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A SIMPLE UNITAL PROJECTIONLESS C^*-ALGEBRA

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1. INTRODUCTION

It has long been an open question whether there exists a simple projectionless C^*-algebra (i.e. a C^*-algebra with no nontrivial projections) [9, p. 18]. In [1], the author constructed a simple nonunital projectionless C^*-algebra. Using the same general method, we now construct a unital projectionless simple C^*-algebra.

The idea is to construct a projectionless unital C^*-algebra Γ with $\mathrm{Prim}(\Gamma)$ a circle, and a unital "twice-around" map $\psi:\Gamma \to \Gamma$. Then, if $A_n \cong \Gamma$ and $\varphi_n:A_n \to A_{n+1}$ is taken to be ψ, $A = \varinjlim (A_n, \varphi_n)$ is the desired algebra.

As in [1], Γ will be a C^*-algebra of the form $\{f:[0, 1] \to B | f(1) = \sigma(f(0))\}$, where B is an appropriate simple unital AF algebra and σ is an appropriate automorphism of B. The important property that σ must have is that $\sigma(p)$ is not equivalent to p, for every nontrivial projection $p \in B$; this property insures that Γ is projectionless. B and σ must have two other technical properties in order that the twice-around embedding can be constructed.

B and σ are constructed in Section 2. Actually what is constructed is the dimension group $K_0(B)$ and the automorphism σ_* of $K_0(B)$. Then in Section 3, the construction of A is described.

Section 4 is a description of some of the structure of A. It is clear from its definition that A is nuclear. Proposition 4.2 shows that A has a unique trace (which of course must be faithful and finite, since A is simple and unital). We also show in Section 4 that $K_0(A) \cong \mathbf{Z}$.

Also in Section 4, some variations on the construction of A are discussed. It is shown in Theorems 4.10 and 4.12 that if X is any compact totally disconnected metric space and N is any countable torsion-free Abelian group, then there is a simple unital projectionless C^*-algebra A_H with $K_0(A_H) \cong H = C(X, \mathbf{Z}) \oplus N$ with the strict ordering from the first coordinate, and with $ET(A_H) \cong X$.

Section 5 concludes with some open questions.

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2. CONSTRUCTION OF B

Since a unital AF algebra D is uniquely determined by the ordered group $K_0(D)$ and an order unit $u \in K_0(D)_+$ [5], we will construct a countable simple Riesz group G with order unit, and apply [4, Theorem 2.2] to get B.

We will construct a simple dimension group G, an order unit $u \in G_+$, and two order-preserving automorphisms σ_* and α_* of G with the following properties:

(1) u is a minimal fixed point for σ_* in G_+;

(2) There is a $v \in G_+$, $v \leqslant u$, with $\sigma_*(v) = u - v$, $\alpha_*(u) = v$;

(3) $\sigma_*^2 \circ \alpha_* = \alpha_* \circ \sigma_*$.

Let $\bar{\mathbf{Q}}$ be the set of 2-adic numbers, $\bar{\mathbf{Z}}$ the set of 2-adic integers. We recall the following properties of $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Z}}$:

(i) $\bar{\mathbf{Q}}$ is a locally compact totally disconnected topological field and $\bar{\mathbf{Z}}$ is a compact open subring.

(ii) The compact open subgroups of the additive group of $\bar{\mathbf{Q}}$ are precisely the sets $2^n\bar{\mathbf{Z}}$, $n \in \mathbf{Z}$. $2^m\bar{\mathbf{Z}} \subseteq 2^n\bar{\mathbf{Z}}$ if and only if $m \geqslant n$. The number $-n$ is called the level of the subgroup $2^n\bar{\mathbf{Z}}$.

(iii) Every compact open subset of $\bar{\mathbf{Q}}$ is a finite disjoint union of cosets of compact open subgroups. Each coset of level k is a disjoint union of two cosets of level $k-1$.

(iv) $\mathbf{Z}$ is dense in $\bar{\mathbf{Z}}$ under the natural injection.

A complete discussion of $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Z}}$ is found in [8, Section 10].

Let μ be the additive Haar measure on $\bar{\mathbf{Q}}$, normalized so that $\mu(\bar{\mathbf{Z}}) = 1$. Let G be the additive group of all integer-valued continuous functions on $\bar{\mathbf{Q}}$ with compact support. (Each function takes only a finite number of values.) Then G is a countable torsion-free Abelian group. Let $\theta(g) = \int g \, d\mu$, and give G the strict ordering from θ, i.e. $G_+ = \{0\} \cup \{g \in G : \theta(g) > 0\}$.

PROPOSITION 2.1. *(G, G_+) is a simple dimension group.*

Proof. θ is a homomorphism from G to $\mathbf{R}$, and $\theta(G)$ consists of the dyadic rationals, which are dense in $\mathbf{R}$, so the Proposition follows from [4, Lemmas 3.1 and 3.2].

Let σ_* and α_* be defined on G by $(\sigma_* g)(x) = g(x-1)$, $(\alpha_* g)(x) = g(x/2)$; then $\sigma_*, \alpha_* \in \operatorname{Aut}(G)$ and $\sigma_*^2 \circ \alpha_* = \alpha_* \circ \sigma_*$. Let $u = \chi_{\bar{\mathbf{Z}}}$, $v = \chi_{2\bar{\mathbf{Z}}}$. Then $\sigma_*(u) = u$,

$\alpha_*(u) = v$, $\sigma_*(v) = u - v$. If g is a function which is invariant under σ_*, then g is constant on cosets of $\bar{\mathbf{Z}}$, and thus $\theta(g)$ is an integer; since $\theta(u) = 1$, it follows that u is a minimal positive fixed point for σ_*.

3 CONSTRUCTION OF A

The group (G, u) constructed in Section 2 is the dimension group of a unique simple unital AF algebra B. By [5, Theorem 4.3], there is an automorphism σ of B implementing σ_* on $G = K_0(B)$. If p is a projection in B with $[p] = v$, then $\sigma(p)$ is equivalent to $1 - p$, so (by composing σ with an inner automorphism if necessary) we may assume $\sigma(p) = 1 - p$. Again by [5, Theorem 4.3], there exists an isomorphism $\alpha: B \to pBp$ implementing α_*. We would like to choose σ and α so that $\sigma^2 \circ \alpha = \alpha \circ \sigma$, but it is not clear this can be done. However, the difficulty can be avoided by using paths of isomorphisms in a manner similar to, but less complicated than, the arguments in [1].

We begin with a theorem of independent interest. This theorem was first stated and independently proved by J. Rosenberg (private communication), but since the nontrivial part of the proof appeared in [1] we will finish the argument.

THEOREM 3.1. *Let D be a unital AF algebra. The following subgroups of* $\mathrm{Aut}(D$ *coincide:*

(1) $\overline{\mathrm{Inn}}(D)$, *the approximately inner automorphisms (the closure of the inner automorphisms* $\mathrm{Inn}(D)$*);*

(2) $\mathrm{Aut}(D)_0$, *the connected component of the identity;*

(3) $\mathrm{Aut}(D)_{\mathrm{p}}$, *the path component of the identity;*

(4) $\mathrm{Id}(D) = \{\beta \in \mathrm{Aut}(D) : \beta_*$ *is the identity on* $K_0(D)\}$.

The topology on $\mathrm{Aut}(D)$ *is the topology of pointwise (norm-) convergence.*

Proof. It is trivial that $\mathrm{Inn}(D) \subseteq \mathrm{Id}(D)$ and $\mathrm{Aut}(D)_{\mathrm{p}} \subseteq \mathrm{Aut}(D)_0$, and the proof of [1, Theorem 2.3] shows that $\mathrm{Id}(D) \subseteq \overline{\mathrm{Inn}}(D) \cap \mathrm{Aut}(D)_{\mathrm{p}}$. If $\gamma \notin \mathrm{Id}(D)$, choose a projection $q \in D$ with $\gamma(q) \nsim q$. The set $U_q = \{\beta \in \mathrm{Aut}(D) : \beta(q) \sim q\}$ is an open-and-closed set in $\mathrm{Aut}(D)$ (actually a subgroup) containing $\mathrm{Id}(D)$ but not γ (since $\{q' \in D : q' \sim q\}$ is an open-and-closed subset of the projections of D by [6, Lemma 1.8]). Hence $\mathrm{Aut}(D)_0 \subseteq \mathrm{Id}(D)$, so $\mathrm{Id}(D) \subseteq \mathrm{Aut}(D)_{\mathrm{p}} \subseteq \mathrm{Aut}(D)_0 \subseteq \mathrm{Id}(D)$. Also, $\mathrm{Id}(D) = \bigcap_q U_q$, so $\mathrm{Id}(D)$ is closed; hence $\overline{\mathrm{Inn}}(D) \subseteq \mathrm{Id}(D) \subseteq \overline{\mathrm{Inn}}(D)$.

Note that $\overline{\mathrm{Inn}}(D) \subseteq \mathrm{Id}(D)$ and $\mathrm{Aut}(D)_0 \subseteq \mathrm{Id}(D)$ hold in any unital C^*-algebra.

We now return to σ and α. $\alpha^{-1} \circ \sigma^2 \circ \alpha \circ \sigma^{-1} \in \mathrm{Id}(B)$, so there exists a continuous path $\beta_t \subseteq \mathrm{Aut}(B)$ from the identity to $\alpha^{-1} \circ \sigma^2 \circ \alpha \circ \sigma^{-1}$. If $\alpha_t = \alpha \circ \beta_t$, α_t is a continuous path of isomorphisms of B onto pBp from $\alpha_0 = \alpha$ to $\alpha_1 = \sigma^2 \circ \alpha \circ \sigma^{-1}$.

Let $\Gamma = \{f : [0, 1] \to B \mid f$ continuous, $f(1) = \sigma(f(0))\}$. Γ is unital, and as in [1, § 1], $\mathrm{Prim}(\Gamma)$ is a circle.

PROPOSITION 3.2. *Γ is projectionless.*

Proof. If $f \in \Gamma$ is a projection, then $f(t)$ is a projection in B for all t, and it follows from the argument of [1, Prop. 1.1] that $f(0) \sim f(1)$. But since u is a minimal fixed point of σ_*, $\sigma(q) \nsim q$ for every nontrivial projection $q \in B$.

We define $\psi : \Gamma \to \Gamma$ by

$$(\psi(f))(t) = \alpha_t\left(f\left(\frac{t}{2}\right)\right) + \sigma^{-1}\left(\alpha_1\left(f\left(\frac{t+1}{2}\right)\right)\right).$$

Since α_t maps B into pBp and $\sigma^{-1} \circ \alpha_1$ maps B into $(1-p)B(1-p)$, it is easily verified that ψ is a $*$-isomorphism of Γ into $C([0,1], B)$.

$$(\psi(f))(0) = \alpha(f(0)) + \sigma^{-1}\left(\alpha_1\left(f\left(\frac{1}{2}\right)\right)\right),$$

$$(\psi(f))(1) = \alpha_1\left(f\left(\frac{1}{2}\right)\right) + \sigma^{-1}(\alpha_1(f(1))) =$$

$$= \alpha_1\left(f\left(\frac{1}{2}\right)\right) + \sigma^{-1}(\alpha_1(\sigma(f(0)))) = \alpha_1\left(f\left(\frac{1}{2}\right)\right) + \sigma(\alpha(f(0))) = \sigma(\psi(f)(0)).$$

So ψ maps Γ into Γ. ψ can be described pictorially as a map into a 2×2 matrix algebra as in [1] if desired.

Now let $A_n = \Gamma$, $\varphi_n = \psi$ regarded as a map from A_n into A_{n+1}, and set $A = \varinjlim A_n$. Since each A_n is unital, and ψ is unital, A is unital, and the arguments in [1, 3.2 and 3.4] show that A is simple and projectionless.

4. FURTHER PROPERTIES OF A

PROPOSITION 4.1. *A is nuclear.*

Proof. Each A_n is clearly nuclear since every primitive subquotient is isomorphic to B.

PROPOSITION 4.2. *A has a unique trace (which is necessarily faithful and finite).*

Proof. Since G has a unique state θ, it follows that B has a unique (normalized) trace τ. Any trace on A_n is of the form τ_μ for some probability measure μ on $[0,1)$, where $\tau_\mu(f) = \int_0^1 \tau(f(t))d\mu(t)$. Since $\tau(p) = \dfrac{1}{2}$, it follows that any trace on A_{n+1}, when restricted to A_n, gives a measure μ with $\mu([0,1/2)) = 1/2$, $\mu([1/2,1)) = 1/2$;

by induction, a trace on A_n extends to A_{n+k} if and only if the corresponding measure μ has $\mu\left(\left[\frac{m}{2^k}, \frac{m+1}{2^k}\right)\right) = 2^{-k}$ for all m. The unique measure with this property for all k is Lebesgue measure λ. Thus τ_λ on each A_n extends to A, and is the only normalized trace on A.

REMARK 4.3. A similar argument shows that the algebra constructed in [1] also has a unique trace, which is finite. However, in this case the measure is not Lebesgue measure since the diagonal blocks are of different sizes; rather, the traces on $A(\sigma_n)$ which extend to $A(\sigma_{n+1})$ are the ones for which the measure satisfies $\mu([0,1/2)) = \mu_n$, $\mu([1/2,1)) = 1 - \mu_n$, where μ_n is the number constructed inductively as the size of the projection q_n. One detail which must be checked is that the traces do not increase in norm when extended, due to the fact that $\varphi_n(A(\sigma_n))$ contains an approximate identity for $A(\sigma_{n+1})$.

EXAMPLE 4.4. If Δ is any metrizable Choquet simplex whose set X of extreme points is compact and 0-dimensional, let G_Δ be the set of all continuous functions from X into G (with the discrete topology). Then G_Δ is a countable torsion-free Abelian group under pointwise operations. Give G_Δ the strict ordering. Let u_Δ and v_Δ be the constant functions on X with value u and v respectively; then u_Δ is an order unit for G_Δ. The states on (G_Δ, u_Δ) are exactly the functions $\hat{p}$ for $p \in \Delta$, where $\hat{p}(f) = \theta(f(p))$; the pure states are the ones for which $p \in X$. Since $\{\theta \circ f : f \in G_\Delta\}$ is dense in $\mathrm{Aff}(\Delta) \cong C(X)$, it follows from [4, Lemma 3.2] that G_Δ is a simple dimension group. Define σ_Δ, $\alpha_\Delta \in \mathrm{Aut}(G_\Delta)$ by $[\sigma_\Delta(f)](x) = \sigma_*(f(x))$, $[\alpha_\Delta(f)](x) = \alpha_*(f(x))$; then, if G_Δ, u_Δ, v_Δ, σ_Δ, α_Δ are used for the construction of A rather than $G, u, v, \sigma_*, \alpha_*$ we obtain a simple unital projectionless C^*-algebra A_Δ whose set of normalized traces is affinely homeomorphic to Δ.

We now describe $K_0(A)$. We will describe $K_0(\Gamma)$ and ψ_*, and then use the fact that $K_0(A) = \varinjlim (K_0(\Gamma), \psi_*)$.

$$M_n(\Gamma) \cong \{f:[0,1] \to M_n(B) \mid f(1) = \sigma(f(0))\},$$

where we use σ also to denote the canonical extension of $\sigma \in \mathrm{Aut}(B)$ to $M_n(B)$. If $f \in M_n(\Gamma)$, then, as in [1, 1.1], $f(0) \sim f(1) = \sigma(f(0))$, and so $[f] = [f(t)] \in K_0(B)$ is independent of t and is an element of $K_0(B)_+$ fixed under σ_* (using the natural identification of $K_0(B)$ with $K_0(M_n(B))$).

LEMMA 4.5. *If f and g are projections of $M_n(\Gamma)$, then $f \sim g$ in $M_n(\Gamma)$ if and only if $[f] = [g]$ in $K_0(B)$.*

Proof. If $f \sim g$ via a partial isometry u, then $f(0) \sim g(0)$ in $M_n(B)$ via $u(0)$, so $[f] = [g]$. Conversely, suppose $[f] = [g]$. There is a unitary $w(0) \in M_n(B)$ with

$w(0)g(0)w(0)^* = f(0)$; since the unitary group of $M_n(B)$ is path-connected, there is a path $w(t)$ of unitaries from $w(0)$ to $\sigma(w(0))$; $w \in M_n(\Gamma)$, and by conjugating g by w we may assume $f(0) = g(0) = q$, so $f(1) = g(1) = \sigma(q)$. There are continuous paths $u(t)$, $v(t)$ of unitaries of $M_n(B)$ such that $f(t) = u(t)qu(t)^*$, $g(t) = v(t)qv(t)^*$, and $u(0) = v(0) = 1$ [3, Lemma 7]. Since $\sigma(q) = u(1)qu(1)^* = v(1)qv(1)^*$, $v(1)^*u(1)$ commutes with q; so by [1, Lemma 2.1] there is a continuous path $z(t)$ of unitaries in the commutant of q in $M_n(B)$ with $z(0) = 1$ and $z(1) = v(1)^*u(1)$. If $h(t) = v(t)z(t)u(t)^*$, then $h(0) = h(1) = 1$, so $h \in M_n(\Gamma)$, and $g(t) = h(t)f(t)h(t)^*$.

I am indebted to L. G. Brown for helpful comments on the proof of 4.5.

It is clear from 4.5 that two projections in $M_n(\Gamma)$ define the same element of $K_0(\Gamma)$ if and only if they are unitarily equivalent in $M_n(\Gamma)$. Thus there is a well-defined injective map $w: K_0(\Gamma) \to K_0(B)$ whose range is contained in the set of fixed points of $K_0(B)$ under σ_*.

PROPOSITION 4.6. *The range of w is exactly the fixed-point subgroup of $K_0(B)$ under σ_*.*

Proof. The group of fixed points of $K_0(B)$ is the set of functions constant on cosets of $\mathbf{Z}$ and thus is positively generated. If $x \in K_0(B)_+$ is a fixed point, for some $n, x \leqslant nu$, so x is the equivalence class of a projection $q \in M_n(B)$ with $\sigma(q) \sim q$. Since the unitary group of $M_n(B)$ is path-connected, there is a path $f(t)$ of projections in $M_n(B)$ with $f(0) = q$, $f(1) = \sigma(q)$; f is then a projection of $M_n(\Gamma)$ with $[f] = x$

We now examine the map ψ_*. If f is a projection in $M_n(\Gamma)$ with $f(0) = q$, then $\psi(f(0)) = \alpha(q) + \sigma^{-1}\left(\alpha_1\left(f\left(\frac{1}{2}\right)\right)\right)$; since $f\left(\frac{1}{2}\right) \sim q$ and $\alpha_{1*} = \alpha_*$, it follows that $\psi_*([f]) = \alpha_*([q]) + \sigma_*^{-1} \circ \alpha_*([q])$, so $\psi_* = \alpha_* + \sigma_*^{-1} \circ \alpha_*$ (restricted to the fixed--point algebra).

LEMMA 4.7. *If $C = m + \mathbf{Z}$, then $\psi_*(\chi_C) = \chi_D$, where $D = 2m + \mathbf{Z}$; if $C = m + 2^{-n}\mathbf{Z}$ for $n > 0$, then $\psi_*(\chi_C) = 2\chi_D$, where $D = 2m + 2^{-n+1}\mathbf{Z}$.*

Proof. Follows easily from the fact that $\alpha_*(\chi_C) = \chi_{2C}$ and $(\sigma_*^{-1} \circ \alpha_*)(\chi_C) = \chi_{2C-1}$.

COROLLARY 4.8. *The image of any element of $K_0(\Gamma)$ under the iterates of ψ_* is eventually a multiple of u.*

THEOREM 4.9. *Every projection in $M_n(A)$ is unitarily equivalent to a (unique) projection of the form* diag$(1, \ldots, 1, 0, \ldots, 0)$*; hence $K_0(A) \cong \mathbf{Z}$.*

Proof. Follows from 4.8 and the remarks preceding 4.5 and 4.6.

THEOREM 4.10. *$K_0(A_\Delta) \cong C(X, \mathbf{Z})$, the integer-valued continuous functions on X, with the strict ordering (where A_Δ and X are as in Example 4.4).*

Proof. Similar to Theorem 4.9.

EXAMPLE 4.11. (a) If Δ is nontrivial, then $K_0(A_\Delta)$ is not a dimension group in the sense of [4]. These are the first known stably finite simple C^*-algebras for which K_0 is not a dimension group. If Δ is an n-simplex, then $K_0(A_\Delta)$ is $\mathbf{Z}^{n+1}$ with the strict ordering.

(b) Let $M = M_2(A_\Delta)$ for a nontrivial Δ. Then M contains many nontrivial projections; in fact, M is generated as an algebra by its projections (as is the 2×2 matrix algebra over any unital C^*-algebra). But in $K_0(M) = K_0(A_\Delta)$ there is only one nonzero positive element (namely the constant function 1) smaller than the equivalence class of the identity (the constant function 2). Since (as in Theorem 4.9) two projections of M have the same image in $K_0(M)$ if and only if they are equivalent in M, it follows that all nontrivial projections in M are equivalent. Thus, if τ is any normalized trace on M and p is any nontrivial projection in M, $\tau(p) = \dfrac{1}{2}$. So all traces on M agree on linear combinations of projections; since M has more than one trace, it follows that linear combinations of projections are not dense in M.

THEOREM 4.12. *Let N be any countable torsion-free Abelian group. Let $H = K_0(A_\Delta) \oplus N$ for some Δ, and give H the strict ordering from the first coordinate (i.e., $g \oplus n \geqslant 0$ iff $g = n = 0$ or $g > 0$). Then there is a separable simple unital stably finite projectionless (nuclear) C^*-algebra A_H with $K_0(A_H) \cong H$ as an ordered group.*

Proof: Let $\bar{\mathbf{Q}}$ be the topological field of 2-adic numbers, $\bar{\mathbf{Z}}$ the 2-adic integers. Let K be the additive group of all continuous functions from $\bar{\mathbf{Q}}$ to N (regarded as a discrete group) with compact support. Let $\bar{\sigma}, \bar{\alpha} \in \mathrm{Aut}(K)$ be defined by $[\bar{\sigma}(f)](x) = f(x+1)$, $[\bar{\alpha}(f)](x) = f(x/2)$. Then $\bar{\sigma}^2 \circ \bar{\alpha} = \bar{\alpha} \circ \bar{\sigma}$. Let $W = G_\Delta \oplus K$, with the strict ordering from the first coordinate; then W is a simple dimension group. Let $\sigma = \sigma_\Delta \oplus \bar{\sigma}$, $\alpha = \alpha_\Delta \oplus \bar{\alpha}$, $u = u_\Delta \oplus 0$, $v = v_\Delta \oplus 0$, and let A_H be the C^*-algebra obtained by applying the construction of A using W, σ, α, u, v in place of G, σ_*, α_*, u, v. By essentially the same proof as in Theorem 4.9, $K_0(A_H)$ will be the inductive limit of the sequence (F_n, ψ_n), where each F_n is the fixed-point subgroup of W under σ and $\psi_n = \alpha + \sigma^{-1} \circ \alpha$, regarded as a map from F_n to F_{n+1}. Therefore $K_0(A_H)$ will be the direct sum of $K_0(A_\Delta)$ and D, where D is the inductive limit of the sequence $(S_n, \bar{\psi}_n)$, where S_n is the fixed-point subgroup of K under $\bar{\sigma}$ and $\bar{\psi}_n = \bar{\alpha} + \bar{\sigma}^{-1} \circ \alpha$, and the ordering on $K_0(A_H)$ will be the strict ordering from the first coordinate. Thus it suffices to show that $D \cong N$. S_n consists of the functions which are constant on cosets of $\bar{\mathbf{Z}}$, and an argument as in Lemma 4.7 shows that each element of S_n under the iterates of $\bar{\psi}_k$ is eventually sent to a function which takes a constant value on $\bar{\mathbf{Z}}$ and vanishes outside, and every such function is left invariant by $\bar{\psi}_k$. This set of functions is clearly isomorphic to N. A_H is stably finite since it has a trace; in fact, $T(A_H) \cong \Delta$.

5. OPEN PROBLEMS AND QUESTIONS

5.1. If D is a C^*-algebra, recall [2] that the stable algebra of D is $D \otimes K$, where K is the C^*-algebra of compact operators; two C^*-algebras D_1 and D_2 are stably isomorphic if $D_1 \otimes K \cong D_2 \otimes K$. A C^*-algebra is *stably projectionless* if its stable algebra is projectionless. Note that a unital C^*-algebra cannot be stably projectionless.

If D is a separable projectionless simple C^*-algebra, there are four possibilities:

(1) D is unital;

(2) D is nonunital and stably projectionless;

(3) D is nonunital, not stably projectionless, and stably isomorphic to a projectionless unital C^*-algebra (i.e., $D \otimes K$ has a minimal projection);

(4) D is nonunital, not stably projectionless, but not stably isomorphic to a projectionless unital C^*-algebra (i.e., $D \otimes K$ has projections but no minimal projections).

The algebra A of this paper is an example of (1), while the algebra of [1] is an example of (2). To get an example of (3), let D be a proper hereditary C^*-subalgebra of A (for example, $D = [xAx]^-$, where x is a positive zero divisor in A). Then by [2] D is stably isomorphic to A and is clearly projectionless and nonunital.

QUESTION 5.1. Can situation (4) occur? At the other extreme, does every nonzero projection of $D \otimes K$ dominate a minimal projection?

5.2. It would be interesting to know which Abelian groups can occur as $K_0(D)$ for a (separable unital) simple C^*-algebra D. If D is stably finite (i.e., $M_n(D)$ is finite in the Murray-von Neumann sense for all n), then $K_0(D)$ has a natural partial ordering [7, Prop. 2.1]. In this case, D is stably isomorphic to a unital projectionless C^*-algebra if and only if $K_0(D)$ has a minimal positive element. Theorems 4.10 and 4.12 show that there are many such groups.

PROBLEM 5.2. Characterize the ordered groups which can occur as $K_0(D)$ for a stably finite separable simple unital C^*-algebra D.

5.3. There are many notions of finiteness for C^*-algebras in the literature, the weakest of which (for unital algebras) is Murray-von Neumann finiteness and the strongest is the existence of a faithful family of finite traces. It is an important (and difficult) problem to reconcile and distinguish between these various notions, particularly for simple unital C^*-algebras, and the existence of projections in the algebras is very relevant.

QUESTION 5.3. Does there exist a simple unital projectionless C^*-algebra which does not have a finite trace? Does there exist one which is not stably finite?

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