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HILBERT C^*-MODULES: THEOREMS OF STINESPRING AND VOICULESCU

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There are two well known technical results in the theory of extensions of C^*-algebras. Stinespring's theorem [11] describes the structure of completely positive maps of a C^*-algebra A into the full operator algebra $\mathcal{L}(\mathcal{H})$. Voiculescu's theorem [12] establishes the existence of the identity element in the semigroup of extensions of the type $0 \to \mathcal{K} \to \mathcal{D} \to A \to 0$ ($\mathcal{K}$ being the algebra of compact operators in $\mathcal{L}(\mathcal{H})$). The theory of general extensions $0 \to \mathcal{K} \otimes B \to \mathcal{D} \to A \to 0$ announced in [13] is based on a suitable generalization of these two theorems (with $\mathcal{L}(\mathcal{H})$ replaced by the multiplier algebra $\mathcal{M}(\mathcal{K} \otimes B)$). In the present paper we give this generalization (Theorems 3 and 6). Besides this, the paper also contains two theorems about Hilbert C^*-modules which are of independent interest (Theorems 1 and 2). In particular, Theorem 2 asserts that every countably generated Hilbert B-module is a direct summand in the canonical Hilbert space over B.

§ 1. NOTATION

1. In what follows all the algebras are C^*-algebras, the homomorphisms are *-homomorphisms, and the ideals are closed and two-sided. All the results are valid for the next three categories of algebras: complex algebras, real algebras, and "real" algebras. A complex algebra is called "real" if it is equipped with an antilinear involution $b \to \bar{b}$ satisfying two conditions: $\overline{b_1 b_2} = \bar{b}_1 \cdot \bar{b}_2$, $\overline{(b^*)} = (\bar{b})^*$. The homomorphisms of such algebras must preserve the "real" involution. Moreover, we suppose that a fixed compact second countable group G acts as a group of automorphisms on all algebras. All homomorphisms are supposed to be equivariant. In the "real" case the group and the action must be "real" (i.e., there exists an involution $g \to \bar{g}$ on G, such that $\overline{g(b)} = \bar{g}(\bar{b})$).

Some explanations are necessary here. C^*-algebras (i.e., algebras of operators) are characterized in the complex case by a usual condition $\|x\|^2 \leqslant \|x^*x\|$, $\forall x$. In the real case however this condition is not sufficient. It must be replaced by $\|x\|^2 \leqslant \|x^*x + y^*y\|$, $\forall x, y$ (cf. [16]). A complexification of a real algebra is a "real" algebra. Conversely, a subalgebra of real elements ($\bar{x} = x$) in a "real" algebra is a real algebra. This implies an isomorphism of the categories of real and "real" algebras, provided that the "real" involution on G is trivial. (In the case of a nontrivial involution on G, the category of "real" algebras does not reduce to the category of real ones.) Note that all the constructions of the paper comply with this isomorphism of real and "real" categories.

2. An element $b \in B$ is called *invariant* if $\bar{b} = b$; $g(b) = b$, $\forall g \in G$. An algebra is called *unital* if it has a unit, 1, which is invariant. The action of G is called *continuous* if the map $G \times B \to B : (g, b) \to g(b)$ is *norm-continuous*. All tensor products of algebras are equipped with the minimal C^*-norm, and $\overline{b_1 \otimes b_2} = \bar{b}_1 \otimes \bar{b}_2$; $g(b_1 \otimes b_2) = g(b_1) \otimes g(b_2)$, $\forall g \in G$.

3. $\mathbb{C}$ is the scalar field (i.e., the algebra $\mathbf{R}$ or $\mathbf{C}$). The "real" involution is the complex conjugation, the action of G is trivial. M_n is the $n \times n$ matrix algebra. The "real" involution is the complex conjugation of all the entries. The action of G is defined by some unitary ("real") representation of G in $\mathbb{C}^n$ which is concretely specified in each case. For every algebra B put $M_n(B) = M_n \otimes B$. By $\{e_{ij}\}$ we denote the standard system of matrix units in M_n. $\mathcal{U}(n)$ is the unitary group of M_n.

4. $\mathcal{M}(B)$ is the multiplier (double centralizer) algebra of B. Recall ([3]) that a pair of maps $T_1, T_2 : B \to B$ is a *multiplier* if $\forall x, y \in B$, $x \cdot T_1(y) = T_2(x) \cdot y$. The *strict topology* on $\mathcal{M}(B)$ is generated by the system of seminorms

$$\{\|T\|_b = \|T_1(b)\| + \|T_2(b)\| \mid b \in B\}$$

where $T = (T_1, T_2) \in \mathcal{M}(B)$. A continuous action of G on B defines an action of G on $\mathcal{M}(B)$: $g(T_i)(b) = g(T_i g^{-1}(b))$, $i = 1, 2$. This action is not, in general, continuous in the norm topology, but only in the strict topology. The "real" involution on $\mathcal{M}(B)$ is defined by $\bar{T}_i(b) = \overline{T_i(\bar{b})}$, $i = 1, 2$.

5. An algebra B with the adjoint (invariant) unit is denoted by $\widetilde{B}$. Every linear map $\varphi : A \to B$ may be (uniquely) extended to the unital map $\varphi : \widetilde{A} \to \widetilde{B}$. If φ is completely positive and $\|\varphi\| \leqslant 1$, then $\widetilde{\varphi}$ is also completely positive ([4]). For a unital B we have $\widetilde{B} \simeq B \oplus \mathbb{C}$, hence there exists a unital projection-homomorphism $p : \widetilde{B} \to B$. Consequently, every linear map $A \to B$ may be continued to $\widetilde{A} \to \widetilde{B} \xrightarrow{p} B$

6. We say that a linear map is *equivariant* if it preserves the G-actions and the "real" involution. Until otherwise specified, every completely positive map is supposed to be equivariant.

7. Ending (or omitting) the proof, we put the sign ▨.

§ 2. HILBERT C^*-MODULES

Recall the definition of a Hilbert module over a C^*-algebra B ([10]).

Definition 1. Let E be a linear space over the field $\mathbb{C}$ equipped with the structure of a right B-module. We suppose that the action of G on B is continuous and

$\lambda(xb) = (\lambda x)b = x(\lambda b)$, $\forall x \in E$, $b \in B$, $\lambda \in \mathbb{C}$. The space E is called a *pre-Hilbert B-module* if there exists a scalar product $E \times E \to B$ satisfying $\forall x, y, z \in E$, $b \in B$, $\lambda \in \mathbb{C}$ the following conditions:

$$1^\circ \quad (x, y + z) = (x, y) + (x, z); \; (x, \lambda y) = \lambda(x, y)$$

$$2^\circ \quad (x, yb) = (x, y)b$$

$$3^\circ \quad (y, x) = (x, y)^*$$

$$4^\circ \quad (x, x) \geqslant 0; \text{ if } (x, x) = 0, \text{ then } x = 0.$$

Moreover, E must be equipped with a linear, norm-continuous G-action (the norm is defined below) and (in the "real" case) with the antilinear involution $x \to \bar{x}$ satisfying $\forall g \in G$, (besides the usual condition $\overline{g(x)} = g(\bar{x})$), also the following:

$$5^\circ \quad g(xb) = g(x)g(b); \quad \overline{xb} = \bar{x}\bar{b}$$

$$6^\circ \quad (g(x), g(y)) = g((x, y)); \; (\bar{x}, \bar{y}) = \overline{(x, y)}.$$

An element $x \in$ E will be called *invariant* if $\bar{x} = x$ and $g(x) = x$, $\forall g \in G$.

LEMMA 1. ([10], [14]). *Put* $\forall x \in E$ $\|x\| = \|(x, x)\|^{1/2}$. *The space* E *with the norm* $\|\cdot\|$ *satisfies all the axioms of a normed space. Moreover,* $\forall x, y \in E, b \in B$, $\|xb\| \leqslant \|x\| \cdot \|b\|$, $\|(x, y)\| \leqslant \|x\| \cdot \|y\|$. *These two inequalities remain valid even if we drop the condition:* $(x, x) = 0 \Rightarrow x = 0$ *in the condition* 4°.

Proof. Only the last inequality $\|(x, y)\| \leqslant \|x\| \cdot \|y\|$ is not obvious. (The triangle inequality for the norm is, as usual, its consequence.) Proving it, we shall consider two cases. If at least one of the norms $\|x\|$, $\|y\|$ is non-zero, say $\|y\| \neq 0$, then our inequality follows from $(x + yb, x + yb) \geqslant 0$ with $b = -(y, x)/\|(y, y)\|$. The case $\|x\| = \|y\| = 0$ follows from the same $(x + yb, x+yb) \geqslant 0$ with $b = -(y, x)$. ▨

DEFINITION 2. A pre-Hilbert B module E is called a *Hilbert B-module* if it is complete with respect to the norm defined in Lemma 1. The *Hilbert direct sum* $\bigoplus_{i \in I} E_i$ is the completion of the corresponding algebraic direct sum by the norm defined by the scalar product $(\bigoplus_i x_i, \bigoplus_i y_i) = \sum_{i \in I} (x_i, y_i)$. We denote $\bigoplus_1^n E$ by E^n.

EXAMPLES.

1) $E = B$, $(x, y) = x^*y$.

1′) $E = B^n$.

2) Let $\{V_i\}$ be a countable collection of finite dimensional Euclidean spaces equipped with a unitary G-action ("real" G-action in the "real" case). We suppose that *every* finite dimensional unitary representation of G occurs (up to isomorphism) an *infinite* number of times in the collection $\{V_i\}$. *The scalar product on* V_i *is assumed*

to be linear in the second argument and antilinear in the first one. Let the scalar product on $V_i \otimes B$ be defined by $(x_1 \otimes b_1, x_2 \otimes b_2) = (x_1, x_2)b_1^* b_2$. The Hilbert direct sum $\mathscr{H}_B = \bigoplus_{i=1}^{\infty} (V_i \otimes B)$ will be called *the Hilbert space over* B. In the case $B = \mathbb{C}$ we get the usual Hilbert space $\mathscr{H}$.

DEFINITION 3. For Hilbert B-modules E_1 and E_2 we denote by $\mathscr{L}(E_1, E_2)$ the set of such maps $T : E_1 \to E_2$ that there exists $T^* : E_2 \to E_1$ satisfying the condition:

$$(T(x), y) = (x, T^*(y)), \ \forall x \in E_1, \ y \in E_2.$$

The action of G and the "real" involution are defined on $\mathscr{L}(E_1, E_2)$ by $g(T)(x) = g(Tg^{-1}(x))$, $g \in G$, $x \in E_1$; $\bar{T}(x) = \overline{T(\bar{x})}$. An element $T \in \mathscr{L}(E_1, E_2)$ is called *equivariant* if $\bar{T} = T$, $g(T) = T$; $\forall g \in G$. Put $\mathscr{L}(E) = \mathscr{L}(E, E)$.

LEMMA 2. ([10]). *Every map $T \in \mathscr{L}(E_1, E_2)$ is a bounded linear B-module map. For $T \in \mathscr{L}(E_1, E_2)$ the operator T^* is uniquely defined and belongs to $\mathscr{L}(E_2, E_1)$. With the norm induced from the space of bounded linear operators on E, $\mathscr{L}(E)$ is a C^*-algebra.*

Proof. The existence of the adjoint easily implies the linearity of T, and the boundedness follows from the Banach-Steinhaus theorem because the family of linear maps

$$\{f_x : E_2 \to B \,|\, x \in E_1, \|x\| \leq 1\}, \ f_x(y) = (Tx, y) = (x, T^*y),$$

is bounded for every fixed $y \in E_2$. Remaining statements may be found in [10]. ▨

LEMMA 3. *Let E_1, E_2, E_3 be Hilbert B-modules. For $x \in E_1, y, z \in E_2$ put $\theta_{x,y}(z) = x \cdot (y, z)$. Then $\theta_{x,y} \in \mathscr{L}(E_2, E_1)$, $\theta^*_{x,y} = \theta_{y,x}$. If $u \in E_2$, $v \in E_3$, $T \in \mathscr{L}(E_2, E_1)$, $S \in \mathscr{L}(E_3, E_2)$, then*

$$T \cdot \theta_{u,v} = \theta_{T(u),v}, \ \theta_{x,y} \cdot S = \theta_{x,S^*(y)}.$$

In particular,

$$\theta_{x,y} \cdot \theta_{u,v} = \theta_{x(y,u),v} = \theta_{x,v(u,y)}.$$

Besides that, $\forall g \in G$ $g(\theta_{x,y}) = \theta_{g(x),g(y)}$; $\bar{\theta}_{x,y} = \theta_{\bar{x},\bar{y}}$. ▨

DEFINITION 4. The closure of the linear span of $\{\theta_{x,y} | x \in E_1, y \in E_2\}$ in $\mathscr{L}(E_2, E_1)$ will be denoted by $\mathscr{K}(E_2, E_1)$. Put $\mathscr{K}(E) = \mathscr{K}(E, E)$, $\mathscr{K} = \mathscr{K}(\mathscr{H})$, $\mathscr{K}_B = \mathscr{K}(\mathscr{H}_B)$.

It follows from Lemma 3 that $\mathscr{K}(E)$ is an ideal in $\mathscr{L}(E)$. The G-action on $\mathscr{K}(E)$ is continuous as it is continuous on E.

LEMMA 4. *If E is a Hilbert B-module, then*

$$\mathscr{L}(E^n) \simeq M_n \otimes \mathscr{L}(E), \ \mathscr{K}(E^n) \simeq M_n \otimes \mathscr{K}(E).$$

Besides that, $\mathscr{K}(B) \simeq B$, $\mathscr{K}(B^n) \simeq M_n \otimes B$, $\mathscr{K}_B \simeq \mathscr{K} \otimes B$. ▨

THEOREM 1. *For an arbitrary Hilbert B-module E the correspondence* $T\in\mathcal{L}(E)\to$ $\to(T_1, T_2)\in\mathcal{M}(\mathcal{K}(E))$ *with* $T_1(\theta_{x,y})=\theta_{T(x),y}$, $T_2(\theta_{x,y})=\theta_{x,T^*(y)}$ $(x, y\in E)$ *defines an isomorphism of* $\mathcal{L}(E)$ *on* $\mathcal{M}(\mathcal{K}(E))$.

Proof. Since for every S from the linear span of $\{\theta_{x,y}\}$ we have $T_1(S)=T\cdot S$, $T_2(S)=S\cdot T$ in $\mathcal{L}(E)$, it follows that $\|T_i(S)\|\leqslant\|T\|\cdot\|S\|$, $i=1,2$. Therefore T_1 and T_2 may be continuously extended on $\mathcal{K}(E)$. It is not difficult to show that $(T_1, T_2)\in\mathcal{M}(\mathcal{K}(E))$. If $T_1(\theta_{x,y})=0$, $\forall x, y\in E$, then $\theta_{T(x),T(x)}=0$, $\forall x$, hence $T=0$. It means that the correspondence is monomorphic. To prove that it is epimorphic, we put for $(T_1, T_2)\in\mathcal{M}(\mathcal{K}(E))$, $x\in E$

$$T(x)=\lim_{\alpha\to+0}T_1(\theta_{x,x})(x)\cdot[(x, x)+\alpha]^{-1}$$

$$T^*(x)=\lim_{\alpha\to+0}[T_2(\theta_{x,x})]^*(x)\cdot[(x, x)+\alpha]^{-1}.$$

The existence of the limits is an easy consequence of the Cauchy criterion and the inequalities:

$$T_1(S)^*\cdot T_1(S)\leqslant\|T_1\|^2\cdot S^*S,$$

$$T_2(S)\cdot T_2(S)^*\leqslant\|T_2\|^2\cdot SS^*\quad(\forall\, S\in\mathcal{K}(E)).$$

Now noticing that $\forall\, x\in E\ \lim_{\alpha\to+0}\theta_{x,x}(x)\cdot[(x, x)+\alpha]^{-1}=x$, we have:

$$(x, T^*(y))=\lim_{\alpha\to+0}(x, [T_2(\theta_{y,y})]^*(y)\cdot[(y, y)+\alpha]^{-1})=$$

$$=\lim_{\alpha\to+0}(T_2(\theta_{y,y})\cdot\theta_{x,x}(x)\cdot[(x, x)+\alpha]^{-1}, y[(y, y)+\alpha]^{-1})=$$

$$=\lim_{\alpha\to+0}(\theta_{y,y}\cdot T_1(\theta_{x,x})(x)\cdot[(x, x)+\alpha]^{-1}, y[(y, y)+\alpha]^{-1})=$$

$$=\lim_{\alpha\to+0}(T_1(\theta_{x,x})(x)\cdot[(x, x)+\alpha]^{-1}, \theta_{y,y}(y)\cdot[(y, y)+\alpha]^{-1})=(T(x), y).$$

It follows that $T\in\mathcal{L}(E)$. To prove that $T_1(\theta_{x,y})=\theta_{Tx,y}$, put $\forall\, z\in E$ $w=T_1(\theta_{x,y})(z)-$ $-\theta_{Tx,y}(z)$. It is easy to check that $\forall\, u, v\in E$ $\theta_{u,v}(w)=0$. Taking $u=v=w$, we get $w=0$. The second relation $T_2(\theta_{x,y})=\theta_{x,T^*y}$ follows in a similar way. ▨

CONSEQUENCE 1. $\mathcal{L}(\mathcal{H}_B)\simeq\mathcal{M}(\mathcal{K}\otimes B)$. ▨

§ 3. THE STABILIZATION THEOREM

DEFINITION 5. A set of elements $\{x_i\}_{i\in I}$ in a Hilbert B-module E is called a *system of generators* for E if the finite sums $\{\sum_k x_{i_k}b_k|\ b_k\in B\}$ are dense in E. A module E having a countable system of generators is called *countably generated.*

Next theorem may be regarded as a generalization of Theorem 4 in [7].

THEOREM 2. *If B is an algebra with a continuous G-action and E is a countably generated Hilbert B-module, then* $E \oplus \mathscr{H}_B \simeq \mathscr{H}_B$.

Proof. Every Hilbert B-module may also be considered as a Hilbert $\widetilde{B}$-module ($x \cdot 1 = x$, $\forall x \in E$). Conversely, if $\widetilde{E}$ is a Hilbert $\widetilde{B}$-module, then denoting the closure of the set of finite sums $\{\sum_j y_j b_j \mid y_j \in E, b_j \in B\}$ by E, we get a Hilbert B-module. It may be easily checked that passing from the B-module to the $\widetilde{B}$-module and then back again to the B-module, we get the initial B-module. Having this in mind, we shall suppose in the following proof that *B is unital.*

According to the definition, $\mathscr{H}_B = \bigoplus_{i=1}^{\infty} V_i \otimes B$. We must represent $E \oplus \mathscr{H}_B$ in the same form $\bigoplus_{i=1}^{\infty} W_i \otimes B$. Let $\{x_k\}$ be a countable system of generators for E and $\{e_k\}$ an orthonormal basis in $\mathscr{H}_B$ constructed from the elements of the spaces V_i Denote by $\{y_j\}$ the sequence of elements in $E \oplus \mathscr{H}_B$ in which every element $x_k \oplus 0$ and every element $0 \oplus e_k$ occur an *infinite* number of times. Proceeding inductively, we suppose that there are already constructed the mutually orthogonal, finite dimensional, G-invariant (and "real"-invariant) linear subspaces $W_1, \ldots, W_n \subset$ $\subset E \oplus \mathscr{H}_B$ satisfying the conditions:

1°. Every W_i has a basis which is orthonormal with respect to the scalar product in $E \oplus \mathscr{H}_B$.

2°. There exists such an integer m, depending on n, that

$$W_1 + \ldots + W_n \subset E_m \stackrel{\text{def}}{=} E \oplus \left(\bigoplus_{i=1}^{m} V_i \otimes B \right).$$

3°. The distance between y_n and $(W_1 + \ldots + W_n) \cdot B$ does not exceed $1/n$.

We shall construct W_{n+1} satisfying the same conditions. Let $\{f_1, \ldots, f_p\}$ be the orthonormal basis in $W_1 + \ldots + W_n$. Put

$$y'_{n+1} = \sum_{j=1}^{p} f_j (f_j, y_{n+1}),$$

$$y''_{n+1} = y_{n+1} - y'_{n+1}.$$

Since either $y_{n+1} \in E$, or $y_{n+1} \in V_i$ for some i, we may suppose that $y''_{n+1} \in E_{m'}$, for some $m' \geqslant m$ (see condition 2°). By the Mostow's theorem on periodic vectors ([9], 2.16) there exists such an element $z \in E_{m'}$, that $\|z - y''_{n+1}\| \leqslant 1/_{2n+2}$ and Gz is contained in a finite dimensional, G-invariant subspace $R \subset E_{m'}$. We may suppose that z and R are both orthogonal to $W_1 + \ldots + W_n$. Replacing R by $R + \bar{R}$, we may also suppose that R is invariant under the "real" involution. Now find such $m'' > m'$ that R is isomorphic as a "real" G-module to $V_{m''}$. Fix an orthonormal basis $\{h_1, \ldots, h_k\} \subset V_{m''}$. By our isomorphism, there is a corresponding basis $\{l_1, \ldots, l_k\} \subset R$. We have:

$$g(h_i) = \sum_{j=1}^{k} T_{ji}(g)h_j ,$$

$$g(l_i) = \sum_{j=1}^{k} T_{ji}(g)l_j \qquad (g \in G),$$

$$\overline{h}_i = h_i, \ \overline{l}_i = l_i,$$

where $T : G \to \mathscr{U}(k)$ is some unitary representation. Since $V_{m''}$ is orthogonal to R, the relation $(l_i, h_j) = 0$ holds $\forall\, i, j$. Let $z = \sum_{i=1}^{k} \alpha_i l_i$, $\alpha = \left(\sum_{i=1}^{k} |\alpha_i|^2\right)^{1/2} + 1$, $l'_i = l_i + h_i/(2n+2)\alpha$. The matrix $L = (L_{ij}) = ((l'_i, l'_j))$ is positive and invertible in the algebra $M_k(B)$ because $(L_{ij}) = ((l_i, l_j)) + [(2n+2)\alpha]^{-2}\cdot(\delta_{ij})$.

Put $\mathscr{D} = (d_{ji}) = L^{-1/2} \in M_k(B)$. Define W_{n+1} to be the linear span of $\{l''_1, \ldots, l''_k\}$ where $l''_i = \sum_{j=1}^{k} l'_j d_{ji}$. Obviously, $W_{n+1} \subset E_{m''}$ and the basis $\{l''_1, \ldots, l''_k\}$ is orthonormal: $((l''_i, l''_j)) = \left(\sum_{p,q=1}^{k} (l'_p d_{pi}, l'_q d_{qj})\right) = \mathscr{D}^* L \mathscr{D} = (\delta_{ij})$. Moreover, W_{n+1} is orthogonal to $W_1 + \ldots + W_n$ since the elements $\{l_i\}$ and $\{h_i\}$ are orthogonal to $W_1 + \ldots + W_n$. A direct computation with the basis $\{l''_i\}$ shows that W_{n+1} is G-invariant and "real". Finally, denoting the distance by ρ, we have:

$$\rho(y_{n+1}, (W_1 + \ldots + W_{n+1})\cdot B) \leqslant$$

$$\leqslant \rho(y''_{n+1}, z) + \rho(z, W_{n+1}\cdot B) \leqslant$$

$$\leqslant 1/(2n+2) + \rho(z, z') \leqslant 1/(n+1),$$

where

$$z = \sum_{i=1}^{k} \alpha_i l_i , \ z' = \sum_{i=1}^{k} \alpha_i l'_i,$$

$$\rho(z, z') \leqslant [(2n+2)\alpha]^{-1} \cdot \left\| \sum_{i=1}^{k} \alpha_i h_i \right\| \leqslant 1/(2n+2).$$

This finishes the induction. Now the properties of the sequences $\{y_j\}$ and $\{W_i\}$ show that we get the required decomposition of $E \oplus \mathscr{H}_B$ into the direct sum. ▨

REMARK 1. If B has a countable approximate unit, then $\mathscr{H}_B$ is countably generated over B.

§ 4. STINESPRING'S THEOREM

THEOREM 3. *Assume that the algebra A is separable, the algebra B has a countable approximate unit, and the G-actions on A and B are continuous. Let $\varphi : A \to \mathcal{M}(\mathcal{K} \otimes B)$ be a completely positive map. Then:*

1) *There exists a homomorphism $\pi : A \to \mathcal{M}(\mathcal{K} \otimes B)$ and an invariant element $V \in \mathcal{M}(\mathcal{K} \otimes B)$ with the property: $\varphi(a) = V^*\pi(a)V$, $\forall\, a \in A$. If A is unital, then π may also be chosen unital.*

2) *If A and φ are unital, there exists such a unital homomorphism*

$$\rho : A \to \mathcal{M}(M_2 \otimes \mathcal{K} \otimes B) = \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B)$$

that $\forall\, a \in A$ $\varphi(a) \oplus 0 = q\rho(a)q$, where $q \in \mathcal{L}(\mathcal{H}_B \oplus \mathcal{H}_B)$ is the projection onto the first direct summand.

For a non-unital algebra A the assertion 2) remains valid if $\|\varphi\| \leqslant 1$.

Proof. First of all we shall consider the case of a unital A. There is a B-valued scalar product on the algebraic tensor product $A \otimes \mathcal{H}_B$:

$$\left(\sum_{i=1}^{n} a_i \otimes x_i,\ \sum_{j=1}^{m} b_j \otimes y_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i, \varphi(a_i^* b_j) y_j).$$

Evidently,

$$\left(\sum_{i=1}^{n} a_i \otimes x_i,\ \sum_{i=1}^{n} a_i \otimes x_i\right) = \sum_{i,j=1}^{n} (x_i, \varphi(a_i^* a_j) x_j) \geqslant 0$$

because the matrix $\sum_{i,j=1}^{n} e_{ij} \otimes a_i^* a_j \in M_n \otimes A$ is positive, and the map $1 \otimes \varphi : M_n \otimes A \to \mathcal{L}(\mathcal{H}_B^n)$ is also positive. Put $\forall\, b \in B$, $g \in G$, $a \in A$:

$$(\sum_i a_i \otimes x_i) \cdot b = \sum_i a_i \otimes x_i b,$$

$$g(\sum_i a_i \otimes x_i) = \sum_i g(a_i) \otimes g(x_i),$$

$$\overline{(\sum_i a_i \otimes x_i)} = \sum_i \bar{a}_i \otimes \bar{x}_i,$$

$$a \cdot (\sum_i a_i \otimes x_i) = \sum_i a a_i \otimes x_i.$$

Then $A \otimes \mathcal{H}_B$ becomes a B-module with a scalar product satisfying Definition 1 except for the condition: $(z, z) = 0 \Rightarrow z = 0$ in the condition 4°. Also we have:

$$(1)\quad \left(a\left(\sum_{i=1}^{n} a_i \otimes x_i\right),\ a\left(\sum_{i=1}^{n} a_i \otimes x_i\right)\right) \leqslant \|a\|^2 \cdot \left(\sum_{i=1}^{n} a_i \otimes x_i,\ \sum_{i=1}^{n} a_i \otimes x_i\right)$$

since the inequality

$$\sum_{i,j=1}^{n} e_{ij} \otimes a_i^* a^* a a_j \leqslant \|a\|^2 \cdot \left(\sum_{i,j=1}^{n} e_{ij} \otimes a_i^* a_j \right)$$

holds in $M_n(A)$ and φ is completely positive.

From Lemma 1 we now derive that

$$\mathcal{N} = \{z \in A \otimes \mathcal{H}_B | (z, z) = 0\}$$

s a B-submodule in $A \otimes \mathcal{H}_B$ with the property: $(x, y) = 0 \ \forall \, x \in A \otimes \mathcal{H}_B, y \in \mathcal{N}$, and from the inequality (1) we obtain that $\mathcal{N}$ is invariant under the action of A. Let E be the completion of $(A \otimes \mathcal{H}_B)/\mathcal{N}$ by the norm $\|z + \mathcal{N}\| = \|(z + \mathcal{N}$ $z + \mathcal{N})\|^{1/2}$. Then E is a Hilbert B-module, and the left action of A defines a unital homomorphism $\pi' : A \to \mathcal{L}(E)$.

Now define the maps $W : \mathcal{H}_B \to E$ and $W^* : A \otimes \mathcal{H}_B \to \mathcal{H}_B$ by the formulas: $W(x) = 1 \otimes x + \mathcal{N}$, $W^* \left(\sum_{i=1}^{n} a_i \otimes x_i \right) = \sum_{i=1}^{n} \varphi(a_i) x_i$. Then

$$\left\| W^* \left(\sum_{i=1}^{n} a_i \otimes x_i \right) \right\|^2 = \left\| \sum_{i,j=1}^{n} (x_i \, , \varphi(a_i^*) \varphi(a_j) x_j) \right\| \leqslant$$

$$\leqslant \|\varphi(1)\| \cdot \left\| \left(\sum_{i=1}^{n} a_i \otimes x_i \, , \ \sum_{i=1}^{n} a_i \otimes x_i \right) \right\|$$

because for the map $\varphi' = 1 \otimes \varphi : M_n \otimes A \to \mathcal{L}(\mathcal{H}_B^n)$ we have that $\varphi'(a^*) \cdot \varphi'(a) \leqslant \leqslant \|\varphi'(1)\| \cdot \varphi'(a^* a)$. (This inequality for an arbitrary completely positive map follows easily from the usual Stinespring's theorem.) Therefore $W^*(\mathcal{N}) = 0$, and W^* may be extended on E by continuity. An easy check shows that $\forall \, x \in \mathcal{H}_B$, $y \in E$, $a \in A$

$$(W(x), y) = (x, W^*(y)) \quad [\text{i.e., } W \in \mathcal{L}(\mathcal{H}_B, E)]$$

and

$$W^* \pi'(a) W = \varphi(a).$$

Take an arbitrary unital homomorphism $\pi_1: A \to \mathcal{L}(\mathcal{H})$ (for example, cf. Lemma 8 in § 6 below). Composing with the embedding $\mathcal{L}(\mathcal{H}) = \mathcal{M}(\mathcal{K}) \otimes \mathbb{C} \subset \mathcal{M}(\mathcal{K} \otimes \otimes B) = \mathcal{L}(\mathcal{H}_B)$, we get a unital homomorphism $\pi_0 : A \to \mathcal{L}(\mathcal{H}_B)$. Finally let $\pi = \pi' \oplus \pi_0 : A \to \mathcal{L}(E \oplus \mathcal{H}_B)$, $i : E \to E \oplus \mathcal{H}'_B$ be the inclusion, $j : E \oplus \mathcal{H}_B \to E$ the projection. Put $V = i \cdot W$, $V^* = W^* \cdot j$. According to Theorem 2 $E \oplus \mathcal{H}_B \simeq \mathcal{H}_B$, and we get the required $V \in \mathcal{L}(\mathcal{H}_B)$ and $\pi : A \to \mathcal{L}(\mathcal{H}_B)$. This proves 1) for a unital A.

If $\varphi(1) = 1$, then $W^* W = 1$ and WW^* is a projection. Hence

$$E \simeq \operatorname{Im}(WW^*) \oplus \operatorname{Im}(1 - WW^*) \simeq \mathcal{H}_B \oplus E'.$$

It follows that

$$E \oplus \mathscr{H}_B \simeq \mathscr{H}_B \oplus (E' \oplus \mathscr{H}_B) \simeq \mathscr{H}_B \oplus \mathscr{H}_B .$$

With this identification V becomes the inclusion $\mathscr{H}_B \oplus 0 \hookrightarrow \mathscr{H}_B \oplus \mathscr{H}_B$ and V^* becomes the projection of $\mathscr{H}_B \oplus \mathscr{H}_B$ onto the first summand. This proves 2).

For a non-unital A notice first that every completely positive map $A \to \mathscr{D}$ is bounded. For complex algebras it follows from the case $\mathscr{D} = \mathbb{C}$ with the use of the Banach-Steinhaus theorem (see the proof of 4.1 in [8]). For real algebras one must pass to complexification.

Now a suitable normalization of φ reduces assertion 1) to the case $\|\varphi\| \leqslant 1$. Replacing φ by $\tilde{\varphi} : \tilde{A} \to \mathscr{L}(\mathscr{H}_B)$ (see point 5 in § 1), we come to the unital case. ▨

§ 5. FACTORABLE MAPS

The terminology in this section is an exception from the rule indicated in point 6 of § 1: we shall not assume here that all completely positive maps are equivariant. Recall that a completely positive map $\varphi : A \to B$ is *factorable* if there exists an integer n and such completely positive maps $\sigma : A \to M_n$, $\tau : M_n \to B$ that $\varphi = \tau\sigma$. The map φ is *nuclear* if it belongs to the point-norm closure of the set of factorable maps. Different special cases of the next lemma are well known ([5]).

LEMMA 5. *For any two algebras A and B there exists a one-to-one correspondence between completely positive maps $\varphi : A \to M_n \otimes B$ and completely positive maps $\psi : M_n \otimes A \to B$ defined in the following way. If $\varphi(a) = \sum_{i,j} e_{ij} \otimes \varphi_{ij}(a)$, $\psi(\sum_{i,j} e_{ij} \otimes a_{ij}) = \sum_{i,j} \psi_{ij}(a_{ij})$, then the correspondence $\varphi \leftrightarrow \psi$ means that $\varphi_{ij}(a) = \psi_{ij}(a)$, $\forall\, a \in A, \forall\, i,j$. If the group G acts on $M_n \otimes A$ by the formula:*

$$g(\sum_{i,j} e_{ij} \otimes a_{ij}) = \sum_{i,j}(\sum_{k,l} T_{ki}(g)\bar{T}_{lj}(g)e_{kl} \otimes g(a_{ij}))$$

and on $M_n \otimes B$ by the formula:

$$g(\sum_{i,j} e_{ij} \otimes b_{ij}) = \sum_{i,j}(\sum_{k,l} \bar{T}_{ki}(g)T_{lj}(g)e_{kl} \otimes g(b_{ij}))$$

then the correspondence $\varphi \leftrightarrow \psi$ takes equivariant maps into equivariant ones. (Here $T : G \to \mathscr{U}(n)$ is a unitary representation and $\bar{T}$ is its complex conjugate representation). ▨

LEMMA 6. *For any two Hilbert B-modules E_1 and E_2 there are isomorphisms of linear spaces $\mathscr{L}(E_2^n, E_1) \simeq \mathscr{L}(E_2, E_1^n)$, $\mathscr{K}(E_2^n, E_1) \simeq \mathscr{K}(E_2, E_1^n)$ which transfer $V : E_2^n \to E_1$, $V(x_1, \ldots, x_n) = \sum_{i=1}^{n} V_i(x_i)$ into $W : E_2 \to E_1^n$, $W(x) = (W_1(x), \ldots, W_n(x))$ iff $V_i(x) = W_i(x)$, $\forall\, x \in E_2, \forall\, i$. Define the completely positive maps*

$$\varphi : \mathscr{L}(E_1) \to \mathscr{L}(E_2^n) = M_n \otimes \mathscr{L}(E_2)$$

and

$$\psi : M_n \otimes \mathscr{L}(E_1) = \mathscr{L}(E_1^n) \to \mathscr{L}(E_2)$$

by the formulas: $\varphi(a) = V^*aV$, $\psi(a) = W^*aW$ *where* $V \in \mathscr{L}(E_2^n, E_1)$, $W \in \mathscr{L}(E_2, E_1^n)$. *Then the correspondence* $V \leftrightarrow W$ *leads to the same correspondence* $\varphi \leftrightarrow \psi$ *as in Lemma* 5. *If* G *acts on* E_1^n *by the formula:*

$$g(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} T_{1i}(g) \cdot g(x_i), \ldots, \sum_{i=1}^{n} T_{ni}(g) \cdot g(x_i) \right)$$

and on E_2^n *by the formula:*

$$g(x_1, \ldots, x_n) = \left(\sum_{i=1}^{n} \overline{T}_{1i}(g) \cdot g(x_i), \ldots, \sum_{i=1}^{n} \overline{T}_{ni}(g) \cdot g(x_i) \right)$$

then the correspondences $V \leftrightarrow W$, $V \to \varphi$, $W \to \psi$ *preserve the property of being equivariant.* (*Here* T *and* $\overline{T}$ *are the same as in Lemma* 5). ▨

LEMMA 7. *Let* A *and* B *be two algebras with continuous* G*-actions and* $\varphi : A \to B$ *an equivariant nuclear map. Then* φ *belongs to the point-norm closure of the set of equivariant factorable maps.*

Proof. Choose the completely positive maps $\sigma : A \to M_m$ and $\tau : M_m \to B$ so that $\|\tau\sigma(a) - \varphi(a)\| \leqslant \varepsilon$ for every a from some bounded subset $X \subset A$. We may suppose that $G \cdot X = X$, $\overline{X} = X$. The G-action on M_m is taken to be trivial. By Lemma 5 τ may be considered as a positive element of $M_m \otimes B$, i. e., $\tau = z^*z$, $z \in M_m \otimes B$. By the Mostow's theorem ([9], 2.16) there exists a periodic element $y \in M_m \otimes B$ which is close enough to z. Denote the linear span of Gy by S_1, and $S_1 + \overline{S}_1$ by S. Let $y_1, \ldots, y_n$ be the "real" basis in S in which the G-action is unitary: $g(y_i) = \sum_{j=1}^{n} T_{ji}(g)\, y_j$, $T : G \to \mathscr{U}(n)$. Then $y = \sum_{i=1}^{n} \alpha_i y_i$ with some $\{\alpha_i\}$.

Define the G-action on M_n by the formula:

$$g(e_{ij}) = \sum_{k,l} T_{ki}\ (g)\, \overline{T}_{lj}(g)\, e_{kl}.$$

An easy check shows that the element $\sum_{i,j=1}^{n} e_{ij} \otimes y_i^* y_j \in M_n \otimes M_m \otimes B$ is invariant. By Lemma 5, there is an equivariant completely positive map $\eta : M_n \otimes M_m \to B$ corresponding to this element. Define a completely positive map

$$\alpha : M_m \to M_n \otimes M_m \quad \text{by } \alpha(e_{kl}) = \sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j e_{ij} \otimes e_{kl}.$$

Then the composition $\eta\cdot\alpha$ coincides with the map defined by the element $y^*y \in M_m \otimes B$. Choosing y so close to z that $\|(\tau - \eta\cdot\alpha)\,\sigma(a)\| \leqslant \varepsilon$, $\forall\, a \in X$, we get $\|\varphi(a) - \eta\alpha\sigma(a)\| \leqslant 2\varepsilon$ for $a \in X$. Finally, put $\forall\, a \in A$ $\xi_1(a) = \int_G g^{-1}\alpha\sigma(ga)\,dg$ (where dg is the normalized Haar measure on G), $\xi(a) = (\xi_1(a) + \overline{\xi_1(a)})/2$. The factorization $\xrightarrow{\xi} M_{nm} \xrightarrow{\eta} B$ is equivariant and $\forall\, a \in X$ $\|\varphi(a) - \eta\xi(a)\| \leqslant 2\varepsilon$. ▨

§ 6. GLIMM'S LEMMA

This section contains the main assertion leading to the Voiculescu's theorem. In everything connected with the Voiculescu's theorem we shall closely follow the scheme suggested by Arveson [1]. From now and on we again assume that all completely positive maps, and in particular *positive functionals* and *states*, are equivariant (see point 6 of § 1). The G-actions on algebras A and B are assumed to be continuous.

By A^G and $\mathscr{H}^G$ we denote the G-invariant elements in the algebra A and the Hilbert space $\mathscr{H}$ respectively. Every homomorphism $\pi : A \to \mathscr{L}(\mathscr{H})$ defines by restriction the homomorphism $\pi^G : A^G \to \mathscr{L}(\mathscr{H}^G)$.

DEFINITION 6. An inclusion $\pi: A \hookrightarrow \mathscr{L}(\mathscr{H})$ is called a *G-inclusion* if $(1 \otimes \pi)^G$: $(\mathscr{L}(\mathbb{C}^n) \otimes A)^G \to \mathscr{L}((\mathbb{C}^n \otimes \mathscr{H})^G)$ is an inclusion for any n and any unitary G-action on $\mathbb{C}^n$.

LEMMA 8.1) *If π^G is an inclusion, then π is also an inclusion.*

2) *The restriction of states from A to A^G defines a one-to-one correspondence between the states of A and the states of A^G.*

3) *If A is separable, then there exists an (unital) inclusion $\pi : A \hookrightarrow \mathscr{L}(H)$ where H is a separable Hilbert space. For any such π*

$$1 \otimes \pi : A \to \mathscr{L}(\mathscr{H} \otimes H) \simeq \mathscr{L}(\mathscr{H})$$

is a G-inclusion.

Proof. To obtain the first two statements we simply use the averaging by G. In 3) we may put $\pi = \bigoplus_{i=1}^{\infty} \pi_i : A \to \mathscr{L}\left(\bigoplus_{i=1}^{\infty} H_i\right)$ where π_i are cyclic representations with $\bigcap_i \operatorname{Ker} \pi_i = 0$. Now denoting $\mathscr{L}(\mathbb{C}^n) \otimes A$ by A and $\mathbb{C}^n \otimes H$ by H, for any inclusion π and $a \neq 0$ in A we can find an orthonormal system $\{x_1, \ldots, x_m\} \subset H$ so that $g(x_i) = \sum_{j=1}^{m} T_{ji}(g)x_j$ and $\pi(a)x_k \neq 0$ for some k. Choose orthonormal $\{y_1, \ldots, y_m\} \subset \mathscr{H}$ so that $g(y_i) = \sum_{j=1}^{m} \overline{T_{ji}(g)}y_j$. Then $z = \sum_{i=1}^{m} y_i \otimes x_i \in (\mathscr{H} \otimes H)^G$ and $(1 \otimes \pi(a))(z) \neq 0$. ▨

Next assertion is an equivariant analogue of Theorem 2.5 of [2] (see also [1]).

LEMMA 9. *Assume that A is separable, $A \hookrightarrow \mathscr{L}(\mathscr{H})$ is a G-inclusion, φ: $A \to M_n = \mathscr{L}(\mathbb{C}^n)$ is completely positive, $\varphi(A \cap \mathscr{K}) = 0$. Then there exists such a sequence of equivariant linear maps $V_i: \mathbb{C}^n \to \mathscr{H}$ that $\forall a \in A$ $\lim_{i\to\infty} \|\varphi(a) - V_i^* a V_i\| = 0$ and $\forall \eta \in \mathscr{H}$ $\lim_{i\to\infty} \|V_i^*(\eta)\| = 0$.*

Proof. First we shall deal with the case of $n = 1$ and the trivial G-action on $\mathbb{C}^1$. Restricting φ to $A^G \subset \mathscr{L}(\mathscr{H}^G)$ and applying Glimm's lemma (cf. [6], 11. 2.1; note that in the real case the proof of Glimm's lemma is the same as in the complex case, and the "real" case may be obtained from the real one), we get such a sequence of invariant vectors $\xi_i \in \mathscr{H}^G$ that $\varphi(a) = \lim_{i\to\infty} (\xi_i, a\xi_i)$, $\forall a \in A^G$; $\lim_{i\to\infty} (\xi_{i,\eta}) = 0$, $\forall \eta \in \mathscr{H}^G$. Obviously the second relation remains valid $\forall \eta \in \mathscr{H}$, and in view of point 2) of Lemma 8 the first relation also extends to all $a \in A$. (Note that the first limit exists $\forall a \in A$ because $(\xi_i, a\xi_i) = (\xi_i, a^G \xi_i)$ where $a^G = \int_G g(a)\, dg$.) Finally we put $V_i(\beta) = \beta\xi_i$, $\beta \in \mathbb{C}$. The case of arbitrary n and arbitrary G-action on $\mathbb{C}^n$ follows by replacing $\varphi : A \to M_n$ with $\psi : M_n \otimes A \to \mathbb{C}$ (Lemma 5) and applying Lemma 6. ▨

THEOREM 4. *Assume that A is separable, $A \hookrightarrow \mathscr{L}(\mathscr{H})$ is a G-inclusion, $\varphi : A/(A \cap \mathscr{K}) \to M_n(B) = \mathscr{K}(B^n)$ is a nuclear map, the G-actions on A and B are continuous, and the G-action on B^n is such as in Lemma 6. Let us consider $\mathscr{L}(\mathscr{H})$ as a subalgebra of scalar operators in $\mathscr{L}(\mathscr{H}_B)$* (cf. the proof of Theorem 3). *Then there exists such a sequence of equivariant elements $V_i \in \mathscr{K}(B^n, \mathscr{H}_B)$ that $\forall a \in A$ $\lim_{i\to\infty} \|\varphi(a) - V_i^* a V_i\| = 0$ and $\forall \eta \in \mathscr{H}_B$ $\lim_{i\to\infty} \|V_i^*(\eta)\| = 0$.*

Proof. The case of an arbitrary n reduces to $n = 1$ in the same way as in Lemma 9. For $n = 1$ it suffices to construct such a sequence of invariant elements $\xi_i \in \mathscr{H}_B$ that $\forall a \in A$ $\lim_{i\to\infty} \|\varphi(a) - (\xi_i, a\xi_i)\| = 0$ and $\forall \eta \in \mathscr{H}_B$ $\lim_{i\to\infty} \|(\xi_i, \eta)\| = 0$. Then we shall be able to define $V_i : B \to \mathscr{H}_B$ and $V_i^* : \mathscr{H}_B \to B$ by $V_i(b) = \xi_i b$, $V_i^*(\eta) = (\xi_i, \eta)$. Evidently, $V_i \in \mathscr{K}(B, \mathscr{H}_B)$ because $V_i = \lim_{\alpha \to +0} \theta_{\xi_i, \delta_i(\alpha)}$ where $\delta_i(\alpha) = (\xi_i, \xi_i)$ $[(\xi_i, \xi_i) + \alpha]^{-1}$.

In the construction of $\{\xi_i\}$ we may suppose in view of Lemma 7 that φ is factorable: $\varphi : A \xrightarrow{\sigma} M_k \xrightarrow{\tau} B$ and $\sigma(A \cap \mathscr{K}) = 0$. The group G acts on $\mathbb{C}^k$ and $M_k = \mathscr{L}(\mathbb{C}^k)$ via some representation $T : G \to \mathscr{U}(k)$. Consider a completely positive map

$$\tilde{\sigma} = 1 \otimes \sigma : A \to M_k \otimes M_k = \mathscr{L}(\mathbb{C}^k \otimes \mathbb{C}^k)$$

where G acts on $\mathbb{C}^k \otimes \mathbb{C}^k$ via $\overline{T} \otimes T$. According to Lemma 9 there exists such a sequence $W_i : \mathbb{C}^k \otimes \mathbb{C}^k \to \mathscr{H}$ that $\forall\, a \in A$ $\tilde{\sigma}(a) = \lim_{i\to\infty} W_i^* a W_i$ and $\forall\, \eta \in \mathscr{H}$ $\lim_{i\to\infty} W_i^*(\eta) = 0$. Fix a coordinate basis $\{e_1, \ldots, e_k\} \subset \mathbb{C}^k$. Let $\sigma(a) = \sum_{p,q=1}^{k} \sigma_{pq}(a) e_{pq}$, $\tau(e_{pq}) = \tau_{pq}$. Then $\varphi(a) = \sum_{p,q=1}^{k} \sigma_{pq}(a)\tau_{pq}$. Define the matrix $(\rho_{pq}) \in M_k(B)$ as a positive square root of the matrix (τ_{pq}). Put $\xi_i = \sum_{p,q=1}^{k} W_i(e_p \otimes e_q)\rho_{pq}$. It can be easily checked that ξ_i is invariant. Finally we have:

$$\lim_{i\to\infty}(\xi_i, a\xi_i) = \lim_{i\to\infty} \sum_{p,q,r,s} (W_i(e_p \otimes e_q)\rho_{pq}, aW_i(e_r \otimes e_s)\rho_{rs}) =$$
$$= \lim_{i\to\infty} \sum_{p,q,r,s} (e_p \otimes e_q, W_i^* a W_i(e_r \otimes e_s))\rho_{pq}^* \rho_{rs} =$$
$$= \sum_{p,q,r,s} (e_p, e_r) \cdot (e_q, \sigma(a) e_s)\rho_{pq}^* \rho_{rs} =$$
$$= \sum_{q,r,s} \sigma_{qs}(a)\rho_{rq}^* \rho_{rs} = \sum_{q,s} \sigma_{qs}(a)\tau_{qs} = \varphi(a). \ ▨$$

§ 7. VOICULESCU'S THEOREM

Recall (cf. [1]) that an approximate unit $\{e_k\}$ for an ideal $B \subset \mathscr{D}$ is called *quasi-central* if $\forall\, d \in \mathscr{D}$ $\lim_k \|e_k d - d e_k\| = 0$. The existence of quasicentral approximate units is proved in [1]. Moreover, if we have any approximate unit $\{u_i\}$ for B, it is possible to construct a quasicentral approximate unit $\{e_k\}$ consisting of the convex linear combinations of elements u_i. If B has a countable approximate unit and $\mathscr{D}/B$ is separable, $\{e_k\}$ may be taken countable.

NOTATION. For $T_1, T_2 \in \mathscr{L}(E)$, $T_1 \sim T_2$ means that $T_1 - T_2 \in \mathscr{K}(E)$.

LEMMA 10. *Assume that algebra A is separable, B has a countable approximate unit, the G-actions on A and B are continuous, and $\varphi : A \to \mathscr{L}(\mathscr{H}_B)$ is a completely positive map. Then there exists a sequence of completely positive maps $\varphi_n : A \to \mathscr{L}(\mathscr{H}_B)$ satisfying the conditions:*

1) $\forall\, n, \forall\, a \in A$ $\varphi_n(a) \sim \varphi(a)$,

2) $\forall\, a \in A$ $\lim_{n\to\infty} \|\varphi_n(a) - \varphi(a)\| = 0$,

3) *In the standard decomposition $\mathscr{H}_B = \bigoplus_{i=1}^{\infty} W_i \otimes B$ there exists $\forall n$ a sequence of integers $\{m_{n,k}\}$ and a sequence of completely positive maps $\psi_{n,k} : A \to \mathscr{K}\left(\bigoplus_{i=1}^{m_{n,k}} W_i \otimes B\right)$ such that $\forall\, a \in A$ $\varphi_n(a) = \sum_{k=1}^{\infty} \psi_{n,k}(a)$ in the sense of strict convergence in $\mathscr{L}(\mathscr{H}_B) = \mathscr{M}(\mathscr{K}_B)$.*

Proof. Choose such a compact set $X \subset A$ that its linear span is dense in A. Also fix $\varepsilon > 0$. It suffices to construct such a completely positive map $\psi : A \to \mathcal{L}(\mathcal{H}_B)$ that $\forall x \in X$ $\psi(x) \sim \varphi(x)$, $\|\psi(x) - \varphi(x)\| \leqslant \varepsilon$, and $\forall a \in A$ $\psi(a) = \sum_{k=1}^{\infty} \psi_k(a)$ as in 3). Let $p_k \in \mathcal{L}(\mathcal{H}_B)$ be the orthogonal projection on $\bigoplus_{i=1}^{k} W_i \otimes B$ and $\{u_j\}$ an invariant, countable, increasing approximate unit in B (an invariant approximate unit may be obtained by averaging). Denote by $\{e_k\}$ an increasing, quasicentral, countable approximate unit in $\mathcal{K}_B \subset \mathcal{D} = \mathrm{Alg}\,(\varphi(A)) + \mathcal{K}_B$ consisting of the convex linear combinations of elements $(1 \otimes u_i) \cdot p_i$. Passing to a subsequence if necessary, $\{e_k\}$ will satisfy

$$\|\varphi(x) \cdot (e_k - e_{k-1})^{1/2} - (e_k - e_{k-1})^{1/2} \cdot \varphi(x)\| \leqslant \varepsilon \cdot 2^{-k},$$

$\forall k \geqslant 1, \forall x \in X$ (here $e_0 = 0$). Put $f_k = (e_k - e_{k-1})^{1/2}$. Then $\forall x \in X$ $\sum_{k=1}^{\infty} \|\varphi(x) f_k - f_k \varphi(x)\| \leqslant \varepsilon$. Therefore the series $\sum_{k=1}^{\infty} (\varphi(x) f_k - f_k \varphi(x)) f_k$ is convergent in norm, its sum belongs to $\mathcal{K}_B$ and the norm of the sum does not exceed ε.

Since $\sum_{k=1}^{\infty} f_k^2$ strictly converges to 1, for any $T \in \mathcal{L}(\mathcal{H}_B)$ the series $\sum_{k=1}^{\infty} f_k T f_k$ is also strictly convergent. In fact, for $T \geqslant 0$, $b \in \mathcal{H}_B$ we have:

$$\left\|\sum_{k=m}^{n} f_k T f_k \cdot b\right\|^2 = \left\| b^* \left(\sum_{k=m}^{n} f_k T f_k\right)^2 b \right\| \leqslant$$

$$\leqslant \left\|\sum_{k=m}^{n} f_k T f_k\right\| \cdot \left\| b^* \left(\sum_{k=m}^{n} f_k T f_k\right) b \right\| \leqslant$$

$$\leqslant \|T\| \cdot \|T\| \cdot \left\|\sum_{k=m}^{n} b^* f_k^2 b\right\| \xrightarrow[n \to \infty]{} 0.$$

In a complex algebra every element T is a linear combination of four positive elements and in the real case the required convergence follows by complexification.

Now put $\forall a \in A$, $\psi_k(a) = f_k \varphi(a) f_k$, $\psi(a) = \sum_{k=1}^{\infty} \psi_k(a)$. Since $\forall k$, $f_k \in \mathcal{K}\left(\bigoplus_{i=1}^{m_k} W_i \otimes B\right)$, for some m_k, and $\forall x \in X, \varphi(x) - \psi(x) = \sum_{k=1}^{\infty} (\varphi(x) f_k - f_k \varphi(x)) f_k$, the map ψ satisfies all the conditions. ▨

Next theorem is a generalization of Theorem 4 in [1].

THEOREM 5. *Assume that the algebra A is unital and separable, B has a countable approximate unit, the G-actions on A and B are continuous, and $A \hookrightarrow \mathcal{L}(\mathcal{H})$ is a unital*

G-inclusion. Also assume that $\forall n$ *every completely positive map* $A/(A \cap \mathscr{K}) \to M_n(B)$ *is nuclear (in particular, it is sufficient to assume at least one of the two algebras A or B to be nuclear). We shall consider* $\mathscr{L}(\mathscr{H})$ *as a subalgebra of scalar operators in* $\mathscr{L}(\mathscr{H}_B)$. *Then for any unital, completely positive map* $\varphi : A/(A \cap \mathscr{K}) \to \mathscr{L}(\mathscr{H}_B)$ *there exists a sequence of invariant elements* $\{V_n\} \subset \mathscr{L}(\mathscr{H}_B)$ *satisfying* $\forall n, \forall a \in A$ *the conditions:*

1) $\varphi(a) \sim V_n^* a V_n$,

2) $\lim_{i \to \infty} \|\varphi(a) - V_i^* a V_i\| = 0$,

3) $V_n^* V_n = 1$.

Proof. Note first that having a sequence $\{V_n\}$ satisfying 1) and 2), it is not difficult to get a sequence satisfying 1) — 3). In fact, since $\varphi(1) = 1$, there exists such a number n_0, that $\forall n > n_0$ $\|(1 - V_n^* V_n\| < 1$ (by 2). The sequence $V_n' = V_{n+n_0} \cdot (V_{n+n_0}^* V_{n+n_0})^{-1/2}$, $n \geq 1$, is the required one.

Choose such a compact set $X \subset A$ that its linear span is dense in A, $X^* = X$, and $1 \in X$. Fix $\varepsilon > 0$. We must construct such an invariant element $\mathscr{U} \in \mathscr{L}(\mathscr{H}_B)$ that $\forall x \in X$, $\varphi(x) \to \mathscr{U}^* x \mathscr{U}$, $\|\varphi(x) - \mathscr{U}^* x \mathscr{U}\| \leq 3\varepsilon$. Let $\psi = \sum_{k=1}^{\infty} \psi_k$ be the map constructed in the proof of Lemma 10 (for the algebra $A/(A \cap \mathscr{K})$ and the map $\varphi : A/(A \cap \mathscr{K}) \to \mathscr{L}(\mathscr{H}_B)$). Using Theorem 4, it is not difficult to define inductively a sequence of invariant elements $\mathscr{U}_i \in \mathscr{K}_B$ in such a way that $\forall x \in X$, $\forall i, j$:

1. $\|\psi_i(x) - \mathscr{U}_i^* x \mathscr{U}_i\| \leq \varepsilon \cdot 2^{-i}$;
2. $\|\mathscr{U}_i^* x \mathscr{U}_j\| \leq \varepsilon \cdot 2^{-i-j}$ for $i \neq j$;
3. the series $\sum_{k=1}^{\infty} \|\mathscr{U}_k^* e_m\|$ is convergent for each element e_m of some countable approximate unit in $\mathscr{H}_B$.

Then $\sum_{k=1}^{\infty} \|\mathscr{U}_k^* \mathscr{U}_k - \psi_k(1)\| \leq \varepsilon$, $\sum_{i \neq j} \|\mathscr{U}_i^* \mathscr{U}_j\| \leq \varepsilon$. The strict convergence of $\sum_{k=1}^{\infty} \psi_k(1)$ implies the strict convergence of $\sum_{k=1}^{\infty} \mathscr{U}_k^* \mathscr{U}_k$. Hence $\forall b \in \mathscr{K}_B$ the series $\sum_{k=1}^{\infty} \mathscr{U}_k b$ is convergent in norm by the Cauchy criterion. Moreover, $\forall b \in \mathscr{K}_B$ the series $\sum_{k=1}^{\infty} \mathscr{U}_k^* b$ is also convergent as it is convergent for $b \in \{e_m\}$. Therefore $\sum_{k=1}^{\infty} \mathscr{U}_k$ converges in the strict topology. Put $\mathscr{U} = \sum_{k=1}^{\infty} \mathscr{U}_k \in \mathscr{L}(\mathscr{H}_B)$. Then $\forall x \in X$

$$\|\varphi(x) - \mathscr{U}^* x \mathscr{U}\| \leq \|\varphi(x) - \psi(x)\| + \sum_{k=1}^{\infty} \|\psi_k(x) - \mathscr{U}_k^* x \mathscr{U}_k\| + \sum_{i \neq j} \|\mathscr{U}_i^* x \mathscr{U}_j\| \leq 3\varepsilon.$$

Since $\forall i, j, k$ $\psi_k(x) \in \mathscr{K}_B$ and $\mathscr{U}_i^* x \mathscr{U}_j \in \mathscr{K}_B$, we get $\varphi(x) \sim \mathscr{U}^* x \mathscr{U}$. ▨

REMARK 2. In the conditions of Theorem 5 put $P_n = V_n V_n^*$. It is easily to verify that if φ is *a homomorphism*, then $\forall\, a \in A, \forall\, n\;\; P_n a \sim aP_n$ and $\lim\limits_{i\to\infty} \|P_i a - aP_i\| = 0$.

Recall the definition of the approximate equivalence ([12]).

DEFINITION 7. Let E_1 and E_2 be Hilbert B-modules. The maps $\varphi_1 : A \to \mathscr{L}(E_1)$, $\varphi_2 : A \to \mathscr{L}(E_2)$ are called *approximately equivalent* if there exists such a sequence of equivariant isometries $u_n \in \mathscr{L}(E_2, E_1)$ that $\forall\, a \in A, \forall\, n\;\; u_n^* \varphi_1(a) u_n \sim \varphi_2(a)$ and $\lim\limits_{i\to\infty} \|u_i^* \varphi_1(a) u_i - \varphi_2(a)\| = 0$.

Suppose that we have fixed such a compact set $X \subset A$ that its linear span is dense in A. The maps $\varphi_1 : A \to \mathscr{L}(E_1)$, $\varphi_2 : A \to \mathscr{L}(E_2)$ are called *ε-intimate* if there exists an equivariant isometry $u \in \mathscr{L}(E_2, E_1)$ such that $\forall\, x \in X\;\; u^*\varphi_1(x)u \sim \varphi_2(x)$ and $\|u^*\varphi_1(x)u - \varphi_2(x)\| \leqslant \varepsilon$.

Next theorem generalizes Theorem 1.3 of [12].

THEOREM 6. *Assume that the algebra A is unital and separable, B has a countable approximate unit, the G-actions on A and B are continuous, and $A \hookrightarrow \mathscr{L}(\mathscr{H})$ is a unital G-inclusion. Also assume that $\forall\, n$ every completely positive map $A/(A \cap \mathscr{K}) \to M_n(B)$ is nuclear (in particular, it is sufficient to assume at least one of the two algebras A or B to be nuclear). We shall consider $\mathscr{L}(\mathscr{H})$ as a subalgebra of scalar operators in $\mathscr{L}(\mathscr{H}_B)$ and denote the composition $A \hookrightarrow \mathscr{L}(\mathscr{H}) \hookrightarrow \mathscr{L}(\mathscr{H}_B)$ by π. Then for any unital homomorphism $\varphi : A/(A \cap \mathscr{K}) \to \mathscr{L}(\mathscr{H}_B)$ the sum $\varphi \oplus \pi : A \to \mathscr{L}(\mathscr{H}_B \oplus \mathscr{H}_B)$ is approximately equivalent to π.*

Proof. It is sufficient to show that $\varphi \oplus \pi$ and π are 6ε-intimate $\forall\, \varepsilon > 0$. Replacing φ by $\varphi' = \bigoplus\limits_1^\infty \varphi$ and applying Theorem 5 and Remark 2, we can find such an invariant element $V \in \mathscr{L}(\mathscr{H}_B)$ that $V^*V = 1$ and $\forall\, x \in X\;\; \varphi'(x) \sim V^*\pi(x)V$, $\|\varphi'(x) - V^*\pi(x)V\| \leqslant \varepsilon$, $VV^*\pi(x) \sim \pi(x)VV^*$, $\|VV^*\pi(x) - \pi(x)VV^*\| \leqslant \varepsilon$. Clearly, $P = VV^*$ is an invariant orthogonal projection, and V is an isometry of $\mathscr{H}_B$ on $\operatorname{Im} P$. Hence $\varphi' : A \to \mathscr{L}(\mathscr{H}_B)$ and $V\varphi'V^* : A \to \mathscr{L}(\operatorname{Im} P)$ are unitary equivalent (i.e., 0-intimate). As φ' and $V^*\pi V$ are ε-intimate, $V\varphi'V^*$ and $VV^*\pi VV^* = P\pi P$ are also ε-intimate. Therefore φ' and $P\pi P$ are ε-intimate. On the other hand π and $P\pi P + (1-P)\pi(1-P)$ are 2ε-intimate because $\forall\, x \in X\;\; P\pi(x) \sim \pi(x)P$, $\|P\pi(x) - \pi(x)P\| \leqslant \varepsilon$. Denoting $(1-P)\pi(1-P) : A \to \mathscr{L}(\operatorname{Im}(1-P))$ by π_0, we conclude that π and $\varphi' \oplus \pi_0$ are 3ε-intimate and $\varphi \oplus \pi$ and $\varphi \oplus \varphi' \oplus \pi_0$ are also 3ε-intimate. Now from the unitary equivalence of $\varphi' \oplus \pi_0$ and $\varphi \oplus \varphi' \oplus \pi_0$ it follows that π and $\varphi \oplus \pi$ are 6ε-intimate. ▨

REMARK 3. It would be desirable to replace in Theorem 6 the inclusion $A \hookrightarrow \mathscr{L}(\mathscr{H}) \hookrightarrow \mathscr{L}(\mathscr{H}_B)$ by arbitrary $\pi : A \hookrightarrow \mathscr{L}(\mathscr{H}_B)$ and $\varphi : A/(A \cap \mathscr{K}) \to \mathscr{L}(\mathscr{H}_B)$ by $\varphi : A/(A \cap \mathscr{K}_B) \to \mathscr{L}(\mathscr{H}_B)$. But this is impossible without some additional assumptions concerning π. Even in the case of $B = \mathscr{C}(\mathscr{Y})$, the algebra of functions on a locally compact space $\mathscr{Y}$ tending to 0 at ∞, Theorem 6 will not be valid without the condition of π being *strictly monomorphic*. We say that π is strictly monomorphic if (denoting by $\omega_y : \mathscr{L}(\mathscr{H}_{\mathfrak{C}(\mathscr{Y})}) \to \mathscr{L}(\mathscr{H})$ the restriction over $y \in \mathscr{Y}$) we have $\forall\, y \in \mathscr{Y}$, $\omega_y(A)/(\omega_y(A) \cap \mathscr{K}) \simeq A/(A \cap \mathscr{K}_{\mathfrak{C}(\mathscr{Y})})$. It is very likely that this condition is not only necessary but also sufficient. For finite dimensional compact $\mathscr{Y}$, this was recently proved in [15].

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