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$C_{\cdot 0}$ CONTRACTIONS: CYCLIC VECTORS, COMMUTANTS AND JORDAN MODELS

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The study of $C_{\cdot 0}$ contractions was initiated by Sz.-Nagy and Foiaş [8]. There they obtained the Jordan model for $C_{\cdot 0}$ contractions whose defect indices are finite. Later on this was generalized to $C_{\cdot 0}$ contractions with at least one finite defect index (cf. [4]). Using this model, Uchiyama was able to characterize the hyperinvariant subspaces of such operators and prove that if the defect indices are not equal they are always reflexive (cf. [10] and [11], resp.). In this paper we will also use this model to explore other properties of $C_{\cdot 0}$ contractions.

In Section 1 we are mainly concerned with the following question: Are the properties of being cyclic and having a commutative commutant equivalent for $C_{\cdot 0}$ contractions? Note that for the more restrictive class of $C_0(N)$ contractions the answer is affirmative (cf. [5] and [6]). We obtain necessary and sufficient conditions for each of these properties. It turns out that for general $C_{\cdot 0}$ contractions these two properties are not equivalent.

The main result in Section 2 is that if two $C_{\cdot 0}$ contractions with finite defect indices are such that one is a quasi-affine transform of the other, then they have the same Jordan model. This is a generalization of the corresponding results for $C_0(N)$ contractions and C_{10} contractions (cf. [6] and [15], resp.). From this we can derive other results for $C_{\cdot 0}$ contractions when they are intertwined by operators which are one-to-one or have dense ranges.

For operators T_1, T_2 on Hilbert spaces $\mathscr{H}_1, \mathscr{H}_2$, $T_1 \overset{\text{i}}{\prec} T_2$ (resp. $T_1 \overset{\text{d}}{\prec} T_2$) denotes that there exists an operator $X: \mathscr{H}_1 \to \mathscr{H}_2$ intertwining T_1, T_2 which is one-to-one (resp. has dense range). $T_1 \overset{\text{ci}}{\prec} T_2$ denotes that there exists a family $\{X_\alpha\}$ of intertwining operators $X_\alpha: \mathscr{H}_1 \to \mathscr{H}_2$ such that each X_α is one-to-one and $\mathscr{H}_2 = \bigvee_\alpha X_\alpha \mathscr{H}_1$. If there is only one operator (called a *quasi-affinity*) in this family, then we say that T_1 is a *quasi-affine transform* of T_2 and denote this by $T_1 \prec T_2$. T_1 is *quasi-similar* to T_2 ($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$. The *multiplicity* μ_T of an operator T on $\mathscr{H}$ is the smallest cardinal number of a set $\mathscr{K}$ of vectors in $\mathscr{H}$ such that $\mathscr{H} = \bigvee_{n=0}^{\infty} T^n \mathscr{K}$. If $T_1 \prec T_2$, then $\mu_{T_1} \geqslant \mu_{T_2}$.

Recall that a contraction T ($\|T\| \leqslant 1$) is *of class* $C_{\cdot 0}$ if $T^{*n}x \to 0$ as $n \to \infty$ for any x. The *defect indices* of T are, by definition, $d_T = \text{rank}(I - T^*T)^{1/2}$ and $d_{T^*} = \text{rank}(I - TT^*)^{1/2}$. If T is a $C_{\cdot 0}$ contraction then $d_T \leqslant d_{T^*}$. For any inner function φ, let $S(\varphi)$ denote the operator defined on $H^2 \ominus \varphi H^2$ by $S(\varphi)f = P(e^{it}f)$ where $f \in H^2 \ominus \varphi H^2$ and P is the (orthogonal) projection onto $H^2 \ominus \varphi H^2$. Let S_l denote the unilateral shift of multiplicity l. An operator of the form $S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$, where $0 \leqslant k < \infty, 0 \leqslant l \leqslant \infty$, φ_j's satisfy $\varphi_j | \varphi_{j-1}$ for $j = 2,3,\ldots,k$ and φ_k is not a constant function, is called a *Jordan operator*. Sz.-Nagy [4] showed that if T is a $C_{\cdot 0}$ contraction with $d_T < \infty$ then there exists a uniquely determined Jordan operator $J = S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$ (called the *Jordan model* of T) such that $J \overset{ci}{\prec} T \prec J$. Moreover, $k \leqslant d_T$ and $l = d_{T^*} - d_T$. In the following, results from Sz.-Nagy and Foiaş' contraction theory will be used extensively. The main reference is their book [7].

1. CYCLIC VECTORS AND COMMUTANTS

We start this section by finding the multiplicity of a Jordan operator.

LEMMA 1.1. *Assume that T is an operator on $\mathscr{H}$ and $\mathscr{K} \subseteq \mathscr{H}$ is an invariant subspace for T. Let $T_1 = PT|\mathscr{K}^\perp$, where P denotes the (orthogonal) projection onto $\mathscr{K}^\perp$. Then $\mu_{T_1} \leqslant \mu_T$.*

Proof. Let $\mathscr{L}$ be a set of vectors in $\mathscr{H}$ such that $\mathscr{H} = \bigvee_{n=0}^{\infty} T^n \mathscr{L}$. It is easily seen that the set $P\mathscr{L}$ of vectors in $\mathscr{K}^\perp$ satisfy $\mathscr{K}^\perp = \bigvee_{n=0}^{\infty} T_1^n P\mathscr{L}$. It follows immediately that $\mu_{T_1} \leqslant \mu_T$.

LEMMA 1.2. *Let $T = S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$ be a Jordan operator. Then $\mu_T = k + l$.*

Proof. Assume that T is acting on $\mathscr{H} = (H^2 \ominus \varphi_1 H^2) \oplus \ldots \oplus (H^2 \ominus \varphi_k H^2) \oplus \underbrace{H^2 \oplus \ldots \oplus H^2}_{l}$. We have $\mu_T \leqslant k + l$ by [12], Lemma 1. To prove $k + l \leqslant \mu_T$, consider the invariant subspace $\mathscr{K} = \underbrace{0 \oplus \ldots \oplus 0}_{k} \oplus \underbrace{\varphi_k H^2 \oplus \ldots \oplus \varphi_k H^2}_{l}$. Since $\mathscr{K}^\perp = (H^2 \ominus \varphi_1 H^2) \oplus \ldots \oplus (H^2 \ominus \varphi_k H^2) \oplus \underbrace{(H^2 \ominus \varphi_k H^2) \oplus \ldots \oplus (H^2 \ominus \varphi_k H^2)}_{l}$, $T_1 \equiv PT|\mathscr{K}^\perp = S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus \underbrace{S(\varphi_k) \oplus \ldots \oplus S(\varphi_k)}_{l}$, where P denotes the projection onto $\mathscr{K}^\perp$. It is known that $\mu_{T_1} = k + l$ (cf. [6]). Therefore Lemma 1.1 implies that $k + l = \mu_{T_1} \leqslant \mu_T$, completing the proof.

Recall that an operator T is *cyclic* if $\mu_T = 1$. The next theorem gives a necessary and sufficient condition for a $C_{\cdot 0}$ contraction with $d_T < \infty$ to be cyclic. Note that characterizations of cyclic $C_0(N)$ and C_{10} contractions are already known (cf. [5], Theorem 2 and [14], Theorem 3.1, resp.).

THEOREM 1.3. *Let T be a $C_{\cdot 0}$ contraction with $d_T < \infty$. Then the following statements are equivalent:*

(1) *T is cyclic;*

(2) *either T is of class $C_0(N)$ and quasi-similar to $S(\varphi)$ for some inner function φ or T is of class C_{10} and quasi-similar to S, the simple unilateral shift.*

Proof. (2) $\Rightarrow$ (1). Obvious.

(1) $\Rightarrow$ (2). Let $J = S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$ be the Jordan model of T. Since $T \prec J$, we have $1 = \mu_T \geqslant \mu_J = k + l$ by Lemma 1.2. Hence either $k = 1$, $l = 0$ or $k = 0$, $l = 1$. In the former case, T is of class $C_0(N)$ and quasi-similar to $S(\varphi_1)$ (cf. [6]). In the latter, $S \overset{\text{ci}}{\prec} T \prec S$, which implies that T is of class C_{10} and quasi-similar to S (cf. [15], Lemma 1 and [14], Theorem 3.1).

COROLLARY 1.4. *Let T be a $C_{\cdot 0}$ contraction with $d_T < \infty$. If T is cyclic, so is T^*.*

Proof. Theorem 1.3 says that T is quasi-similar to $S(\varphi)$ or S. Hence T^* is quasi-similar to $S(\varphi^\sim)$ or S^*, where $\varphi^\sim(\lambda) = \overline{\varphi(\bar{\lambda})}$ for $|\lambda| < 1$. In either case, T^* must be cyclic.

The converse of the preceding assertion is certainly false as the example $T = S \oplus S$ shows, where S denotes the simple unilateral shift (cf. [2], Problem 126).

Let $\{T\}'$ and $\{T\}''$ denote the *commutant* and the *double commutant* of an operator T. The next theorem gives a necessary and sufficient condition that a $C_{\cdot 0}$ contraction with $d_T < \infty$ satisfy $\{T\}' = \{T\}''$.

THEOREM 1.5. *Let T be a $C_{\cdot 0}$ contraction on $\mathscr{H}$ with $d_T < \infty$. Then the following statements are equivalent:*

(1) $\{T\}' = \{T\}''$;

(2) *either T is of class $C_0(N)$ and cyclic or T is of class C_{10} with $d_{T^*} - d_T = 1$.*

Proof. (1) $\Rightarrow$ (2). Let $J = S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$ be the Jordan model of T acting on the space $\mathscr{K}$. By [4], Theorem 3, there exist one-to-one operators $X_1, X_2 : \mathscr{K} \to \mathscr{H}$ and Y_1, Y_2: $\mathscr{H} \to \mathscr{K}$ which intertwine J, T, satisfy $\mathscr{H} = X_1\mathscr{K} \vee \vee X_2\mathscr{K}$, $\mathscr{K} = Y_1\mathscr{H} \vee Y_2\mathscr{H}$ and are such that $Y_1X_1 = \eta_1(J)$, $Y_2X_2 = \eta_2(J)$ for some functions η_1, η_2 in H^∞ with $\eta_1 \wedge \eta_2 = 1$.

We first show that $\{J\}' = \{J\}''$. Let $V, W \in \{J\}'$. Obviously, $X_1VY_1, X_2VY_2 \in \in \{T\}' = \{T\}''$. Note that we may assume that $d_T < d_{T^*}$ for otherwise T is of class

$C_0(N)$ whence (1) implies that T is cyclic (cf. [5] and [6]). Then $\{T\}'' = \{\varphi(T): :\varphi \in H^\infty\}$ (cf. [11], Theorem 1). Say, $X_1VY_1 = \varphi_1(T)$ and $X_2VY_2 = \varphi_2(T)$, where $\varphi_1, \varphi_2 \in H^\infty$. We have

$$\eta_1(J)V\eta_1(J) = Y_1X_1VY_1X_1 = Y_1\varphi_1(T)X_1 = Y_1X_1\varphi_1(J) = \eta_1(J)\varphi_1(J).$$

Since $\eta_1(J)$ is one-to-one, $V\eta_1(J)=\varphi_1(J)$. Hence

$$VW\eta_1(J)=V\eta_1(J)W=\varphi_1(J)W=W\varphi_1(J) = WV\eta_1(J).$$

Similarly, $VW\eta_2(J) = WV\eta_2(J)$. By Beurling's theorem $\eta_1 \wedge \eta_2 = 1$ implies that $\mathcal{K} = \eta_1(J)\mathcal{K} \vee \eta_2(J)\mathcal{K}$ whence $VW = WV$ on $\mathcal{K}$ for any $V, W \in \{J\}'$. This shows that $\{J\}' = \{J\}''$ as asserted. From this we can easily deduce that $k = 1$, $l = 0$ or $k = 0$, $l = 1$. Since by assumption $d_T < d_{T^*}$, the former case cannot occur. Hence T must be of class C_{10} and $d_{T^*} - d_T = l = 1$.

(2) $\Rightarrow$ (1). It suffices to consider the case when T is a C_{10} contraction with $d_{T^*} - d_T = 1$. Then the Jordan model of T is S, the simple unilateral shift. Let $V \in \{T\}'$ and let X_1, X_2, Y_1 and Y_2 be as above. Since Y_1VX_1, $Y_2VX_2 \in \{S\}' = \{\varphi(S):\varphi \in H^\infty\}$, (1) follows from [11], Lemma 6.

COROLLARY 1.6. *Let T be a $C_{\cdot 0}$ contraction with $d_T < \infty$. If T is cyclic, then $\{T\}' = \{T\}''$.*

The first example of a non-cyclic operator T such that $\{T\}' = \{T\}''$ was constructed by Deddens [1]. From Theorems 1.3 and 1.5 we can easily construct C_{10} contractions with similar properties. One such example is in [8], pp. 321—322.

2. JORDAN MODELS

Recall that a contraction T is *completely non-unitary* (*c.n.u.*) if there is no non-trivial reducing subspace for T on which T is unitary. $C_{\cdot 0}$ contractions are always c.n.u..

LEMMA 2.1. *Let T_1, T_2 be c.n.u. contractions acting on $\mathcal{H}_1$, $\mathcal{H}_2$, respectively, and let $\varphi \in H^\infty$.*

(1) *If $T_1 \prec T_2$, then $T_1|\overline{\varphi(T_1)\mathcal{H}_1} \prec T_2|\overline{\varphi(T_2)\mathcal{H}_2}$.*

(2) *If $T_1 \overset{ci}{\prec} T_2$, then $T_1|\overline{\varphi(T_1)\mathcal{H}_1} \overset{ci}{\prec} T_2|\overline{\varphi(T_2)\mathcal{H}_2}$.*

Proof. We only prove (1) and leave the rest to the readers. Let $X: \mathcal{H}_1 \to \mathcal{H}_2$ be a quasi-affinity intertwining T_1, T_2. Since $X\varphi(T_1) = \varphi(T_2)X$, we have $X\overline{\varphi(T_1)\mathcal{H}_1} \subseteq \overline{\varphi(T_2)\mathcal{H}_2}$. To complete the proof it suffices to show that $\overline{X\varphi(T_1)\mathcal{H}_1} = \overline{\varphi(T_2)\mathcal{H}_2}$. For any $x \in \overline{\varphi(T_2)\mathcal{H}_2}$ and $\varepsilon > 0$, there exist $y \in \mathcal{H}_2$, $z \in \mathcal{H}_1$ such that $\|\varphi(T_2)y - x\| \leqslant \varepsilon$ and $\|Xz - y\| \leqslant \varepsilon$. Hence

$$\|X\varphi(T_1)z-x\|=\|\varphi(T_2)Xz - x\| \leqslant \|\varphi(T_2)Xz - \varphi(T_2)y\|+\|\varphi(T_2)y - x\| \leqslant \|\varphi\|_\infty\varepsilon + \varepsilon.$$

Since ε is arbitrary, this proves our assertion.

The next theorem is the major step in proving our main result. It says that for a $C_{\cdot 0}$ contraction T the only Jordan operator of which T is a quasi-affine transform is its Jordan model. Unfortunately, the proof works only for those whose defect indices are finite.

THEOREM 2.2. *Let T be a $C_{\cdot 0}$ contraction on $\mathscr{H}$ with $\mathrm{d}_T \leqslant \mathrm{d}_{T^*} < \infty$. Assume that $J = S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$ is a Jordan operator acting on $\mathscr{K}$ such that $T \prec J$. Then J is the Jordan model of T.*

Proof. Let $J' = S(\psi_1) \oplus \ldots \oplus S(\psi_p) \oplus S_r$ be the Jordan model of T acting on $\mathscr{K}'$. Then we have $J' \overset{\text{ci}}{\prec} T \prec J'$. Let $X: \mathscr{K}' \to \mathscr{H}$ and $Y: \mathscr{H} \to \mathscr{K}'$ be, respectively, a one-to-one operator and a quasi-affinity which intertwine J', T and such that $YX = \eta(J')$, $XY = \eta(T)$ for some function η in H^∞ satisfying $\eta \wedge \psi_1\varphi_1 = 1$. (The existence of such intertwining operators follows from the proof of [3], Theorem 2.) Since $X: \mathscr{K}' \to \overline{X\mathscr{K}'} = \overline{XY\mathscr{H}} = \overline{\eta(T)\mathscr{H}}$ is a quasi-affinity intertwining J' and $T|\overline{\eta(T)\mathscr{H}}$, we have $J' \prec T|\overline{\eta(T)\mathscr{H}}$. By Lemma 2.1, $T|\overline{\eta(T)\mathscr{H}} \prec J|\overline{\eta(J)\mathscr{K}}$ and since $\eta \wedge \varphi_1 = 1$, $J|\overline{\eta(J)\mathscr{K}}$ is unitarily equivalent to J (cf. [8], pp. 315—316). We conclude that $J' \prec J$ and so $p \leqslant k$, $r \leqslant l$ and $\psi_j | \varphi_j$ for $j = 1, 2, \ldots, p$ (cf. [8], Theorem 4). On the other hand, by Lemma 1.2 we have $p + r = \mu_{J'} \geqslant \mu_J = k + l$. This, together with $p \leqslant k$, $r \leqslant l$, implies that $p = k$ and $r = l$.

Next we prove that $\psi_j = \varphi_j$ for $j = 1, 2, \ldots, p$. From $J' \prec J$, we have $J'|\overline{\varphi(J')\mathscr{K}'} \prec J|\overline{\varphi(J)\mathscr{K}}$ for any $\varphi \in H^\infty$. In particular, let $\varphi = \psi_t$, $t = 1, 2, \ldots, p$. Since $J'|\overline{\psi_t(J')\mathscr{K}'} \cong S(\xi_1) \oplus \ldots \oplus S(\xi_p) \oplus S_r$ and $J|\overline{\psi_t(J)\mathscr{K}} \cong S(\tau_1) \oplus \ldots \oplus S(\tau_p) \oplus S_r$, where $\xi_j = \psi_j/(\psi_t \wedge \psi_j)$ and $\tau_j = \varphi_j/(\psi_j \wedge \varphi_j)$ for $j = 1, 2, \ldots, p$, we infer that

$$S(\xi_1) \oplus \ldots \oplus S(\xi_p) \oplus S_r \prec S(\tau_1) \oplus \ldots \oplus S(\tau_p) \oplus S_r.$$

Note that $\xi_j = 1$ for $j = t, t+1, \ldots, p$. Hence

$$S(\xi_1) \oplus \ldots \oplus S(\xi_{t-1}) \oplus S_r \prec S(\tau_1) \oplus \ldots \oplus S(\tau_p) \oplus S_r.$$

We claim that τ_t must be a constant function. Assume the contrary. Let m and n be, respectively, the largest integers not greater than $t-1$ and p such that ξ_m and τ_n are not constant functions. Then

$$S(\xi_1) \oplus \ldots \oplus S(\xi_m) \oplus S_r \prec S(\tau_1) \oplus \ldots \oplus S(\tau_n) \oplus S_r$$

and, by assumption, $n \geqslant t$. Hence by Lemma 1.2, $t - 1 + r \geqslant m + r \geqslant n + r \geqslant t + r$, which leads to a contradiction. Thus $\tau_t = \varphi_t/(\psi_t \wedge \varphi_t)$ is a constant function as asserted. Therefore $\varphi_t | \psi_t$ for each t. This, together with $\psi_j | \varphi_j$ for $j = 1, 2, \ldots, p$, shows that $\psi_j = \varphi_j$ for all j. Hence $J = J'$ is the Jordan model of T.

THEOREM 2.3. *Let T_1, T_2 be $C_{\cdot 0}$ contractions with $\mathrm{d}_{T_1} \leqslant \mathrm{d}_{T_1^*} < \infty$ and $\mathrm{d}_{T_2} < \infty$ and assume that $T_1 \prec T_2$. Then T_1, T_2 have the same Jordan model.*

Proof. Let J be the Jordan model of T_2. Then $T_1 \prec T_2 \prec J$. Theorem 2.2 implies that J is also the Jordan model of T_1.

The preceding theorem generalizes the corresponding results for $C_0(N)$ contractions and C_{10} contractions (cf. [6], Corollary 1 and [15], Lemma 4, resp.).

COROLLARY 2.4. *Let T_1, T_2 be $C_{\cdot 0}$ contractions with $\mathrm{d}_{T_1} \leqslant \mathrm{d}_{T_1^*} < \infty$ and $\mathrm{d}_{T_2} < \infty$ and assume that $T_1 \prec T_2$. Then T_1 is reflexive if and only if T_2 is.*

Recall that an operator T is *reflexive* if $\mathrm{AlgLat}T = \mathrm{Alg}T$, where $\mathrm{AlgLat}T$ and $\mathrm{Alg}T$ denote the (weakly closed) algebra of operators which leave every invariant subspace of T invariant and the (weakly closed) algebra generated by T and I, respectively.

Proof. From Theorem 2.3, $T_1 \prec T_2$ implies that $\mathrm{d}_{T_1^*} - \mathrm{d}_{T_1} = \mathrm{d}_{T_2^*} - \mathrm{d}_{T_2} \equiv d$. We consider two cases:

(i) If $d > 0$, then both T_1 and T_2 are reflexive by [11], Theorem 2.

(ii) If $d = 0$, then both T_1 and T_2 are $C_0(N)$ contractions. Hence $T_1 \prec T_2$ implies that $T_1 \sim T_2$ (cf. [6]) and thus T_1 is reflexive if and only if T_2 is (cf. [13], Theorem 2).

In the rest of this paper, we consider questions which are similar in nature to those in [9]. More specifically, if T_1, T_2 are $C_{\cdot 0}$ contractions, we ask

(i) whether $T_1 \overset{\mathrm{i}}{\prec} T_2$ and $T_1 \overset{\mathrm{d}}{\prec} T_2$ imply $T_1 \prec T_2$ and

(ii) whether $T_1 \overset{\mathrm{d}}{\prec} T_2$ and $T_2 \overset{\mathrm{d}}{\prec} T_1$ imply $T_1 \sim T_2$.

It turns out that both answers are positive if the defect indices of T_1, T_2 are finite. We start with the following lemma.

LEMMA 2.5. *Let T_1, T_2 be $C_{\cdot 0}$ contractions on $\mathscr{H}_1, \mathscr{H}_2$ with $\mathrm{d}_{T_1}, \mathrm{d}_{T_2} < \infty$.*

(1) *If $T_1 \overset{\mathrm{i}}{\prec} T_2$, then $\mathrm{d}_{T_1^*} - \mathrm{d}_{T_1} \leqslant \mathrm{d}_{T_2^*} - \mathrm{d}_{T_2}$.*

(2) *If $T_1 \overset{\mathrm{d}}{\prec} T_2$, then $\mathrm{d}_{T_2^*} - \mathrm{d}_{T_2} \leqslant \mathrm{d}_{T_1^*} - \mathrm{d}_{T_1}$.*

Proof. (1) Let J_1, J_2 be the Jordan models of T_1, T_2, respectively. $T_1 \overset{\mathrm{i}}{\prec} T_2$ implies that $J_1 \overset{\mathrm{i}}{\prec} J_2$. Hence, by [8], Theorem 4, $\mathrm{d}_{T_1^*} - \mathrm{d}_{T_1} \leqslant \mathrm{d}_{T_2^*} - \mathrm{d}_{T_2}$.

(2) We may assume that $\mathrm{d}_{T_1^*} < \infty$ for otherwise the assertion is trivial. Let $X: \mathscr{H}_1 \to \mathscr{H}_2$ be an operator intertwining T_1, T_2 with dense range and let $T_1 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ be the triangulation of T_1 on $\mathscr{H}_1 = \ker X \oplus \overline{\mathrm{range} X^*}$. It is easily

seen that T_4 is a $C_{\cdot 0}$ contraction with $d_{T_4} \leqslant d_{T_4^*} < \infty$ and $T_4 \prec T_2$. Therefore, by Theorem 2.3, $d_{T_4^*} - d_{T_4} = d_{T_2^*} - d_{T_2}$.

Next let $\Theta_{T_1} = \Theta_4\Theta_3$ be the regular factorization of the characteristic function of T_1 corresponding to $T_1 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ (cf. [7], p. 288) and let n be the dimension of the intermediate space of $\Theta_{T_1} = \Theta_4\Theta_3$. Since Θ_{T_1} is an inner function, so are Θ_3 and Θ_4 (cf. [7], p. 299). Thus $d_{T_1} \leqslant n \leqslant d_{T_1^*}$ (cf. [7], p. 190). Note that the characteristic function of T_4 is the purely contractive part of Θ_4 (cf. [7], p. 289). It follows that $d_{T_4} = n - m$ and $d_{T_4^*} = d_{T_1^*} - m$ for some m, $0 \leqslant m \leqslant \min\{n, d_{T_1^*}\}$. Hence $d_{T_4^*} - d_{T_4} = d_{T_1^*} - n \leqslant d_{T_1^*} - d_{T_1}$ and we conclude that $d_{T_2^*} - d_{T_2} \leqslant \leqslant d_{T_1^*} - d_{T_1}$.

Note that in the preceding lemma if $S(\psi_1) \oplus \ldots \oplus S(\psi_p) \oplus S_r$ and $S(\varphi_1) \oplus \ldots \ldots \oplus S(\varphi_k) \oplus S_l$ denote the Jordan models of T_1 and T_2, respectively, then under (1) we even have $p \leqslant k$ and $\psi_j | \varphi_j$ for $j = 1, \ldots, p$ (cf. [8], Theorem 4). But under (2) the corresponding assertion, that is, $k \leqslant p$ and $\varphi_j | \psi_j$ for $j = 1, \ldots, k$, is in general false as the following example shows.

EXAMPLE 2.6. Let T_1 be the simple unilateral shift on H^2 and $T_2 = S(\varphi)$ for some inner function φ. Let $X: H^2 \to H^2 \ominus \varphi H^2$ be defined by $Xf = Pf$ for $f \in H^2$, where P denotes the (orthogonal) projection onto $H^2 \ominus \varphi H^2$. Then it is easily seen that X intertwines T_1, T_2 and has dense range. Thus $T_1 \overset{d}{\prec} T_2$ but the assertion above is certainly false.

In a sense this example exhibits the worst which can happen in this situation. Indeed, if the unilateral shift parts of the Jordan models of T_1 and T_2 are the same, then their $C_0(N)$ parts will also be the same. In order to prove this, we need another lemma.

LEMMA 2.7. *Let T be a $C_{\cdot 0}$ contraction with $d_T < \infty$ and let* $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ *be the (unique) triangulation of type* $\begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix}$. *If the Jordan model of T is $S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$, then the Jordan models of T_1 and T_2 are $S(\varphi_1) \oplus \ldots \ldots \oplus S(\varphi_k)$ and S_l, respectively.*

Proof. Let $J_1 = S(\varphi_1) \oplus \ldots \oplus S(\varphi_k)$ and $J_2 = S_l$. Note that any injection intertwining contractions induces an injection intertwining their $C_{0\cdot}$ parts. Hence $J_1 \oplus J_2 \overset{ci}{\prec} T \prec J_1 \oplus J_2$ implies that $J_1 \overset{i}{\prec} T_1 \overset{i}{\prec} J_1$. It follows that $T_1 \sim J_1$ (cf. [9], Theorem 1) whence J_1 is the Jordan model of T_1.

As for T_2, let $\Theta_T = \Theta_{*e}\Theta_{*i}$ be the $*$-canonical factorization of the characteristic function Θ_T of T, which corresponds to the triangulation $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ (cf. [7],

p. 300), and let n be the dimension of its intermediate space. T_1 is of class C_{00} implies that Θ_{*i} is inner from both sides whence $n = d_T$ (cf. [7], p. 257). Since the characteristic function of T_2 is the purely contractive part of Θ_{*e}, we infer that $d_{T_2} = n - m$ and $d_{T_2^*} = d_{T^*} - m$ for some m. Thus the Jordan model of the C_{10} contraction T_2 is S_r, where $r = d_{T_2^*} - d_{T_2} = d_{T^*} - n = d_{T^*} - d_T = l$. This completes the proof.

LEMMA 2.8. *Let T_1, T_2 be $C_{\cdot 0}$ contractions on $\mathscr{H}_1$, $\mathscr{H}_2$ with finite defect indices and let $S(\psi_1) \oplus \dots \oplus S(\psi_p) \oplus S_r$, $S(\varphi_1) \oplus \dots \oplus S(\varphi_k) \oplus S_r$ be their Jordan models, respectively. If $T_1 \overset{d}{\prec} T_2$, then $k \leqslant p$ and $\varphi_j | \psi_j$ for $j = 1, \dots, k$.*

Proof. Let $X: \mathscr{H}_1 \to \mathscr{H}_2$ be an operator intertwining T_1, T_2 with dense range and let $T_1 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ be the triangulation on $\mathscr{H}_1 = \ker X \oplus \overline{\operatorname{range} X^*}$ with the corresponding factorization $\Theta_{T_1} = \Theta_4 \Theta_3$. As in the proof of Lemma 2.5 (2), $T_4 \prec T_2$ implies that T_4, T_2 have the same Jordan model. In particular,

$$d_{T_4^*} - d_{T_4} = d_{T_2^*} - d_{T_2} = r = d_{T_1^*} - d_{T_1}.$$

As before, $d_{T_4^*} - d_{T_4} = d_{T_1^*} - n$, where n is the dimension of the intermediate space of $\Theta_{T_1} = \Theta_4 \Theta_3$. It follows that $d_{T_1} = n$ and therefore Θ_3 is an inner function implies that it is inner from both sides (cf. [7], p. 190). Hence T_3 is of class C_{00} (cf. [7], p. 257). Let $T_4 = \begin{bmatrix} T_5 & * \\ 0 & T_6 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix}$. We have

$$T_1 = \begin{bmatrix} T_3 & * & * \\ 0 & T_5 & * \\ 0 & 0 & T_6 \end{bmatrix}.$$

Since both T_3 and T_5 are of class C_{00}, from the corresponding regular factorization of $T_7 \equiv \begin{bmatrix} T_3 & * \\ 0 & T_5 \end{bmatrix}$ it can be easily derived that T_7 is also of class C_{00}. Hence $T_1 = \begin{bmatrix} T_7 & * \\ 0 & T_6 \end{bmatrix}$ is of type $\begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix}$. By Lemma 2.7, $S(\psi_1) \oplus \dots \oplus S(\psi_p)$ and $S(\varphi_1) \oplus \dots \oplus S(\varphi_k)$ are the Jordan models of T_7 and T_5, respectively. Since for any inner function φ, $S(\varphi)^* \cong S(\varphi^\sim)$, where $\varphi^\sim(\lambda) = \overline{\varphi(\bar{\lambda})}$ for $|\lambda| < 1$, we infer that $S(\varphi_1^\sim) \oplus \dots \oplus S(\varphi_k^\sim) \sim T_5^* \overset{i}{\prec} T_7^* \sim S(\psi_1^\sim) \oplus \dots \oplus S(\psi_p^\sim)$. This implies that $k \leqslant p$ and $\varphi_j^\sim | \psi_j^\sim$ for $j = 1, \dots, k$ or $\varphi_j | \psi_j$ for all j (cf. [8], Theorem 4).

Now we are ready to answer the two questions posed before Lemma 2.5. The next theorem answers question (i). For $C_0(N)$ and C_{10} contractions, the assertion is already known to be true (cf. [9], Theorem 1 and [15], remark after Theorem 5).

THEOREM 2.9. *Let T_1, T_2 be $C_{\cdot 0}$ contractions on $\mathscr{H}_1$, $\mathscr{H}_2$ with* $d_{T_1} \leqslant d_{T_1^*} < \infty$ *and* $d_{T_2} < \infty$. *If* $T_1 \overset{\mathrm{i}}{\prec} T_2$ *and* $T_1 \overset{\mathrm{d}}{\prec} T_2$, *then* $T_1 \prec T_2$. *Moreover, any operator* $X: \mathscr{H}_1 \to \mathscr{H}_2$ *which intertwines* T_1, T_2 *and has dense range must be one-to-one.*

Proof. Let $S(\psi_1) \oplus \ldots \oplus S(\psi_p) \oplus S_r$ and $S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$ be the Jordan models of T_1 and T_2, respectively. Lemma 2.5 implies that $r = d_{T_1^*} - d_{T_1} = d_{T_2^*} - d_{T_2} = l$. Then by Lemma 2.8, $k \leqslant p$ and $\varphi_j | \psi_j$ for $j = 1, \ldots, k$. But $T_1 \overset{\mathrm{i}}{\prec} T_2$ implies that $p \leqslant k$ and $\psi_j | \varphi_j$ for $j = 1, \ldots, p$. Thus $p = k$ and $\psi_j = \varphi_j$ for all j. Let $T_1 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ be the triangulation on $\mathscr{H}_1 = \ker X \oplus \overline{\operatorname{range} X^*}$ and let $T_4 = \begin{bmatrix} T_5 & * \\ 0 & T_6 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix}$. As in the proof of Lemma 2.8, $T_7 \equiv \begin{bmatrix} T_3 & * \\ 0 & T_5 \end{bmatrix}$ and T_5 have the same Jordan model $S(\psi_1) \oplus \ldots \oplus S(\psi_p)$. Hence T_3 must act on a zero space (cf. [6], Corollary 2), that is, $\ker X = \{0\}$. We conclude that X is one-to-one and $T_1 \prec T_2$ as asserted.

COROLLARY 2.10. *Let T_1, T_2 be $C_{\cdot 0}$ contractions with finite defect indices. Assume that T_1, T_2 have the same Jordan model and $X: \mathscr{H}_1 \to \mathscr{H}_2$ intertwines T_1, T_2 with dense range. Then X is one-to-one.*

Proof. Let J be the common Jordan model of T_1 and T_2. Then $T_1 \prec J \overset{\mathrm{i}}{\prec} T_2$ and the conclusion follows from Theorem 2.9.

We conclude this paper with the answer to question (ii) posed before Lemma 2.5.

THEOREM 2.11. *Let T_1, T_2 be $C_{\cdot 0}$ contractions with finite defect indices. Assume that* $T_1 \overset{\mathrm{d}}{\prec} T_2$ *and* $T_2 \overset{\mathrm{d}}{\prec} T_1$. *Then* $T_1 \sim T_2$.

Proof. Let $S(\psi_1) \oplus \ldots \oplus S(\psi_p) \oplus S_r$ and $S(\varphi_1) \oplus \ldots \oplus S(\varphi_k) \oplus S_l$ be the Jordan models of T_1 and T_2, respectively. Lemma 2.5 (2) implies that $r = d_{T_1^*} - d_{T_1} = d_{T_2^*} - d_{T_2} = l$. Then by Lemma 2.8, $p = k$ and $\psi_j = \varphi_j$ for all j. Thus T_1, T_2 have the same Jordan model. The conclusion now follows from Corollary 2.10.

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