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ON DISTRIBUTION SEMI-GROUPS OF SUBNORMAL OPERATORS

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§ 1. INTRODUCTION

Let X be a Banach space and A a closed, densely defined operator in X with domain $D(A)$; an $\mathscr{L}(X)$-valued distribution $\mathscr{E}$ with the support contained in $[0, +\infty)$ is said to be a *regular distribution semi-group* (R.D.S.G., in short) of generator A if $\mathscr{E}$ and A satisfy the equations:

(1.1) $$(A - \mathrm{d}/\mathrm{d}t)*\mathscr{E} = \delta \otimes I_X \quad \text{and} \quad \mathscr{E}*(A - \mathrm{d}/\mathrm{d}t) = \delta \otimes I_{D(A)}.$$

An R.D.S.G. $\mathscr{E}$ is said to be an *exponential distribution semi-group* of type $\leqslant \omega$ (E.D.S.G., in short) if $\mathscr{E}$ satisfies the following condition:

there exists a real ω such that $\mathrm{e}^{-\xi t}\mathscr{E}$ is an $\mathscr{L}(X)$-valued tempered distribution, for any $\xi > \omega$.

Distribution semi-groups were defined and studied by J. L. Lions in [7].

Let us denote $Y = \bigcap_{n=1}^{\infty} D(A^n)$ and endowe Y with the Fréchet topology determined by the norms $\|x\|_n = \sum_{j=0}^{n} \|A^j x\|$.

Then the following conditions on the operator A are equivalent:

(i) A is the generator of a R.D.S.G.;

(ii) the resolvent $R(\lambda; A)$ exists in a logarithmic region of the form

$$\Lambda = \{\lambda \in \mathbf{C};\ \operatorname{Re}\lambda \geqslant \alpha \log|\operatorname{Im}\lambda| + \beta,\ \operatorname{Re}\lambda \geqslant \gamma\}$$

where $\alpha, \beta \geqslant 0$, $\gamma \in \mathbf{R}$ are some given constants, and satisfies

$$\|R(\lambda; A)\| \leqslant p(|\lambda|), \quad \lambda \in \Lambda$$

$p(\lambda)$ beeing a polynom with positive coefficients.

(iii) the resolvent set $\rho(A)$ is not empty and the restriction of A to Y, A_Y, is the generator of a locally-equicontinuous semi-group $\{U_t\}_{t\geqslant 0}$ of class (C_0) in Y.

The equivalence (i) $\Leftrightarrow$ (ii) was proved by J. Chazarain in [2] and the equivalence (i) $\Leftrightarrow$ (iii) was obtained by T. Ushijima in [11].

The R.D.S.G. $\mathscr{E}$ and the semi-group $\{U_t\}_{t\geqslant 0}$ generated by A_Y may be expressed in terms of the resolvent $R(\lambda; A)$ as follows:

$$\mathscr{E}(\varphi) = \int_{\partial \Lambda} \widetilde{\varphi}(\lambda) R(\lambda; A) \mathrm{d}\lambda \quad \text{for each } \varphi \in \mathscr{D} \tag{1.2}$$

where $\mathscr{D}$ is the space of all indefinitely differentiable functions on the real line with compact support, $\widetilde{\varphi}(\lambda) = \int_{\mathbf{R}} \mathrm{e}^{\lambda t}\varphi(t)\mathrm{d}t$, and $\partial\Lambda$ is the boundary of Λ;

$$U_t x = \lim_{h \to 0+} (I - hA)^{-[t/h]} x \quad \text{for each } x \in Y, \tag{1.3}$$

where the convergence is uniform with respect to t in every finite interval in $[0, +\infty)$ (see [8] and [9]).

Moreover, we recall that:

$$\mathscr{E}(\varphi) = \int_{\mathbf{R}} \varphi(t) U_t x \mathrm{d}t \quad \text{for each } x \in Y,\ \varphi \in \mathscr{D} \tag{1.4}$$

(see [11]) and that the semi-group property is given by $\mathscr{E}(\varphi * \psi) = \mathscr{E}(\varphi)\mathscr{E}(\psi)$ for $\varphi, \psi \in \mathscr{D}_0 = \{\varphi \in \mathscr{D},\ \mathrm{supp}\, \varphi \subset [0, +\infty)\}$.

Further, the following conditions on the operator A are equivalent:

(i′) A is the generator of an E.D.S.G. of type $\leqslant \omega$;

(ii′) the resolvent $R(\lambda; A)$ exists for $\mathrm{Re}\,\lambda > \omega$ and satisfies

$$\|R(\lambda; A)\| \leqslant p(|\lambda|), \quad \mathrm{Re}\,\lambda > \omega$$

for a polynom $p(\lambda)$;

(iii′) the resolvent set $\rho(A)$ is not void and $A_Y - \omega$ is the generator of an equicontinuous semi-group of class (C_0) in Y.

J. L. Lions proved [7] that (i′) $\Leftrightarrow$ (ii′) and D. Fujiwara [4] got that (i′) $\Leftrightarrow$ (iii′).

Finally we recall that in [3] C. Foiaş studied distribution semi-groups of normal operators and obtained the following result:

if $\mathscr{E}$ is an E.D.S.G. *of normal operators in a Hilbert space, then $\mathscr{E}$ is an ordinary continuous semi-group of normal operators.*

After these preliminaries on distribution semi-groups, we shall now give some elementary facts on subnormal operators.

Let X be a Hilbert space; a linear operator T with domain $D(T)$ is called subnormal if there exists a larger Hilbert space H containing X and a normal operator N in H which extends T (one says also that H reduces T). This definition is due to P. R. Halmos [5]. J. Bram [1] showed that a bounded operator on X is subnormal if and only if for every finite sequence of vectors $x_0, x_1, \ldots, x_n$ in X holds:

$$\sum_{i,j=0}^{n} < T^i x_j, T^j x_i > \geqslant 0.$$

He also proved that if N is a minimal normal extension of the bounded subnormal operator T, then $\|T\| = \|N\|$ and $\sigma(N) \subset \sigma(T)$. ($\sigma(N)$ denotes the spectrum of N.)

Let $\{T_t\}_{t \geqslant 0}$ be a continuous semi-group of bounded subnormal operators in X; then there exists a Hilbert space $H \supset X$ and a continuous semi-group of normal operators in H extending $\{T_t\}_{t \geqslant 0}$.

This theorem was first proved by T. Ito [6]; recently a short, different proof was obtained by E. Nussbaum [10].

Adapting coveniently the method from [10], we shall extend in this note the above theorem to distribution semi-groups of subnormal operators and as a corollary we shall get a generalization of Foiaş' result to E.D.S.G. of subnormal operators in a Hilbert space.

§ 2. MAIN RESULT

In all this paragraph X will be a Hilbert space.

PROPOSITION. *Let $\{T_t\}_{t \geqslant 0}$ be a continuous semi-group of bounded subnormal operators in X; then for each $\varphi \in \mathscr{D}$, the operator*

$$\mathscr{E}(\varphi) = \int_0^{+\infty} \varphi(t) T_t \mathrm{d}t$$

is a subnormal operator.

Proof. By a result from [10], Proposition 2, for each $a > 0$ and each continuous function $f:[0, a] \to X$, we have:

$$\int_0^a \int_0^a \langle T_t f(s), T_s f(t) \rangle \mathrm{d}t \mathrm{d}s \leqslant 0.$$

On the other hand, using the semi-group property, it is easy to prove that $\mathscr{E}(\varphi \circledast \psi) =$ $= \mathscr{E}(\varphi)\mathscr{E}(\psi)$, for each $\varphi, \psi \in \mathscr{D}$, where $\varphi \circledast \psi(t) = \int_0^t \varphi(s)\psi(t-s)\mathrm{d}s$. Then, $x_0, x_1, \ldots$ $\ldots, x_n$ beeing arbitrary $n + 1$ vectors in X, it follows that:

$$\sum_{i,j=0}^{n} \langle \mathscr{E}^i(\varphi)x_j, \mathscr{E}^j(\varphi)x_i \rangle =$$

$$= \sum_{i,j=0}^{n} \langle \mathscr{E}(\varphi_i)x_j, \mathscr{E}(\varphi_j)x_i \rangle =$$

$$= \int_0^{na} \int_0^{na} \sum_{i,j=0}^{n} \langle \varphi_i(t) T_t x_j, \varphi_j(s) T_s x_i \rangle \mathrm{d}t \mathrm{d}s =$$

$$= \int_0^{na} \int_0^{na} \langle T_t f(s), T_s f(t) \rangle \mathrm{d}t \mathrm{d}s \geqslant 0$$

where $\varphi_i = \underbrace{\varphi \circledast \varphi \circledast \ldots \circledast \varphi}_{i}, f(t) = \sum_{i=0}^{n} \varphi_i(t) x_i$ and a is such that $\mathrm{supp}\varphi \subset [-a, a]$.

q.e.d

We shall say that the R.D.S.G. $\mathscr{E}$ is a distribution semi-group of subnormal operators on X if for every $\varphi \in \mathscr{D}$, $\mathscr{E}(\varphi)$ is a subnormal operator on X.

By the above proposition, it is a quite natural generalization of the notion of continuous semi-group of bounded subnormal operators.

Then we have the following

LEMMA. *The generator A of a* R.D.S.G. *of subnormal operators is a subnormal operator.*

Proof. We start by using some arguments from [2] to get a convenient form of the resolvent $R(\lambda; A)$.

Let $0 < a < a'$ and $\theta \in \mathscr{D}$ such that $\theta(t) \equiv 1$ for $t \in [0, a]$ and $\theta(t) \equiv 0$ for $t \notin [-1, a']$. Denote $\theta_\lambda(t) = e^{-\lambda t}\theta(t)$, $\lambda \in \mathbf{C}$. Then, using the first equation from (1.1), we get:

$$(A - \lambda)\mathscr{E}(\theta_\lambda) = I - \mathscr{E}(e^{-\lambda t}\theta'(t)).$$

Put $\psi_\lambda(t) = e^{-\lambda t}\theta'(t)$; then in [2] it is proved that for λ belonging to some logarithmic region Λ, $\|\mathscr{E}(\psi_\lambda)\| \leqslant 1/2$, that is

$$[I - \mathscr{E}(\psi_\lambda)]^{-1} = \sum_{n=0}^{\infty} \mathscr{E}^n(\psi_\lambda) = \sum_{n=0}^{\infty} \mathscr{E}^n(\psi_\lambda^+) =$$

$$= \sum_{n=0}^{\infty} \mathscr{E}(\underbrace{\psi_\lambda^+ * \ldots * \psi_\lambda^+}_{n\text{-times}}) = \lim_{k\to\infty} \mathscr{E}(\varphi_{\lambda,k}),$$

where $\varphi_{\lambda,k} = \sum_{n=0}^{k} \underbrace{\psi_\lambda^+ * \ldots * \psi_\lambda^+}_{n\text{-times}} \in \mathscr{D}_0$ and $\psi_\lambda^+ = \begin{cases} \psi_\lambda & \text{on } [0, +\infty) \\ 0 & \text{on } (-\infty, 0) \end{cases}$. Finally we get using in the same way the second equation from (1.1) that $R(\lambda; A)$ exists in a logarithmic region Λ and it is given by

$$R(\lambda; A) = \mathscr{E}(\theta_\lambda)[I - \mathscr{E}(\psi_\lambda)]^{-1} =$$

$$= \mathscr{E}(\theta_\lambda) \lim_{k\to\infty} \mathscr{E}(\varphi_{\lambda,k}) = \lim_{k\to\infty} \mathscr{E}(\theta_\lambda^+)\cdot\mathscr{E}(\varphi_{\lambda,k}) =$$

$$= \lim_{k\to\infty} \mathscr{E}(\Phi_{\lambda,k}),$$

where $\Phi_{\lambda,k} = \theta_\lambda^+ * \varphi_{\lambda,k} \in \mathscr{D}_0$.

(We used the fact proved in [7], that putting for $\psi \in \mathscr{D}$, $\mathscr{E}(\psi^+)\mathscr{E}(\varphi)x = \mathscr{E}(\psi^+ * \varphi)x$, $\varphi \in \mathscr{D}_0$, $x \in X$, we get a closable densely defined operator such that $\overline{\mathscr{E}(\psi^+)} = \mathscr{E}(\psi)$.)

As for each $\lambda \in \Lambda$ and $k \in \mathbf{N}$, the operator $\mathscr{E}(\Phi_{\lambda,k})$ is subnormal, it is clear that $R(\lambda; A)$ is a subnormal operator on X.

Let $\lambda_0 \in \Lambda$ be fixed and let N_{λ_0} be a minimal normal extension of $R(\lambda_0; A)$ acting on a Hilbert space H. Then $N_{\lambda_0}^{-1}$ exists by an argument given in [10], Proposition 3; we give it for completness. If $\mathcal{N}_{\lambda_0} = \mathcal{N}(N_{\lambda_0})$ is the null space of N_{λ_0}, then $\mathcal{N}_{\lambda_0} = \mathcal{N}(N_{\lambda_0}^*)$ and $\mathcal{N}_{\lambda_0}^{\perp} = \overline{\mathcal{R}(N_{\lambda_0})} \supset \mathcal{R}(N_{\lambda_0}) = X$ ($\mathcal{R}$ denotes the range). Since $\mathcal{N}_{\lambda_0}^{\perp}$ reduces N_{λ_0} and N_{λ_0} is a minimal normal extension of $R(\lambda_0; A)$, $\mathcal{N}_{\lambda_0}^{\perp} = H$ and therefore $\mathcal{N}_{\lambda_0} = \{0\}$. Hence $N_{\lambda_0}^{-1}$ exists, is closed and densely defined and is a minimal normal extension of $\lambda_0 - A$. Hence $N = \lambda_0 - N_{\lambda_0}^{-1}$ is a minimal normal extension of A. q.e.d.

We can now give the

THEOREM. *Let $\mathcal{E}$ be a* R.D.S.G. *of bounded subnormal operators in a Hilbert space X; then there exists a Hilbert space H containing X and a* R.D.S.G. *$\tilde{\mathcal{E}}$ of normal operators in H such that $\tilde{\mathcal{E}}(\varphi)_X = \mathcal{E}(\varphi)$, for each $\varphi \in \mathcal{D}$.*

Proof. Let N be a minimal normal extension of A, associated as in the Lemma to a fixed $\lambda_0 \in \Lambda$, acting in a Hilbert space $H \supset X$. Then $\sigma((\lambda_0 - N)^{-1}) \subset \sigma((\lambda_0 - A)^{-1})$ and by the spectral mapping theorem, it results that $\sigma_e(N) \subset \sigma_e(A)$ (σ_e is the extended spectrum). Hence $\sigma(N) \subset \sigma(A)$, whence $\rho(A) \subset \rho(N)$. So $\rho(N)$ contains the logarithmic region Λ and for $\lambda \in \Lambda$ holds:

$$\|R(\lambda; N)\| = \|R(\lambda; A)\| \leqslant p(|\lambda|).$$

Therefore, by the equivalence (i) $\Leftrightarrow$ (ii), N is the generator of a R.D.S.G. of normal operators in H, given by (1.2):

$$\tilde{\mathcal{E}}(\varphi) = \int_{\Lambda} \tilde{\varphi}(\lambda) R(\lambda; N) \mathrm{d}\lambda, \quad \varphi \in \mathcal{D},$$

where by the Lemma, $R(\lambda; N)$ is normal. It is clear that each $\tilde{\mathcal{E}}(\varphi)$ extends $\mathcal{E}(\varphi)$, $\varphi \in \mathcal{D}$. q.e.d.

COROLLARY. *Let $\mathcal{E}$ be an* E.D.S.G. *of subnormal operators in a Hilbert space X; then $\mathcal{E}$ is given by an usual continuous semi-group of bounded subnormal operators in X.*

Proof. Let N be a minimal normal extension of the generator A of $\mathcal{E}$ acting on the Hilbert space $H \supset X$. Then, by a similar argument as in the above theorem, it results that $R(\lambda; N)$ exists for $\mathrm{Re}\lambda > \omega$, where ω is the type of $\mathcal{E}$ and it is majorized by a polynom. Hence, by the equivalence (i′) $\Leftrightarrow$ (ii′), N is the generator of an E.D.S.G. of normal operators in H, $\tilde{\mathcal{E}}$. By the result of C. Foiaş, $\tilde{\mathcal{E}}$ is given by an ordinary continuous semi-group $\{T_t\}_{t \geqslant 0}$ of normal operators in H.

Let $x \in Y = \bigcap_{n=0}^{\infty} D(A^n)$ and $\{U_t\}_{t \geqslant 0}$ the locally-equicontinuous semi-group of class (C_0) generated by A_Y in Y (see (iii)); then by (1.3) we have:

$$U_t x = \lim_{h \to 0+} (I - hA)^{-[t/h]} x =$$
$$= \lim_{h \to 0+} (I - hN)^{-[t/h]} x = T_t x.$$

As Y is dense in X, it is clear that $\{U_t\}_{t \geqslant 0}$ is a continuous semi-group of subnormal operators on X which by (1.4) coincide, in the distributional sense, with $\mathscr{E}$.
q.e.d.

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