C_n^-$. ▨

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