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TOPOLOGICAL DIRECT INTEGRALS OF LEFT HILBERT ALGEBRAS. I

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INTRODUCTION

In some papers it was shown (see [8], [17]) how the theory of standard von Neumann algebras may be used in order to improve some results in the "separable" theory of direct integrals which goes back to J. von Neumann. In this paper we are concerned in the following question. Using the theory of standard von Neumann algebras, is it possible to develop a "non separable" theory of direct integrals such that within this theory some of the more delicate problems may be solved, including the questions on the types of the associated von Neumann algebras. This question is also motivated by mathematical physics where decompositions of representations of non separable C^*-algebras are considered (see [3]), and in this context standard von Neumann algebras play a natural role (see [19]).

Using Hilbert algebras only, the author started to develop such a theory in [11]. We also want to mention the paper of H. Halpern [7] where the theory of standard von Neumann algebras has been used in order to decompose type III von Neumann algebras with a cyclic and separating vector.

This paper is divided into three sections. First we introduce the notion of a continuous field of left Hilbert algebras and of the corresponding direct integral with respect to a positive Radon measure. Moreover we investigate the behavior with respect to direct integral decomposition of some characteristic operators. In Section 2 we prove that the type of the semifinite portion of the associated von Neumann algebras reduces to the components. Throughout in Section 2 we assume a countability condition to be satisfied which is the analogue of (H8) in [11], 1.8. We will treat the corresponding problem for the type-III-portion in a subsequent paper. Finally in Section 3 we show that the theory applies to the central decomposition of a state φ on some C^*-algebra $\mathscr{A}$ with unit if φ satisfies the KMS-boundary condition with respect to some strongly continuous one parameter group of automorphisms of $\mathscr{A}$.

For the general theory of topological direct integrals of Hilbert spaces we refer the reader to [4] and [10]. For the theory of standard von Neumann algebras we refer the reader to [18], [16] and [12].

1. CONTINUOUS FIELDS OF LEFT HILBERT ALGEBRAS

Let Ω be a locally compact space and let μ be a positive Radon measure on Ω. Furthermore let $\{\mathfrak{A}_\xi\}_{\xi\in\Omega}$ be a family of non trivial left Hilbert algebras (l. H. a.'s), i. e. $\mathfrak{A}_\xi \neq \{0\}$ holds for any $\xi \in \Omega$. We denote the characteristic objects associated with $\mathfrak{A}_\xi$ as usual, we only add ξ as an index. For Δ_ξ (modular operator), S_ξ, F_ξ, J_ξ, $\{\sigma_t^\xi\}_{t\in\mathbf{R}}$ (modular automorphism group), π_ξ (left regular representation of $\mathfrak{A}_\xi$) see [18] and for P_ξ, Q_ξ, R_ξ, T_ξ, $\mathscr{K}_\xi$ see [12].

DEFINITION 1.1. $(\{\mathfrak{A}_\xi\}_{\xi\in\Omega}, \Lambda)$ is called *a continuous field of l. H. a.'s* if Λ is a subset of $\prod_{\xi\in\Omega} \mathfrak{A}_\xi$ which satisfies the following conditions:

(L1) Λ is a linear subspace of $\prod_{\xi\in\Omega} \mathfrak{A}_\xi$.

(L2) For any $x \in \Lambda$ the function $\xi \mapsto \|x(\xi)\|$ belongs to $C_c(\Omega)$, $C_c(\Omega)$ being the set of all continuous functions on Ω whose support is compact.

(L3) For any $x, y \in \Lambda$ the vector field $\xi \mapsto x(\xi)y(\xi)$ belongs to Λ.

(L4) For any $x \in \Lambda$ the vector field $\xi \mapsto x(\xi)^\#$ belongs to Λ.

(L5) For any $x \in \Lambda$ and $f \in C_c(\Omega)$ the vector field $\xi \mapsto f(\xi)x(\xi)$ belongs to Λ.

(L6) For any $\xi \in \Omega$ the real linear space generated by the set $\{x(\xi)^\# x(\xi) \,|\, x \in \Lambda\}$ is dense in $\mathscr{K}_\xi$.

(L7) For any $x \in \Lambda$ the operator field $\xi \mapsto \pi_\xi(x(\xi))$ is bounded, i.e. $\sup_{\xi\in\Omega} \|\pi_\xi(x(\xi))\| < \infty$ holds.

(L8) The operator field $\xi \mapsto J_\xi$ is continuous with respect to Λ.

For any $\xi \in \Omega$ we denote by $\mathscr{H}_\xi$ the Hilbert space which is the completion of $\mathfrak{A}_\xi$.

REMARKS. (1) The remarks (1), (2), (3), (4) to Definition 1.1 in [11] remain valid in our situation.

(2) In [10] we considered some properties which operator fields may have, such as "continuity" and "strong measurability" (see [10], 1.10). In this paper we assume these notions to be extended to fields of operators which are linear with respect to the real numbers only. In particular condition (L8) makes sense. Similarly we extend the notion of a "decomposable" operator (see [10], 1.5) or a "strongly decomposable" operator (see [10], 1.10). Observe that all statements in [10] remain valid in this more general setting, except [10], 2.8.

(3) If Λ satisfies the conditions (L1), (L3), (L4) then condition (L6) is equivalent to the following (see [12], p. 219):

(L6)′ For any $\xi \in \Omega$ the set $\{(x(\xi), x(\xi)^\#) \,|\, x \in \Lambda\}$ is dense in the graph of the operator S_ξ.

By [18], 5.1, we know that $\mathfrak{A}_\xi^0 := \{x(\xi) | x \in A\}$ is a l. H. a. and by (L6) we infer from [13], 1.3, that $\mathfrak{A}_\xi^{0\prime\prime} = \mathfrak{A}_\xi^{\prime\prime}$ holds for any $\xi \in \Omega$.

In the applications which we have in view (see Section 3) it is not so trivial to see that condition (L8) is satisfied. However, as the following proposition shows, there are several possibilities to substitute (L8) by some equivalent conditions. First we need a technical lemma.

LEMMA 1.2. (a) *Let $K \subseteq \Omega$ be a compact subset and let $L \subseteq \prod_{\xi \in \Omega} \mathscr{H}_\xi$ be a subset such that any $x \in L$ is continuous on K and the set $\{x(\xi) | x \in L\}$ is dense in $\mathscr{H}_\xi$ for any $\xi \in K$. Furthermore let $\xi \mapsto Z(\xi)$ be an operator field of (real or complex) linear operators which is bounded on K such that for any $x \in L$ the vector field $\xi \mapsto Z(\xi)x(\xi)$ is continuous on K. Then $\xi \mapsto Z(\xi)$ is also continuous on K.*

(b) *Let $K \subseteq \Omega$ be compact and let $\xi \mapsto Z(\xi)$ be a bounded field of normal operators which is continuous on K such that $\xi \mapsto Z(\xi)^*$ is also continuous on K. Furthermore let $M := (\bigcup_{\xi \in \Omega} \mathrm{Sp}(Z(\xi)))^-$ and let $f \in C(M)$. Then the operator field $\xi \mapsto f(Z(\xi))$ is continuous on K.*

Proof. (a) Let L_0 be the real linear space which is generated by all vector fields of the form $\xi \mapsto f(\xi)x(\xi)$, where $x \in L$ and $f \in C_c(\Omega)$ is real valued. As in [4], p. 81, Proposition 6 one can see that any vector field x which is continuous on K is the uniform limit on K of some sequence of vector fields in L_0. Since $\xi \mapsto Z(\xi)$ is bounded on K and $\xi \mapsto Z(\xi)x(\xi)$ is continuous on K for any $x \in L_0$ we conclude as in [10], 1.4, that our assertion is true.

(b) Since Z is bounded the set M is compact. Let $z \mapsto p_1(z, \bar{z}), z \mapsto p_2(z, \bar{z}), \ldots$ be a sequence of polynomials which converges to f uniformly on M. For any $n \in \mathbf{N}$ the operator field $\xi \mapsto p_n(Z(\xi), Z(\xi)^*)$ is continuous on K. Furthermore we obtain for any $n \in \mathbf{N}$ and $a \in A$

$$\|p_n(Z(\xi), Z(\xi)^*)a(\xi) - f(Z(\xi))a(\xi)\| \leqslant \|p_n - f\|_M \|a(\xi)\| \leqslant$$

$$\leqslant \|p_n - f\|_M \sup_{\xi \in \Omega} \|a(\xi)\|$$

if $\xi \in \Omega$. Since $\lim_{n \to \infty} \|p_n - f\|_M = 0$ holds our assertion follows from this.

PROPOSITION 1.3. *Suppose that the conditions* (L1) *up to* (L6) *are satisfied. Then any two of the following statements are equivalent:*

(1) *$\xi \mapsto P_\xi$ is continuous.*
(2) *$\xi \mapsto Q_\xi$ is continuous.*
(3) *$\xi \mapsto R_\xi$ is continuous.*
(4) *$\xi \mapsto \Delta_\xi^{it}$ is continuous for any $t \in \mathbf{R}$.*
(5) *$\xi \mapsto \mathscr{C}(\Delta_\xi)$ is continuous ($\mathscr{C}(\Delta_\xi)$ = Cayley transform of Δ_ξ).*
(6) *$\xi \mapsto J_\xi$ is continuous.*

Proof. (1) $\Leftrightarrow$ (2): $\xi \mapsto P_\xi$ is continuous if and only if $\xi \mapsto iP_\xi$ is continuous. Since $iP_\xi = Q_\xi i$ holds for any $\xi \in \Omega$, $\xi \mapsto iP_\xi$ is continuous if and only if $\xi \mapsto Q_\xi$ is continuous.

(2) $\Rightarrow$ (3): Since $R_\xi = P_\xi + Q_\xi$ holds for any $\xi \in \Omega$ our assertion follows from the implication (2) $\Rightarrow$ (1).

(3) $\Rightarrow$ (4): Suppose that $\xi \mapsto R_\xi$ is continuous. Then $\xi \mapsto 2 - R_\xi$ is also continuous. It is sufficient to show that for any $t \in \mathbf{R}$ the operator fields $\xi \mapsto R_\xi^{it}$ and $\xi \mapsto (2 - R_\xi)^{it}$ are continuous (see [12], 3.2). Let $t \in \mathbf{R}$ be given. By 1.2(b) the operator field $\xi \mapsto f(R_\xi)$ is continuous for any $f \in C([0, 2])$. Let $\mathscr{F} := \{f \in C([0, 2]) \mid f|_{]0,2[} \in C_c(]0,2[)\}$. Since for any $f \in \mathscr{F}$ the function $\lambda \mapsto f(\lambda)\lambda^{it}$ belongs to $\mathscr{F}$ too, for any $f \in \mathscr{F}$ and $x \in \Lambda$ the vector field $\xi \mapsto R_\xi^{it} f(R_\xi)x(\xi)$ is continuous. Let $f_1, f_2, \ldots$ be a monotonely increasing sequence of positive functions in $\mathscr{F}$ such that $\sup_{n \in \mathbf{N}} f_n(\lambda) = \lambda$ holds for any $\lambda \in]0, 2[$. Since the spectral measure of R_ξ is concentrated on the interval $]0, 2[$ (see [12], p. 194) the sequence $\{f_n(R_\xi)\}_{n \in \mathbf{N}}$ converges to R_ξ in the strong operator topology. In particular the set $\{f_n(R_\xi)x(\xi) \mid n \in \mathbf{N}, x \in \Lambda\}$ is dense in $\{R_\xi x(\xi) \mid x \in \Lambda\}$. Since R_ξ is injective and positive the set $R_\xi(\mathscr{H}_\xi)$ is dense in $\mathscr{H}_\xi$. Hence the set $\{R_\xi x(\xi) \mid x \in \Lambda\}$ is also dense in $\mathscr{H}_\xi$. We conclude that the set $\{f(R_\xi)x(\xi) \mid f \in \mathscr{F}, x \in \Lambda\}$ is dense in $\mathscr{H}_\xi$. Now we obtain from 1.2(a) that $\xi \mapsto R_\xi^{it}$ is a continuous operator field. Similarly one can see that $\xi \mapsto (2 - R_\xi)^{it}$ is a continuous operator field.

(4) $\Rightarrow$ (5): Suppose that $\xi \mapsto \Delta_\xi^{it}$ is continuous for any $t \in \mathbf{R}$. First we want to show that the following is true.

(i) For any $f \in C_c(\mathbf{R}^+)$ the operator field $\xi \mapsto f(\Delta_\xi)$ is continuous.

Let $f \in C_c(\mathbf{R}^+)$ be given. Furthermore let $I := [\alpha, \beta] \subseteq \mathbf{R}^+$ be some compact interval which contains the support of f. Finally let $t_0 \in \mathbf{R}^+$ be chosen so small that the function $\lambda \mapsto \lambda^{it_0}$ on I is injective. Let us denote this function by g. g is a homeomorphism from I onto $D := g(I)$. We define a function h on $S^1 := \{\lambda \in \mathbf{C} \mid |\lambda| = 1\}$ as follows

$$h(\lambda) := \begin{cases} f \circ g^{-1}(\lambda) & \text{if } \lambda \in D \\ 0 & \text{if } \lambda \in S^1 \setminus D. \end{cases}$$

Since the support of f is contained in I the function h is continuous and we have $h(\Delta_\xi^{it_0}) = f(\Delta_\xi)$ for any $\xi \in \Omega$. By 1.2(b) the operator field $\xi \mapsto h(\Delta_\xi^{it_0})$ is continuous. Thus our assertion (i) follows from this.

Since the Cayley transform maps $\mathbf{R}^+$ homeomorphic onto $\Gamma := \{\lambda \in S^1 \mid \operatorname{Im} \lambda < 0\}$ we infer from (i) that

(ii) For any $f \in C_c(\Gamma)$ the operator field $\xi \mapsto f(\mathscr{C}(\Delta_\xi))$ is continuous.

Since the spectral measure of $\mathscr{C}(\Delta_\xi)$ is concentrated on Γ, $\mathscr{C}(\Delta_\xi)$ is the limit of some strongly convergent sequence of operators of the form $f(\mathscr{C}(\Delta_\xi))$, where $f \in C_c(\Gamma)$. From this it follows that the set $\{f(\mathscr{C}(\Delta_\xi))x(\xi) \mid f \in C_c(\Gamma), x \in \Lambda\}$ is dense in $\mathscr{H}_\xi$. Since the function $\lambda \mapsto \lambda f(\lambda)$ belongs to $C_c(\Gamma)$ we obtain from (ii) and 1.2(a) that $\xi \mapsto \mathscr{C}(\Delta_\xi)$ is continuous.

(5) $\Rightarrow$ (6): Suppose that $\xi \mapsto \mathscr{C}(\Delta_\xi)$ is continuous. Let $\mathscr{E}$ be the linear subspace of $C_c(\mathbf{R})$ which is generated by all functions of the form $f * g$, where $f, g \in C_c(\mathbf{R})$ ($*$ denotes the convolution of functions on $\mathbf{R}$). Let Λ_0 be the linear subspace of $\prod_{\xi \in \Omega} \mathscr{H}_\xi$ which is generated by all vector fields of the form $\xi \mapsto f(\log \Delta_\xi)x(\xi)$, where $f \in \mathscr{E}$, $x \in \Lambda$. Since $\xi \mapsto \mathscr{C}(\Delta_\xi)$ is continuous we infer from 1.2(b) that any $x \in \Lambda_0$ is continuous. From [18], Lemma 10.1, we obtain

(iii) For any $x \in \Lambda_0$ we have $x(\xi) \in \mathfrak{A}_\xi''$ if $\xi \in \Omega$, and $\xi \mapsto S_\xi(x(\xi))$ is contained in Λ_0.

Furthermore we obtain (see the proof of [18], Theorem 10.1)

(iv) For any $x \in \Lambda_0$ and $\alpha \in \mathbf{C}$ we have $x(\xi) \in \mathscr{D}(\Delta_\xi^\alpha)$ if $\xi \in \Omega$, and $\xi \mapsto \Delta_\xi^\alpha x(\xi)$ is contained in Λ_0.

Since $x(\xi) \in \{f(\log \Delta_\xi)x(\xi) | f \in \mathscr{E}\}^-$ holds for any $x \in \Lambda$ and since $\{x(\xi) | x \in \Lambda\}$ is dense in $\mathscr{H}_\xi$ we obtain in addition

(v) $\{x(\xi) | x \in \Lambda_0\}$ is dense in $\mathscr{H}_\xi$.

Since $J_\xi x(\xi) = S_\xi \Delta_\xi^{-1/2} x(\xi)$ holds for any $x \in \Lambda_0$ (see [12], p. 219, Proposition (3)) we infer from (iii) and (iv) that $\xi \mapsto J_\xi x(\xi)$ belongs to Λ_0. From this and from (vi) as well as from 1.2(a) we obtain that $\xi \mapsto J_\xi$ is continuous.

(6) $\Rightarrow$ (1): Suppose that $\xi \mapsto J_\xi$ is continuous. Let Λ_1 be the real vector space of vector fields which is generated by vector fields of the form $\xi \mapsto x(\xi)^\# x(\xi)$, where $x \in \Lambda$. Using [12], 2.3(1), we infer from (L6)

(vi) $\{x(\xi) + iJ_\xi y(\xi) | x, y \in \Lambda_1\}$ is dense in $\mathscr{H}_\xi$ for any $\xi \in \Omega$.

Moreover for any $x, y \in \Lambda_1$ and $\xi \in \Omega$ the following identity holds

$$P_\xi(x(\xi) + iJ_\xi y(\xi)) = x(\xi).$$

From this and from (vi) as well as from 1.2(a) we infer that $\xi \mapsto P_\xi$ is continuous.

DEFINITION 1.4. For any continuous field of l. H. a.'s $(\{\mathfrak{A}_\xi\}_{\xi \in \Omega}, \Lambda)$ let $\mathfrak{A}_\xi^* := J_\xi \mathfrak{A}_\xi$ for any $\xi \in \Omega$, and let $\Lambda^* := \{\xi \mapsto J_\xi x(\xi) | x \in \Lambda\}$. (It is easy to see that $(\{\mathfrak{A}_\xi^*\}_{\xi \in \Omega}, \Lambda^*)$ is also a continuous field of l. H. a.'s.) We call $(\{\mathfrak{A}_\xi^*\}_{\xi \in \Omega}, \Lambda^*)$ *the dual continuous field of l. H. a.'s associated with* $(\{\mathfrak{A}_\xi\}_{\xi \in \Omega}, \Lambda)$.

For the remainder of this section we assume that $(\{\mathfrak{A}_\xi\}_{\xi \in \Omega}, \Lambda)$ is an arbitrary continuous field of l. H. a.'s. As in [10] let H be the set of all square integrable vector fields in $\prod_{\xi \in \Omega} \mathscr{H}_\xi$ and let $\mathscr{H} = \int^\oplus \mathscr{H}_\xi d\mu(\xi)$ be the Hilbert space consisting of all classes of equivalent vector fields in H.

PROPOSITION 1.5. *The set* $\mathfrak{A}$ *consisting of all classes in* $\mathscr{H}$ *which contain at least one representative in* Λ *is a* l. H. a. *which is dense in* $\mathscr{H}$. *The operations of* $\mathfrak{A}$ *are defined canonically.*

Proof. As in [11], 1.5, one can see that the set $\mathfrak{A}$ is an involutive algebra which is dense in $\mathscr{H}$. Also as in [11], 1.5, one can show that the conditions (1), (2), (3) in [12], 5.1, are satisfied. We still have to verify the condition in [12], 5.5. Let $\mathscr{K}$ be the real vector space which is generated by all elements of the form $\tilde{x}^\# \tilde{x}$, where $x \in \Lambda$.

Let $x \in H$ such that $\tilde{x} \in \mathscr{K} \cap i\mathscr{K}$. Then there are two sequences, $\{x_n\}_{n\in\mathbf{N}}$ and $\{y_n\}_{n\in\mathbf{N}}$ respectively, such that x_n as well as y_n is a real linear combination of vector fields of the form $\xi \mapsto x(\xi)^{\#}x(\xi)$, where $x \in A$, and in addition the following holds

$$\lim_{n\to\infty} x_n(\xi) = \lim_{n\to\infty} iy_n(\xi) = x(\xi) \quad \text{almost everywhere (a.e.).}$$

Since for any $\xi \in \Omega$ and $n \in \mathbf{N}$ the components $x_n(\xi)$ and $y_n(\xi)$ belong to $\mathscr{K}_\xi$ we obtain from this that $x(\xi) \in \mathscr{K}_\xi \cap i\mathscr{K}_\xi$ holds a. e. Since $\mathfrak{A}_\xi$ satisfies the condition in [12], 5.5 we infer that $x(\xi) = 0$ a. e. Hence $\tilde{x} = 0$ holds.

DEFINITION 1.6. The l. H. a. $\mathfrak{A}$ in 1.5 is called *the direct integral of the continuous field* $(\{\mathfrak{A}_\xi\}_{\xi\in\Omega}, A)$ *with respect to the measure* μ. We also write $\int^{\oplus} \mathfrak{A}_\xi \, d\mu(\xi)$ instead of $\mathfrak{A}$.

Similarly we define $\mathfrak{A}^* := \int^{\oplus} \mathfrak{A}_\xi^* \, d\mu(\xi)$ to be the direct integral of the dual continuous field $(\{\mathfrak{A}_\xi^*\}_{\xi\in\Omega}, A^*)$ with respect to μ. According to the notation which was introduced in [18] and [12] we denote the objects associated with $\mathfrak{A}$ by Δ, S, F, J, σ_t $(t \in \mathbf{R})$, π, P, Q, R, T, $\mathscr{K}$.

PROPOSITION 1.7. *The following identities hold:* $J = \int^{\oplus} J_\xi \, d\mu(\xi)$, $P = \int^{\oplus} P_\xi \, d\mu(\xi)$, $Q = \int^{\oplus} Q_\xi \, d\mu(\xi)$, $R = \int^{\oplus} R_\xi \, d\mu(\xi)$, $\Delta^{it} = \int^{\oplus} \Delta_\xi^{it} \, d\mu(\xi)$, *for any* $t \in \mathbf{R}$, $\mathscr{C}(\Delta) = \int^{\oplus} \mathscr{C}(\Delta_\xi) \, d\mu(\xi)$.

Proof. In order to prove the identity $J = \int^{\oplus} J_\xi \, d\mu(\xi)$ it is sufficient to show that the operator $J_0 := \int^{\oplus} J_\xi \, d\mu(\xi)$ satisfies the conditions in [12], 2.3. J_0 is selfadjoint as a real linear operator. For, we have for any $x, y \in A$

$$\operatorname{Re}(J_0\tilde{x}, \tilde{y}) = \operatorname{Re}\int (J_\xi x(\xi), y(\xi)) \, d\mu(\xi) = \int \operatorname{Re}(J_\xi x(\xi), y(\xi)) \, d\mu(\xi) =$$

$$= \int \operatorname{Re}(x(\xi), J_\xi y(\xi)) \, d\mu(\xi) = \operatorname{Re}\int (x(\xi), J_\xi y(\xi)) \, d\mu(\xi) = \operatorname{Re}(\tilde{x}, J_0\tilde{y}).$$

Furthermore J_0^2 is orthogonal. For, we have

$$J_0^2 = \int^{\oplus} J_\xi^2 \, d\mu(\xi) = \int^{\oplus} \mathrm{Id}_\xi \, d\mu(\xi) = \mathrm{Id}.$$

Next we want to show that $J_0\mathscr{K} = \mathrm{i}\mathscr{K}^\perp$ holds. Let $x \in H$ such that $\tilde{x} \in \mathscr{K}$. Then we have $x(\xi) \in \mathscr{K}_\xi$ a. e. Furthermore let $y \in H$ such that $\tilde{y} \in \mathscr{K}$. Since $\mathrm{i}y(\xi) \in \mathrm{i}\mathscr{K}_\xi$ holds a. e. and $J_\xi(\mathscr{K}_\xi) = \mathrm{i}\mathscr{K}_\xi^\perp$ is satisfied for any $\xi \in \Omega$ we obtain

$$\mathrm{Re}(J_0\tilde{x}, \mathrm{i}\tilde{y}) = \int \mathrm{Re}(J_\xi x(\xi), \mathrm{i}y(\xi))\, \mathrm{d}\mu(\xi) = 0.$$

From this we infer that $J_0(\mathscr{K}) \subseteq \mathrm{i}\mathscr{K}^\perp$ holds. Hence, in order to prove the identity $J_0(\mathscr{K}) = \mathrm{i}\mathscr{K}^\perp$ it is sufficient to show that $\mathscr{K} + \mathrm{i}J_0(\mathscr{K})$ is dense in $\mathscr{H}$. Let Λ_1 be the real vector space which is generated by all vector fields of the form $\xi \mapsto x(\xi)^{\#}x(\xi)$, where $x \in \Lambda$. Let $\Lambda_1^* := \{\xi \mapsto \mathrm{i}J_\xi x(\xi) | x \in \Lambda_1\}$ and $\Lambda_2 := \Lambda_1 + \Lambda_1^*$. Λ_2 is a real linear subspace of H. For any $x \in \Lambda_2$ and for any real valued function $f \in C_c(\Omega)$ the vector field $\xi \mapsto f(\xi)x(\xi)$ belongs to Λ_2 too. Furthermore $\{x(\xi) | x \in \Lambda_2\}$ is dense in $\mathscr{H}_\xi$. Hence we conclude that a continuous vector field y is the limit of some uniformly convergent sequence of vector fields in Λ_2 if the function $\xi \mapsto \|y(\xi)\|$ has a compact support (see the proof of [4], p. 81, Proposition 6). In particular $\tilde{y}$ lies in the closure of the set $\{\tilde{x} | x \in \Lambda_2\}$. Hence we obtain that $\{\tilde{x} | x \in \Lambda_2\}$ is dense in $\mathscr{H}$. Since $\{\tilde{x} | x \in \Lambda_1\} \subseteq \mathscr{K}$ holds our assertion follows from this.

We still have to show that $\mathrm{Re}(J_0\varkappa, \varkappa) \geqslant 0$ holds for any $\varkappa \in \mathscr{K}$. (Since J_0 is conjugate linear we then also know that $\mathrm{Re}(J_0\varkappa, \varkappa) \leqslant 0$ holds for any $\varkappa \in \mathrm{i}\mathscr{K}$.) Let $x \in H$ such that $\tilde{x} \in \mathscr{K}$. Since $x(\xi) \in \mathscr{K}_\xi$ a. e. and thus $\mathrm{Re}(J_\xi x(\xi), x(\xi)) \geqslant 0$ holds a. e. we obtain

$$\mathrm{Re}(J_0\tilde{x}, \tilde{x}) = \int \mathrm{Re}(J_\xi x(\xi), x(\xi))\, \mathrm{d}\mu(\xi) \geqslant 0.$$

Hence we conclude that $J_0 = J$ holds.

For any $x \in \Lambda_1$ and $y \in \Lambda_1^*$ we have $P(\tilde{x} + \tilde{y}) = \tilde{x}$. Since $x(\xi) = P_\xi(x(\xi) + y(\xi))$ holds for any $\xi \in \Omega$ and since $\{\tilde{x} | x \in \Lambda_1\}$ is dense in $\mathscr{K}$ as well as $\{\tilde{x} | x \in \Lambda_1^*\}$ is dense in $\mathscr{K}^\perp$ we obtain that $\xi \mapsto P_\xi$ is a decomposition of P.

Since $\mathrm{i}P = Q\mathrm{i}$ and $\mathrm{i}P_\xi = Q_\xi\mathrm{i}$ holds for any $\xi \in \Omega$ we obtain that $\xi \mapsto Q_\xi$ is a decomposition of Q.

Since $R = P + Q$ holds as well as $R_\xi = P_\xi + Q_\xi$ for any $\xi \in \Omega$, $\xi \mapsto R_\xi$ is a decomposition of R.

By [10], 2.8, for any $f \in C([0,2])$, in particular for any $f \in \mathscr{I}_0 := \{f \in C([0,2]) | f(0) = 0\}$ the operator field $\xi \mapsto f(R_\xi)$ is a decomposition of $f(R)$. If f lies in $\mathscr{I}_0$ then the function $\lambda \mapsto f(\lambda)\lambda^{\mathrm{i}t}$ lies in $\mathscr{I}_0$ too for any $t \in \mathbf{R}$. Hence for any $x \in \Lambda$ and for any $\in \mathscr{I}_0$ the vector field $\xi \mapsto R_\xi^{\mathrm{i}t} f(R_\xi)x(\xi)$ is a decomposition of $R^{\mathrm{i}t}f(R)\tilde{x}$, if $t \in \mathbf{R}$. In particular $\xi \mapsto R_\xi^{\mathrm{i}t} R_\xi x(\xi)$ is a decomposition of $R^{\mathrm{i}t}R\tilde{x}$ for any $x \in \Lambda$, $t \in \mathbf{R}$. Since R is injective and positive $R(\mathscr{H})$ is dense in $\mathscr{H}$. By [10], 1.7, it follows from this that $\xi \mapsto R_\xi^{\mathrm{i}t}$ is a decomposition of $R^{\mathrm{i}t}$ for any $t \in \mathbf{R}$. Similarly one can see that $\xi \mapsto (2 - R_\xi)^{\mathrm{i}t}$ is a decomposition of $(2 - R)^{\mathrm{i}t}$ for any $t \in \mathbf{R}$. Hence we conclude that $\xi \mapsto \Delta_\xi^{\mathrm{i}t}$ is a decomposition of $\Delta^{\mathrm{i}t}$ for any $t \in \mathbf{R}$ (see [12], 3.2).

For some $f \in C_c(\mathbf{R}^+)$ let $t_0 \in \mathbf{R}$ and $h \in C(S^1)$ be chosen as in the proof of the implication (4) $\Rightarrow$ (5) in 1.3. Then the following identities hold, $h(\Delta^{it_0}) = f(\Delta)$ and $h(\Delta_\xi^{it_0}) = f(\Delta_\xi)$ for any $\xi \in \Omega$. Since $\xi \mapsto \Delta_\xi^{it_0}$ is a decomposition of Δ^{it_0} this implies that $\xi \mapsto f(\Delta_\xi)$ is a decomposition of $f(\Delta)$ (see [10], 2.8(b)). Since this is true for any $f \in C_c(\mathbf{R}^+)$ and the Cayley transform maps $\mathbf{R}^+$ homeomorphic onto $\Gamma := \{\lambda \in \mathbf{C} \mid \operatorname{Im} \lambda < 0\}$, the operator field $\xi \mapsto f(\mathscr{C}(\Delta_\xi))$ is a decomposition of $f(\mathscr{C}(\Delta))$ for any $f \in C_c(\Gamma)$. Since the set $\{f(\mathscr{C}(\Delta))\widetilde{x}\ f \in C_c(\Gamma),\ x \in A\}$ is dense in $\mathscr{H}$ and the function $\lambda \mapsto \lambda f(\lambda)$ belongs to $C_c(\Gamma)$ if f does, we infer from [10], 1.7, that $\xi \mapsto \mathscr{C}(\Delta_\xi)$ is a decomposition of $\mathscr{C}(\Delta)$.

COROLLARY 1.8. *The following identity holds,* $\mathfrak{A}^{*\prime} = \mathfrak{A}''$.

Proof. The assertion follows immediately from the fact that $J\mathfrak{A} = \mathfrak{A}^*$ holds.

THEOREM 1.9. (see [11], 1.7) *For any element $a \in H$ the following is true*

(a) *$\widetilde{a}$ is left-bounded with respect to* $\mathfrak{A}$ (see [18], Definition 14.1) $\Leftrightarrow$ *$a(\xi)$ is left-bounded with respect to $\mathfrak{A}_\xi$* a. e. *and the operator field $\xi \mapsto \pi_\xi(a(\xi))$ is essentially bounded.*

(b) *If $\widetilde{a}$ is left-bounded then the operator field $\xi \mapsto \pi_\xi(a(\xi))$ is a strongly measurable decomposition of $\pi(\widetilde{a})$.*

Proof. (a) $\Leftarrow$: Suppose that $a(\xi)$ is left-bounded and that $\|\pi_\xi(a(\xi))\| \leq m$ holds a. e. for some $m > 0$. Then we obtain for any $x \in A^*$

$$\|\pi'_\xi(x(\xi))a(\xi)\| = \|\pi_\xi(a(\xi))x(\xi)\| \leq m\|x(\xi)\| \quad \text{a. e. .}$$

From this we infer $\|\pi'(\widetilde{x})\widetilde{a}\| \leq m\|\widetilde{x}\|$. Since $\mathfrak{A}^*$ is dense in $\mathscr{H}$ this implies that $\widetilde{a}$ is left-bounded.

$\Rightarrow$: We need the following result on l. H. a.'s (see [6]):

(+) Let $\mathfrak{B}$ be a l. H. a. and let $\mathscr{L}$ be the completion of $\mathfrak{B}$. Then for an element x in $\mathscr{L}$ the following two conditions are equivalent

(1) x is left-bounded with respect to $\mathfrak{B}$.

(2) There is some sequence $\{x_n\}_{n \in \mathbf{N}}$ in $\mathfrak{B}$ such that

$$\lim_{n \to \infty} \|x - x_n\| = 0 \quad \text{and} \quad \sup_{n \in \mathbf{N}} \|\pi(x_n)\| < \infty \quad \text{holds.}$$

Now we suppose that $\widetilde{a}$ is left-bounded. By (+) we can find some sequence $\{x_n\}_{n \in \mathbf{N}}$ in A such that

$$\lim_{n \to \infty} \int \|x_n(\xi) - a(\xi)\|^2 \mathrm{d}\mu(\xi) = 0 \quad \text{and} \quad \sup_{n \in \mathbf{N}} \|\pi(\widetilde{x}_n)\| < \infty \quad \text{holds.}$$

By [6] we can choose $\{x_n\}_{n \in \mathbf{N}}$ in such a manner that $\sup_{n \in \mathbf{N}} \|\pi(\widetilde{x}_n)\| \leq \|\pi(\widetilde{a})\|$ holds. Chosing a suitable subsequence of $\{x_n\}_{n \in \mathbf{N}}$ instead of $\{x_n\}_{n \in \mathbf{N}}$ we can arrange the situation such that in addition for some subset $M \subseteq \Omega$ whose measure is zero the following is satisfied

(i) $$\lim_{n\to\infty} \|x_n(\xi) - a(\xi)\| = 0 \qquad \text{holds for any } \xi \in M^c.$$

By extending M by a certain set of measure zero we can assume that the following is true (see [10], 1.11(b))

(ii) $$\sup_{n\in\mathbf{N}} \|\pi_\xi(x_n(\xi))\| \leqslant \|\pi(\tilde{a})\| \qquad \text{holds for any } \xi \in M^c.$$

From (i), (ii) and (+) we obtain that $a(\xi)$ is left-bounded for any $\xi \in M^c$. Moreover it follows from (i) and (ii) that the sequence $\{\pi_\xi(x_n(\xi))\}_{n\in\mathbf{N}}$ converges strongly to $\pi_\xi(a(\xi))$ for any $\xi \in M^c$. Using (ii) again we deduce from this for any $\xi \in \Omega$ and $y \in \mathscr{H}_\xi$

$$\|\pi_\xi(a(\xi))y\| = \lim_{n\to\infty} \|\pi_\xi(x_n(\xi))y\| \leqslant \sup_{n\in\mathbf{N}} \|\pi_\xi(x_n(\xi))y\| \leqslant$$

$$\leqslant \sup_{n\in\mathbf{N}} \|\pi_\xi(x_n(\xi))\| \, \|y\| \leqslant \|\pi(\widetilde{a})\| \, \|y\|$$

i.e., $\|\pi_\xi(a(\xi))\| \leqslant \|\pi(\widetilde{a})\|$ holds for any $\xi \in M^c$.

(b) Let M be chosen as above. Furthermore let $Z(\xi) := \pi(a(\xi))$ if $\xi \in M^c$ and $Z(\xi) := 0$ if $\xi \in M$. Since for any $y \in A^*$ the operator field $\xi \mapsto \pi'_\xi(y(\xi))$ is a decomposition of $\pi'(\widetilde{y})$ the following holds

$$\pi'(\widetilde{y})\widetilde{a}(\xi) = \pi'_\xi(y(\xi))a(\xi) \qquad \text{a. e. for any } y \in A^*.$$

This condition is equivalent to the following

$$\pi(\widetilde{a})\widetilde{y}(\xi) = Z(\xi)y(\xi) \qquad \text{a. e. foı any } y \in A^*.$$

By [10], 1.7 we obtain from this that $\xi \mapsto Z(\xi)$ is a decomposition of $\pi(\widetilde{a})$.

It remains to show that $\xi \mapsto Z(\xi)$ is strongly measurable. Let $K \subseteq \Omega$ be a compact subset and let $\varepsilon > 0$ be given. By [4], p. 90, théorème de Lusin, there is a compact subset $K_\varepsilon \subseteq K \cap M^c$ such that $\mu(K \setminus K_\varepsilon) \leqslant \varepsilon$ and a is continuous on K_ε Since for any $y \in A^*$ the operator field $\xi \mapsto \pi'_\xi(y(\xi))$ is continuous on K_ε the vector field $\xi \mapsto Z(\xi)y(\xi)$ is continuous on K_ε for any $y \in A^*$. (Observe that the identity $Z(\xi)y(\xi) = \pi'_\xi(y(\xi))a(\xi)$ holds for any $\xi \in K_\varepsilon$.) Since $\xi \mapsto Z(\xi)$ is bounded on K_ε we infer from this that $\xi \mapsto Z(\xi)$ is continuous on K_ε (see 1.2(a)). Since the compact subset $K \subseteq \Omega$ and $\varepsilon > 0$ have been chosen arbitrarily we conclude that Z is strongly measurable.

COROLLARY 1.10. *For any element $a \in H$ the following holds.*
$\widetilde{a} \in \mathfrak{A}''$ $\Leftrightarrow$ *The following conditions are satisfied*

(α) $a(\xi) \in \mathfrak{A}''_\xi$ a. e.

(β) *The vector field $\xi \mapsto a(\xi)^\#$ belongs to H.*

(γ) *The operator field $\xi \mapsto \pi_\xi(a(\xi))$ is essentially bounded.*

Proof. $\Rightarrow$: Let $b \in H$ such that $\widetilde{b} = \widetilde{a}^{\#}$. By 1.9(*a*) the elements $a(\xi)$ and $b(\xi)$ are left-bounded a. e. and $\xi \mapsto \pi_\xi(a(\xi))$ is essentially bounded. Since $\pi(\widetilde{b}) = \pi(\widetilde{a})^*$ holds we obtain from 1.9(*b*) as well as from [10], 1.11(c) and 1.12, that $\pi_\xi(a(\xi))^* = \pi_\xi(b(\xi))$ a. e. . This implies that $a(\xi)$ lies in $\mathfrak{A}_\xi''$ a. e. and $a(\xi)^{\#} = b(\xi)$ holds a. e. . Thus we have shown that the conditions (α), (β), (γ) are fulfilled.

$\Leftarrow$: Let us denote the vector field $\xi \mapsto a(\xi)^{\#}$ by $a^{\#}$. By 1.9(a) the elements $\widetilde{a}$ and $\widetilde{a^{\#}}$ are left-bounded. By 1.9(b) we obtain for any $x, y \in \Lambda^*$

$$(\pi(\widetilde{a})\widetilde{x}, \widetilde{y}) = \int (\pi_\xi(a(\xi))x(\xi), y(\xi))\, \mathrm{d}\mu(\xi) =$$

$$= \int (x(\xi), \pi_\xi(a(\xi)^{\#})y(\xi))\, \mathrm{d}\mu(\xi) =$$

$$= \int (x(\xi), \pi_\xi'(y(\xi))a(\xi)^{\#})\, \mathrm{d}\mu(\xi) =$$

$$= (\widetilde{x}, \pi'(\widetilde{y})\widetilde{a^{\#}}) = (\widetilde{x}, \pi(\widetilde{a^{\#}})\widetilde{y}).$$

This implies that the identity $\pi(\widetilde{a})^* = \pi(\widetilde{a^{\#}})$ is true, i. e., $\widetilde{a}$ is bounded and we have $\widetilde{a}^{\#} = \widetilde{a^{\#}}$.

2. REDUCTION OF THE SEMIFINITE PORTION OF THE VON NEUMANN ALGEBRA $\mathscr{L}(\mathfrak{A})$

Throughout in this section we keep the notation of Section 1 and we assume in addition that the following condition is satisfied:

(L9) There is a sequence $\{x_n\}_{n \in \mathbf{N}}$ of vector fields in H such that the set $\{Zx_n(\xi) | Z \in \mathscr{L}(\mathfrak{A}_\xi)', n \in \mathbf{N}\}$ is total in $\mathscr{H}_\xi$ locally a. e..

Just as in the case of continuous fields of Hilbert algebras we obtain the following result (see [11], 1.8).

THEOREM 2.1. *The von Neumann algebra* $\mathscr{L}(\mathfrak{A})$ *is regularly decomposable (*i. e. *every operator in* $\mathscr{L}(\mathfrak{A})$ *is strongly decomposable). Furthermore* $\mathscr{L}(\mathfrak{A})$ *is countably decomposable in the sense of* [2], 1.1.2.

REMARK. (1) Since the von Neumann algebra $\mathfrak{z}$ of all diagonalisable operators is contained in $\mathscr{L}(\mathfrak{A})$, $\mathfrak{z}$ is also countably decomposable. Since $\mathfrak{z}$ is $*$-isomorphic to $L^\infty(\Omega, \mu)$ (see [10], 1.14) we may assume μ to be a σ-finite regular Borel measure. In particular a subset of Ω has locally the measure zero if and only if it has the measure zero.

(2) Just as in [11], 1.9, one can show that for any $\mathscr{Z} = \int^{\oplus} Z(\xi)\, \mathrm{d}\mu(\xi) \in \mathscr{L}(\mathfrak{A})$, we have $Z(\xi) \in \mathscr{L}(\mathfrak{A}_\xi)$ a. e..

LEMMA 2.2. *Let* $\mathscr{Z}_1 = \int^{\oplus} Z_1(\xi)\,d\mu(\xi)$, $\mathscr{Z}_2 = \int^{\oplus} Z_2(\xi)\,d\mu(\xi), \ldots$ *be a sequence of operators in* $\mathscr{L}(\mathfrak{A})$ *which converges strongly to some operator* $\mathscr{Z} = \int^{\oplus} Z(\xi)\,d\mu(\xi)$. *Then there is a subsequence* $\{\mathscr{Z}_{n_i}\}_{i\in\mathbf{N}}$ *of* $\{\mathscr{Z}_n\}_{n\in\mathbf{N}}$ *such that* $\{Z_{n_i}(\xi)\}_{i\in\mathbf{N}}$ *converges strongly to* $Z(\xi)$ a. e..

The proof of this lemma is quite similar to the verification of the corresponding result in [11], 3.3. Thus we omit the proof here.

LEMMA 2.3. *Let* $\mathscr{Z} = \int^{\oplus} Z(\xi)\,d\mu(\xi) \in \mathscr{L}(\mathfrak{A})$ *be a normal operator and let* $f\colon \mathrm{Sp}(\mathscr{Z}) \to \mathbf{C}$ *be a bounded function of first Baire class*, i. e., *there is a sequence* $f_1, f_2, \ldots$ *of functions in* $C(\mathrm{Sp}(\mathscr{Z}))$ *which converges pointwise to* f. *Then the identity* $f(\mathscr{Z}) = \int^{\oplus} f(Z(\xi))\,d\mu(\xi)$ *holds.*

Proof. By [10], 2.8(a) we have $\mathrm{Sp}(Z(\xi)) \subseteq \mathrm{Sp}(\mathscr{Z})$ a. e. and $Z(\xi)$ is normal a. e.. In particular $f(Z(\xi))$ and $f_n(Z(\xi))$ is well defined a. e. for any $n \in \mathbf{N}$. We may assume that the sequence $\{f_n\}_{n\in\mathbf{N}}$ is uniformly bounded. By [10], 2.8(b) we have $f_n(\mathscr{Z}) = \int^{\oplus} f_n(Z(\xi))\,d\mu(\xi)$ for any $n \in \mathbf{N}$. Now the sequence $\{f_n(\mathscr{Z})\}_{n\in\mathbf{N}}$ converges strongly to $f(\mathscr{Z})$ and $\{f_n(Z(\xi))\}_{n\in\mathbf{N}}$ converges strongly to $f(Z(\xi))$ a. e.. By 2.2 there is a subsequence $\{f_{n_i}\}_{i\in\mathbf{N}}$ of $\{f_n\}_{n\in\mathbf{N}}$ such that $\{f_{n_i}(Z(\xi))\}_{i\in\mathbf{N}}$ converges strongly to $G(\xi)$ a. e. if $f(\mathscr{Z}) = \int^{\oplus} G(\xi)\,d\mu(\xi)$ holds. Since $\{f_{n_i}(Z(\xi))\}_{i\in\mathbf{N}}$ also converges strongly to $f(Z(\xi))$ a. e. the identity $f(Z(\xi)) = G(\xi)$ must be valid a. e..

REMARK. Let us mention that the assertion of 2.3 is even true if f is any bounded measurable function on $\mathrm{Sp}(\mathscr{Z})$. Since we do not need this general result in the sequel we suppress its proof.

LEMMA 2.4. *Let* $\varphi(\varphi_\xi)$ *be the weight associated to* $\mathfrak{A}$ $(\mathfrak{A}_\xi)$ *which is defined on* $\mathscr{L}(\mathfrak{A})^+(\mathscr{L}(\mathfrak{A}_\xi)^+)$ *for* $\xi \in \Omega$ (see [1] *or* [16], 10.16) *and let* $\mathscr{I}$ *be an involutive ideal in* $\mathscr{L}(\mathfrak{A})$. *Moreover assume that for* $\mathscr{G} = \int^{\oplus} G(\xi)\,d\mu(\xi) \in \mathscr{L}(\mathfrak{A})$ *the weight* $\mathscr{L}(\mathfrak{A}) \ni \mathscr{Z}^+ \mapsto \varphi(\mathscr{G}^*\mathscr{Z}\mathscr{G})$ *takes only finite values on the positive elements of* $\mathscr{I}$. *Then for any* $\mathscr{Z} = \int^{\oplus} Z(\xi)\,d\mu(\xi) \in \overline{\mathscr{I}}^+$ ($\overline{\mathscr{I}}$ *is the strong closure of* $\mathscr{I}$ *in* $\mathscr{L}(\mathfrak{A})$) *the function* $\xi \mapsto \varphi_\xi(G(\xi)^*Z(\xi)G(\xi))$ *is measurable and* $\varphi(\mathscr{G}^*\mathscr{Z}\mathscr{G}) = \int \varphi_\xi(G(\xi)^*Z(\xi)G(\xi))\,d\mu(\xi)$ *holds.*

Proof. First we will show

(i) If $\varphi(\mathscr{Z}) < \infty$ holds for some $\mathscr{Z} = \int^{\oplus} Z(\xi)\, d\mu(\xi) \in \mathscr{L}(\mathfrak{A})^{+}$ then the function $\xi \mapsto \varphi_{\xi}(Z(\xi))$ is measurable and $\varphi(\mathscr{Z}) = \int \varphi_{\xi}(Z(\xi))\, d\mu(\xi)$ holds.

Let $\mathscr{Z} = \int^{\oplus} Z(\xi)\, d\mu(\xi) \in \mathscr{L}(\mathfrak{A})^{+}$ be chosen such that $\varphi(\mathscr{Z}) < \infty$ holds. Then there is some vector field $a \in H$ such that $\tilde{a}$ is left-bounded and $\pi(\tilde{a}) = \mathscr{Z}^{1/2}$ holds. By 1.9(a) $a(\xi)$ is left-bounded a. e. and by 1.9(b) we have $\pi_{\xi}(a(\xi)) = Z(\xi)^{1/2}$ a. e. . Clearly the function $\xi \mapsto (a(\xi), a(\xi))$ is measurable and by the definition of the weights φ and φ_{ξ} $(\xi \in \Omega)$ we obtain

$$\varphi(\mathscr{Z}) = (\tilde{a}, \tilde{a}) = \int (a(\xi), a(\xi))\, d\mu(\xi) = \int \varphi_{\xi}(Z(\xi))\, d\mu(\xi).$$

Hence (i) is satisfied.

From (i) we obtain for any $\mathscr{Z} = \int^{\oplus} Z(\xi)\, d\mu(\xi) \in \mathscr{I}^{+}$ that

$$\varphi(\mathscr{G}^{*}\mathscr{Z}\mathscr{G}) = \int \varphi_{\xi}(G(\xi)^{*}Z(\xi)G(\xi))\, d\mu(\xi).$$

Now let $\mathscr{Z} = \int^{\oplus} Z(\xi)\, d\mu(\xi) \in \bar{\mathscr{I}}^{+}$ be chosen arbitrarily. Since $\mathscr{L}(\mathfrak{A})$ is countably decomposable we infer from [2], p. 43, Corollary 5 and [2], p. 31, Corollary to Proposition 1, that there is a monotonely increasing sequence $\mathscr{Z}_1 = \int^{\oplus} Z_1(\xi)\, d\mu(\xi)$, $\mathscr{Z}_2 = \int^{\oplus} Z_2(\xi)\, d\mu(\xi)$, ... of positive operators in $\mathscr{I}$ which converges strongly to $\mathscr{Z}$. By 2.2 we can assume that $\{Z_n(\xi)\}_{n \in \mathbf{N}}$ converges strongly to $Z(\xi)$ a. e. . Hence the sequence $\{\varphi_{\xi}(G(\xi)^{*}Z_n(\xi)G(\xi))\}_{n \in \mathbf{N}}$ is monotonely increasing a. e. and $\varphi_{\xi}(G(\xi)^{*}Z(\xi)G(\xi)) = \sup_{n \in \mathbf{N}} \varphi_{\xi}(G(\xi)^{*}Z_n(\xi)G(\xi))$ holds a. e. . By (i) the function $\xi \mapsto \varphi_{\xi}(G(\xi)^{*}Z_n(\xi)G(\xi))$ is measurable. Hence the function $\xi \mapsto \varphi_{\xi}(G(\xi)^{*}Z(\xi)G(\xi))$ is also measurable and from (i) we conclude by Lebesgue's monotone convergence theorem

$$\varphi(\mathscr{G}^{*}\mathscr{Z}\mathscr{G}) = \lim_{n \to \infty} \varphi(\mathscr{G}^{*}Z_n\mathscr{G}) = \int \lim_{n \to \infty} \varphi_{\xi}(G(\xi)^{*}Z_n(\xi)G(\xi))\, d\mu(\xi) =$$

$$= \int \varphi_{\xi}(G(\xi)^{*}Z(\xi)G(\xi))\, d\mu(\xi).$$

THEOREM 2.5. *Let $\mathscr{E} = \int^{\oplus} E(\xi) d\mu(\xi)$ be the maximal central semifinite projection in $\mathscr{L}(\mathfrak{A})$ (we may assume that $E(\xi)$ is a central projection in $\mathscr{L}(\mathfrak{A}_{\xi})$ for any $\xi \in \Omega$).*

Furthermore let τ be a positive semifinite normal and faithful trace on $\mathscr{L}(\mathfrak{A})^+_{\mathscr{E}}$. Then $E(\xi)$ is a semifinite projection a. e. *and for any $\xi \in \Omega$ there is a positive semifinite normal trace τ_ξ on $\mathscr{L}(\mathfrak{A}_\xi)^+_{E(\xi)}$ such that the following holds:*

(1) *τ_ξ is faithful* a. e.;

(2) *For any $\mathscr{Z} = \int^{\oplus} Z(\xi)\,\mathrm{d}\mu(\xi) \in \mathscr{L}(\mathfrak{A})^+_{\mathscr{E}}$ the function $\xi \mapsto \tau_\xi(Z(\xi))$ is measurable and we have $\tau(\mathscr{Z}) = \int \tau_\xi(Z(\xi))\,\mathrm{d}\mu(\xi)$.*

Proof. Let φ (φ_ξ, $\xi \in \Omega$) be defined as in 2.4. Let ψ be the following weight $\mathscr{L}(\mathfrak{A})^+ \ni \mathscr{Z} \mapsto \varphi((\mathrm{Id} - \mathscr{E})\mathscr{Z}) + \tau(\mathscr{E}\mathscr{Z})$. ψ is normal semifinite faithful and invariant with respect to the modular automorphism group $\{\sigma_t\}_{t\in\mathbf{R}}$. By [9], 5.12, there is a selfadjoint positive and regular operator $\mathscr{G}$ which is affiliated with the von Neumann algebra $\mathscr{L}(\mathfrak{A})^\sigma = \{\mathscr{Z} \in \mathscr{L}(\mathfrak{A}) | \sigma_t(\mathscr{Z}) = \mathscr{Z}$ for any $t \in \mathbf{R}\}$ such that $\psi = \varphi(\mathscr{G}^{-1}\cdot)$ holds (see [9], p. 62). For any $t \in \mathbf{R}$ we set $\mathscr{U}_t := \mathscr{G}^{it}$. Since the restriction of ψ on $\mathscr{L}(\mathfrak{A})^+_{\mathscr{E}}$ is a trace (hence the modular automorphism group of $\psi | \mathscr{L}(\mathfrak{A})_{\mathscr{E}} = \tau$ is trivial) we infer from [9], 4.6,

(i) $\mathscr{U}_t \mathscr{Z} \mathscr{U}_t^* = \Delta^{it} \mathscr{Z} \Delta^{-it}$ holds for any $\mathscr{Z} \in \mathscr{L}(\mathfrak{A})_{\mathscr{E}}$ and $t \in \mathbf{R}$.

Since all projections of the spectral measure of $\mathscr{G}$ belong to $\mathscr{L}(\mathfrak{A})$ the Cayley transform of $\mathscr{G}$, $\mathscr{C}(\mathscr{G})$, belongs also to $\mathscr{L}(\mathfrak{A})$. Let $\mathscr{C}(\mathscr{G}) = \int^{\oplus} V(\xi)\,\mathrm{d}\mu(\xi)$. Since $\mathscr{C}(\mathscr{G})$ is the Cayley transform of a non singular selfadjoint and positive operator the numbers 1 and -1 are not contained in the point spectrum of $\mathscr{C}(\mathscr{G})$ (see [14], 13.19(b)), i. e., $\chi_{\{-1,1\}}(\mathscr{C}(\mathscr{G})) = 0$ holds ($\chi_{\{-1,1\}}$ is the characteristic function of the set $\{-1, 1\}$ on S^1) and the spectrum of $\mathscr{C}(\mathscr{G})$ is contained in $\Gamma \cup \{-1, 1\}$ ($\Gamma = \{\lambda \in S^1 | \mathrm{Im}\,\lambda < 0\}$). By 2.3 we have $\chi_{\{-1,1\}}(\mathscr{C}(\mathscr{G})) = \int^{\oplus} \chi_{\{-1,1\}}(V(\xi))\mathrm{d}\mu(\xi)$. Hence $\chi_{\{-1,1\}}(V(\xi))$ vanishes a. e., or equivalently, the numbers -1 and 1 are not contained in the point spectrum of $V(\xi)$ a. e. . Furthermore we infer from [10], 2.8(a), that the spectrum of $V(\xi)$ is contained in $\Gamma \cup \{-1,1\}$ a. e. . Thus we obtain from [14], 13.19 that $V(\xi)$ is the Cayley transform of some non singular selfadjoint and positive operator. Hence there is a sequence $K_1 \subseteq K_2 \subseteq \ldots \subseteq \Omega$ of compact subsets such that the following conditions are satisfied :

(a) The support of $\mu | K_n$ is K_n for any $n \in \mathbf{N}$ and $\mu(M^c) = 0$ holds for $M := \bigcup_{n=1}^{\infty} K_n$.

(b) $\xi \mapsto E(\xi)$ is continuous on K_n for any $n \in \mathbf{N}$.

(c) $\xi \mapsto V(\xi)$ is continuous on K_n for any $n \in \mathbf{N}$ and for any $\xi \in M$ the operator $V(\xi) \in \mathscr{L}(\mathfrak{A}_\xi)$ is the Cayley transform of some non singular selfadjoint and positive operator G_ξ.

We set $G_\xi := \mathrm{Id}_\xi$ if $\xi \in M^c$. Since $\{f \circ \mathscr{C} | f \in C_c(\Gamma)\} = C_c(\mathbf{R}^+)$ holds ($\mathscr{C}(t) = (t - i)/(t + i)$) and $f \circ \mathscr{C}(G_\xi) = f(V(\xi))$ holds for any $f \in C_c(\Gamma)$, $\xi \in M$ we infer from (c) (see 1.2(b))

(ii) For any $f \in C_c(\mathbf{R}^+)$ and for any $n \in \mathbf{N}$ the operator field $\xi \mapsto f(G_\xi)$ is continuous on K_n.

Since $\mathscr{C}(\mathscr{G}) = \int^\oplus \mathscr{C}(G_\xi)\,\mathrm{d}\mu(\xi)$ holds we obtain from [10], 2.8(b),

(iii) $f(\mathscr{G}) = \int^\oplus f(G_\xi)\,\mathrm{d}\mu(\xi)$ holds for any $f \in C_c(\mathbf{R}^+)$.

Since for any $f \in C_c(\mathbf{R}^+)$ and any $t \in \mathbf{R}$ the function $\lambda \mapsto \lambda^{it} f(\lambda)$ belongs also to $C_c(\mathbf{R}^+)$ we infer from (ii) that for any $f \in C_c(\mathbf{R}^+)$, $t \in \mathbf{R}$ and $a \in \Lambda$ the vector field $\xi \mapsto G_\xi^{it} f(G_\xi) a(\xi)$ is continuous on K_n for any $n \in \mathbf{N}$. Since for any $\xi \in \Omega$ the set $\{f(G_\xi)a(\xi) | f \in C_c(\mathbf{R}^+), a \in \Lambda\}$ is dense in $\mathscr{H}_\xi$ we obtain from this (see 1.2(a))

(iv) $\xi \mapsto G_\xi^{it}$ is continuous on K_n for any $n \in \mathbf{N}$, $t \in \mathbf{R}$.

Since the set $\{f(\mathscr{G})\varkappa | f \in C_c(\mathbf{R}^+), \varkappa \in \mathfrak{A}\}$ is dense in $\mathscr{H}$ it follows from (iii), (iv), and [10], 1.7,

(v) $\mathscr{U}_t = \int^\oplus G_\xi^{it}\,\mathrm{d}\mu(\xi)$ holds for any $t \in \mathbf{R}$.

For any $\xi \in \Omega$, $t \in \mathbf{R}$ we set $U_t(\xi) := G_\xi^{it}$. From (a), (b), (i), (iv), (v) as well as from the fact that $\Delta^{it} = \int^\oplus \Delta_\xi^{it}\,\mathrm{d}\mu(\xi)$ holds and $\xi \mapsto \Delta_\xi^{it}$ is continuous on K_n for any $n \in \mathbf{N}$ we obtain for any $\xi \in M$, $t \in \mathbf{R}$, $a \in \Lambda$

$$U_t(\xi)E(\xi)\pi_\xi(a(\xi))U_t(\xi)^* = \Delta_\xi^{it}E(\xi)\pi_\xi(a(\xi))\Delta_\xi^{-it}.$$

Since $\{E(\xi)\pi_\xi(a(\xi)) | a \in \Lambda\}$ is strongly dense in $\mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)}$ for any $\xi \in \Omega$ we obtain from this

(vi) $U_t(\xi)ZU_t(\xi)^* = \Delta_\xi^{it}Z\Delta_\xi^{-it}$ holds for any $\xi \in M$, $t \in \mathbf{R}$ and $Z \in \mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)}$.

By [9], 7.4 this implies that $E(\xi)$ is a semifinite projection for any $\xi \in M$. For $n \in \mathbf{N}$ let

$$f_n : \mathbf{R}^+ \ni \lambda \mapsto \lambda(1 + \lambda/n)^{-1}.$$

For any $n \in \mathbf{N}$ $f_n(\mathscr{G}^{-1})$ ($f_n(G_\xi^{-1})$) is a bounded positive operator and the sequence $\{f_n(\mathscr{G}^{-1})\}_{n\in\mathbf{N}}$ ($\{f_n(G_\xi^{-1})\}_{n\in\mathbf{N}}$) is monotonely increasing ($\xi \in \Omega$). As in the proof of (v) we can see that the following is true

(vii) $f_n(\mathscr{G}^{-1}) = \int^\oplus f_n(G_\xi^{-1})\mathrm{d}\mu(\xi)$, holds for any $n \in \mathbf{N}$.

For any $n \in \mathbf{N}$ we define a weight on $\mathscr{L}(\mathfrak{A})_\mathscr{E}^+$ and $\mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)}^+$ respectively as follows

$$\tau^{(n)} : \mathscr{L}(\mathfrak{A})_\mathscr{E}^+ \ni \mathscr{Z} \mapsto \varphi(f_n(\mathscr{G}^{-1})\mathscr{Z}\ ,$$

$$\tau_\xi^{(n)} : \mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)}^+ \ni \mathscr{Z} \mapsto \varphi_\xi(f_n(G_\xi^{-1})Z) \qquad \text{if } \xi \in \Omega.$$

The sequence $\{\tau^{(n)}\}_{n\in\mathbf{N}}$ is monotonely increasing and by [9], p. 62, for any $\mathscr{Z} \in \mathscr{L}(\mathfrak{A})_\mathscr{E}^+$ we have $\lim_{n\to\infty} \tau^{(n)}(\mathscr{Z}) = \tau(\mathscr{Z})$. Similarly for any $\xi \in \Omega$ the sequence $\{\tau_\xi^{(n)}\}_{n\in\mathbf{N}}$ is monotonely increasing and by (vi) as well as by the proof of [9], 7.4, for any $\xi \in M$ there is a positive semifinite normal and faithful trace τ_ξ on $\mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)}^+$ such that $\lim_{n\to\infty} \tau_\xi^{(n)}(Z) = \tau_\xi(Z)$ holds for any $Z \in \mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)}^+$. We set $\tau_\xi := 0$ if $\xi \in M^c$. There is an ideal $\mathscr{I}$ in $\mathscr{L}(\mathfrak{A})_\mathscr{E}$ which is strongly dense in $\mathscr{L}(\mathfrak{A})_\mathscr{E}$ such that $\mathscr{I}^+ = \{\mathscr{Z} \in \mathscr{L}(\mathfrak{A})_\mathscr{E} | \tau(\mathscr{Z}) < \infty\}$

holds. Since $\tau^{(n)}$ is dominated by τ the weight $\tau^{(n)}$ is finite on $\mathscr{I}^+$. Thus we obtain from (vii) and 2.4 for any $\mathscr{Z} = \int^{\oplus} Z(\xi)\,\mathrm{d}\mu(\xi) \in \mathscr{L}(\mathfrak{A})^+_{\mathscr{E}}$ and $n \in \mathbf{N}$

$$\tau^{(n)}(\mathscr{Z}) = \int \tau^{(n)}_{\xi}(Z(\xi))\,\mathrm{d}\mu(\xi).$$

Moreover $\xi \mapsto \tau_{\xi}(Z(\xi))$ is measurable and from Lebesgue's monotone convergence theorem we conclude

$$\tau(\mathscr{Z}) = \lim_{n\to\infty} \tau^{(n)}(\mathscr{Z}) = \lim_{n\to\infty} \int \tau^{(n)}_{\xi}(Z(\xi))\,\mathrm{d}\mu(\xi) =$$

$$= \int \lim_{n\to\infty} \tau^{(n)}_{\xi}(Z(\xi))\,\mathrm{d}\mu(\xi) = \int \tau_{\xi}(Z(\xi))\,\mathrm{d}\mu(\xi).$$

COROLLARY 2.6. *Let* $\mathscr{E}$, $E(\xi)$, τ *and* τ_{ξ} *be chosen as in* 2.5. *Furthermore let* $\mathscr{B} = \int^{\oplus} B(\xi)\,\mathrm{d}\mu(\xi) \in \mathscr{L}(\mathfrak{A})_{\mathscr{E}}$ *be a projection such that* $\tau(\mathscr{B}) < \infty$. *Then* $B(\xi)$ *is a finite projection in* $\mathscr{L}(\mathfrak{A}_{\xi})_{B(\xi)}$ a. e. .

Proof. From $\tau(\mathscr{B}) = \int \tau_{\xi}(B(\xi))\,\mathrm{d}\mu(\xi) < \infty$ we infer that $\tau_{\xi}(B(\xi)) < \infty$ holds a. e. . Since τ_{ξ} is a faithful trace a. e. this implies that $B(\xi)$ is a finite projection in $\mathscr{L}(\mathfrak{A}_{\xi})_{E(\xi)}$ a. e. .

Using 2.6 one can prove now the following theorem in the same manner as the corresponding assertions in [11], 3.1, 3.5, 3.6, 3.8.

THEOREM 2.7. *Let* $\mathscr{E} = \int^{\oplus} E(\xi)\,\mathrm{d}\mu(\xi)$ *be the maximal central projection in* $\mathscr{L}(\mathfrak{A})$ *such that* $\mathscr{L}(\mathfrak{A})_{\mathscr{E}}$ *is a von Neumann algebra of type* I, *type* II, *finite type, properly infinite type respectively. Then* $\mathscr{L}(\mathfrak{A}_{\xi})_{E(\xi)}$ *is a von Neumann algebra of type* I, *type* II, *finite type, properly infinite type respectively* a. e. .

3. APPLICATION: THE CENTRAL DECOMPOSITION OF KMS-STATES

Let $\mathscr{A}$ be a C^*-algebra with unit e and let $\{\sigma_t\}_{t\in\mathbf{R}}$ be a one parameter group of automorphisms of $\mathscr{A}$ such that for any $a \in \mathscr{A}$ the function $t \mapsto \sigma_t(a)$ is continuous with respect to the uniform topology on $\mathscr{A}$. A state φ is said to satisfy the KMS-condition if one of the following statements (KMS I) or (KMS II) respectively is satisfied, which are known to be equivalent (see [5], p. 26)

(KMS I) For any $a, b \in \mathscr{A}$ there is some bounded continuous function on $\{z \in \mathbf{C} \mid 0 \leqslant \operatorname{Im} z \leqslant 1\}$ such that F is holomorphic in $\{z \in \mathbf{C} \mid 0 < \operatorname{Im} z < 1\}$ and for any $t \in \mathbf{R}$ the following holds

$$F(t) = \varphi(b\sigma_t(a)), \quad F(t + \mathrm{i}) = \varphi(\sigma_t(a)b).$$

(KMS II) For any $a, b \in \mathscr{A}$ and for any test function $\hat{f}$ the following holds

$$\int f(t-\mathrm{i})\varphi(b\sigma_t(a))\,\mathrm{d}t = \int f(t)\varphi(\sigma_t(a)b)\,\mathrm{d}t,$$

where $f(z) = \int \hat{f}(s)\,\mathrm{e}^{\mathrm{i}sz}\,\mathrm{d}s$ for any $z \in \mathbf{C}$.

A state φ which satisfies the KMS-condition is called a KMS-state. Let $\mathscr{S}_1$ be the set of all KMS-states on $\mathscr{A}$. By using (KMS II) it was shown in [3], Theorem 1.1, that $\mathscr{S}_1$ is a simplex, the topology on $\mathscr{S}_1$ being the weak-$*$-topology. Moreover it was shown there that the extremal points of $\mathscr{S}_1$ are exactly the factorial states in $\mathscr{S}_1$ and for any $\psi \in \mathscr{S}_1$ the support of the corresponding central measure μ is contained in $\mathscr{S}_1$. The restriction of μ to $\mathscr{S}_1$ coincides with the maximal measure with respect to the simplex $\mathscr{S}_1$. Hence we may consider μ as a measure on $\mathscr{S}_1$.

For any $\varphi \in \mathscr{S}_1$ let π_φ be the corresponding GNS-representation on the Hilbert space $\mathscr{H}_\varphi$ and for any $a \in \mathscr{A}$ let a_φ be the canonical image of a in $\mathscr{H}_\varphi$. By using (KMS I) it was shown in [19], Proposition 1.3, that e_φ is a separating vector for $\pi_\varphi(\mathscr{A})''$ for any $\varphi \in \mathscr{S}_1$. Let $\mathfrak{A}_\varphi$ be the l. H. a. corresponding to $\pi_\varphi(\mathscr{A})''$ and e_φ (see [18], Theorem 12.1), and let Δ_φ be the modular operator associated with $\mathfrak{A}_\varphi$. Then for any $a \in \mathscr{A}$, $t \in \mathbf{R}$ we have $\pi_\varphi(\sigma_t(a)) = \Delta_\varphi^{\mathrm{i}t}\pi_\varphi(a)\Delta_\varphi^{-\mathrm{i}t}$ (see [19], Proposition 1.3). Let Λ be the linear subspace of $\prod_{\varphi \in \mathscr{S}_1} \mathfrak{A}_\varphi$ which is generated by all vector fields of the form $\mathscr{S}_1 \ni \varphi \mapsto$ $\mapsto f(\varphi)a_\varphi$, where $f \in C(\mathscr{S}_1)$, $a \in \mathscr{A}$.

THEOREM 3.1. (1) *$(\{\mathfrak{A}_\varphi\}_{\varphi \in \mathscr{S}_1}, \Lambda)$ is a continuous field of l. H. a.'s.*

(2) *Let $\psi \in \mathscr{S}_1$ and let μ be the corresponding central measure. Then there is an isomorphism Φ from the Hilbert space $\mathscr{H}_\psi$ onto $\mathscr{H}^\psi = \int^\oplus \mathscr{H}_\varphi\,\mathrm{d}\mu(\varphi)$ such that the following holds:*

(a) *Φ^{-1} maps the l. H. a. $\mathfrak{A}^\psi := \int^\oplus \mathfrak{A}_\varphi\,\mathrm{d}\mu(\varphi)$ isomorphic onto a subalgebra $\mathfrak{A}_\psi^0$ of $\mathfrak{A}_\psi$ and $\mathfrak{A}_\psi^{0\prime\prime} = \mathfrak{A}_\psi$ holds.*

(b) *The set $\{\Phi \circ \mathscr{Z} \circ \Phi^{-1} \mid \mathscr{Z} \in \pi_\psi(\mathscr{A})'' \cap \pi_\psi(\mathscr{A})'\}$ coincides with the set of all diagonalisable operators.*

Proof. (1) Obviously the conditions (L1) up to (L5) and (L7) are satisfied. By using [2], p. 43, Theorem 3, one can see that (L6) is also valid. By 1.2(a) it is clear that condition (4) in 1.3 is satisfied. By 1.3 this implies that condition (L8) is satisfied. Thus we have shown that $(\{\mathfrak{A}_\varphi\}_{\varphi \in \mathscr{S}_1}, \Lambda)$ is a continuous field of l. H. a.'s.

(2) Using [15], 3.1.3, one can show as in [11], 4.1, that there is a unique isomorphism Φ which maps the Hilbert space $\mathscr{H}_\psi$ onto $\mathscr{H}^\psi$ such that for any $a \in \mathscr{A}$ the image of the element $a_\psi \in \mathscr{H}_\psi$ with respect to Φ is the element in $\mathscr{H}^\psi$ corresponding to the vector field $\varphi \mapsto a_\varphi$ and in addition (b) is satisfied. Moreover it is seen that the

restriction of Φ^{-1} to $\mathfrak{A}^{\psi}$ is an isomorphism of the l. H. a.'s $\mathfrak{A}^0_{\psi}$ and $\mathfrak{A}^{\psi}$. Since $\Phi^{-1}(\mathfrak{A}^{\psi}) \supseteq \{a_{\psi} | a \in \mathscr{A}\}$ holds it is clear that $\mathfrak{A}^{0\prime\prime}_{\psi} = \mathfrak{A}_{\psi}$ is valid.

Since in our situation the condition (L9) is satisfied the results of Section 2 are available here.

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