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A NOTE ON POSITIVE OPERATORS

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In the present note, we give a generalization of the elementary fact that a complex number z is a nonnegative real number, if $|z| \leqslant \mathrm{Re}(z)$.

Let T be a bounded linear operator on a Hilbert space. We denote $(T + T^*)/2$ by $\mathrm{Re}(T)$ and the positive square root of T^*T by $|T|$. The generalization is the following.

THEOREM 1. *If* $|T| \leqslant \mathrm{Re}(T)$, *then* T *is positive.*

The above theorem gives a characterization of positive operators. In what follows we shall prove a stronger result, from which Theorem 1 can be derived immediately.

THEOREM 2. *Let* $T = VP$, *where* T, V, P *are bounded linear operators on a Hilbert space with* $P \geqslant 0$ *and* V *being power bounded (i.e.,* $\|V^n\| \leqslant k$ *for a fixed* k *and* $n = 1, 2\ldots$*). If* $P \leqslant \mathrm{Re}(T)$, *then* $T = P$.

Note that Theorem 1 follows from Theorem 2 by considering the polar decomposition of T. An immediate consequence of Theorem 2 is

COROLLARY 3. *If* V *is a power bounded operator and* $\mathrm{Re}(V) \geqslant I$, *then* $V = I$.

We remark that the hypothesis $\mathrm{Re}(V) \geqslant I$ in the above Corollary cannot be replaced by a weaker condition such as $|\mathrm{Re}(V)| \geqslant I$. For example, let V be a 2×2 matrix of the form $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $|\mathrm{Re}(V)| = I$, and $V^2 = 0$. Now we proceed to prove Theorem 2. Firstly, we show the following lemma.

LEMMA 4. *Suppose that* P, V *are operators on a Hilbert space* $\mathscr{H}$, *and* P *is positive. If* $P \leqslant \mathrm{Re}(VP)$, *then* $P \leqslant VPV^*$. *Furthermore, if* $P \leqslant \mathrm{Re}(VP)$ *and* $P = VPV^*$, *then* $VP = P$.

Proof. For each vector x in $\mathscr{H}$, we have

$$\begin{aligned} &\langle Px, x\rangle \leqslant \langle \mathrm{Re}(VP)x, x\rangle = \mathrm{Re}\,\langle VPx, x\rangle \leqslant \\ &\leqslant |\langle VPx, x\rangle| \leqslant \langle Px, x\rangle^{1/2}\langle PV^*x, V^*x\rangle^{1/2} \end{aligned} \tag{1}$$

by applying Schwarz's inequality to the positive semi-definite form $(x, y) \to \langle Px, y\rangle$ $(x, y \in \mathscr{H})$ (in obtaining the last inequality in (1)). Hence $\langle Px, x\rangle \leqslant \langle VPV^*x, x\rangle$ for all x in $\mathscr{H}$, that is, $P \leqslant VPV^*$. In addition to $P \leqslant \mathrm{Re}(VP)$, if $P = VPV^*$ is assumed, then (1) yields

$$\langle Px, x\rangle = \mathrm{Re}\langle VPx, x\rangle = |\langle VPx, x\rangle| = \langle VPx, x\rangle$$

for all x in $\mathscr{H}$, and hence $P = VP$. Q.E.D.

Proof of Theorem 2. Since $VPV^* - P \geqslant 0$ by Lemma 4, it follows that $V(VPV^* - P)V^* \geqslant 0$, that is, $V^2P(V^*)^2 \geqslant VPV^*$. Repeating the process n times, we have $V^{n+1}P(V^*)^{n+1} \geqslant V^nP(V^*)^n$.

Thus, $\{V^nP(V^*)^n \mid n = 1, 2 \dots\}$ is an increasing sequence of positive operators. This sequence is bounded, since V is power bounded. Therefore it converges to a positive operator on $\mathscr{H}$, say Q, in the strong operator topology. Note that

$$VQV^* = V(\lim_{n\to\infty} V^nP(V^*)^n)V^* =$$

$$= \lim_{n\to\infty} V^{n+1}P(V^*)^{n+1} = Q.$$

From $P \leqslant (VP + PV^*)/2$, we have

$$V^nP(V^*)^n \leqslant [V^n(VP + PV^*)(V^*)^n]/2 =$$

$$= [V(V^nP(V^*)^n) + (V^nP(V^*)^n)V^*]/2.$$

By letting n tend to ∞, we have

$$Q \leqslant (VQ + QV^*)/2 = \mathrm{Re}(VQ).$$

It follows from the second statement of Lemma 4 that $Q = VQ$. Since $P \leqslant Q$, it follows that the range of P is contained in the range of Q, and hence $P = VP = T$. Q.E.D.

We conjecture that Theorem 1 can be improved as follows:

CONJECTURE. *If T is a bounded operator and $|T| \leqslant |\mathrm{Re}(T)|$, then T is hermitian.*

The conjecture holds if the underlying Hilbert space is finite dimensional.

THEOREM 5. *If T is a (square) matrix and $|T| \leqslant |\mathrm{Re}(T)|$, then $T = T^*$.*

Proof. Here we give two different proofs which lead to further development in two different directions.

First Proof. Write $T = A + iB$, where A, B are hermitian matrices. Thus $\mathrm{Re}(T) = A$, and $T^*T = A^2 + B^2 + i(AB - BA)$. We show $B = 0$. Let the trace function on the matrix algebra be denoted by tr. From $|T| \leqslant |A|$, we have

$$\operatorname{tr}(|A|^2 - |T|^2) = \operatorname{tr}(|A|^2 - |T|^2) + \operatorname{tr}(|A|\,|T|) - \operatorname{tr}(|T|\,|A|) =$$

(2) $$= \operatorname{tr}([|A| - |T|]\,[|A| + |T|]) =$$

$$= \operatorname{tr}([|A| - |T|]^{1/2}\,[|A| + |T|]\,[|A| - |T|]^{1/2}) \geqslant 0,$$

which is equivalent to

$$\operatorname{tr}(A^2) = \operatorname{tr}(|A|^2) \geqslant \operatorname{tr}(T^*T) = \operatorname{tr}(A^2 + B^2 + \mathrm{i}(AB - BA)).$$

Hence $0 \geqslant \operatorname{tr}(B^2)$. Therefore $B = 0$.

Second Proof. As above write $T = A + \mathrm{i}B$, where A, B are hermitian. Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of A with $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$. Let $\mathcal{M}_k$ be the eigenspace corresponding to λ_k, $k = 1, \ldots, n$. Notice that $|\lambda_1| = \|A\| \leqslant \|T\|$, and

$$\|T\| = \|\,|T|\,\| \leqslant \|\,|\operatorname{Re}(T)|\,\| = \|A\|.$$

Thus $|\lambda_1| = \|A\| = \|T\|$. Let x be in $\mathcal{M}_1$ and $\|x\| = 1$. Then

$$\lambda_1^2 = \|T\|^2 \geqslant \langle T^* Tx, x\rangle = \langle A^2 x, x\rangle + \langle B^2 x, x\rangle = \lambda_1^2 + \|Bx\|^2.$$

From this it follows that $Bx = 0$. Hence $\mathcal{M}_1$ is a reducing subspace of T on which $T = \lambda_1$. By induction, we can show that each $\mathcal{M}_k$ reduces T and $T|_{\mathcal{M}_k} = \lambda_k$. Therefore $T = T^*$. Q.E.D.

REMARKS 1. In [2] it is shown that if $|T|^2 \leqslant (\operatorname{Re}(T))^2$, then $T = T^*$. This result is a weaker statement than our conjecture.

2. The conjecture can be shown true if T is in a finite von Neumann algebra by simply following the first proof of Theorem 5. Since (2) in that proof is also true for any normal tracial state of a finite von Neumann algebra, and totality of normal tracial states separates positive elements in a finite von Neumann algebra ([3], Cor 2.4.7.), the same conclusion $B = 0$ can be obtained.

3. Following the second proof of Theorem 5 we can show the conjecture holds if T is a compact operator.

4. The conjecture holds under the condition that the largest spectral projection E of $\operatorname{Re}(T)$ with $\operatorname{Re}(T)E \geqslant 0$ reduces $|T|$. In fact, let $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be the matrix decomposition with $A = ETE$, $B = ET(I - E)$, $C = (I - E)TE$, and $D = (I - E)T(I - E)$. From the above condition we have

$$\operatorname{Re}(T) = \begin{pmatrix} \operatorname{Re}(A) & 0 \\ 0 & \operatorname{Re}(D) \end{pmatrix}$$

with $\mathrm{Re}(A) \geqslant 0$, $\mathrm{Re}(D) \leqslant 0$, and

$$|T|^2 = \begin{pmatrix} |A|^2 + |C|^2 & 0 \\ 0 & |B|^2 + |D|^2 \end{pmatrix}.$$

From $|\mathrm{Re}(T)| \geqslant |T|$ we obtain $\mathrm{Re}(A) \geqslant (|A|^2 + |C|^2)^{1/2} \geqslant |A|$. (For the last inequality see [1], Thm. 2.5). It follows from this and Theorem 1 that $A \geqslant 0$, and $C = 0$. Similarly, we can show $D \leqslant 0$ and $B = 0$. Therefore, $T = T^*$.

Finally, we show an analoguous result to Theorem 1 for σ-continuous linear functionals on a von Neumann algebra. Let $\mathscr{R}$ be a von Neumann algebra and φ be a σ-continuous linear functional on $\mathscr{R}$ with the polar decomposition $\varphi = U \cdot |\varphi|$, where $U \cdot |\varphi|(A) = |\varphi|(AU)$ for all A in $\mathscr{R}$, and U is a partial isometry in $\mathscr{R}$ with U^*U equal to the support of the positive linear functional $|\varphi|$ ([3], Thm. 1.14.4). In this notation we have the following

THEOREM 6. *If* $|\varphi| \leqslant \mathrm{Re}(\varphi)$, *the real part of* φ, *then* $\varphi \geqslant 0$.

Proof. By the assumption $|\varphi| \leqslant \mathrm{Re}(\varphi)$ we have

$$\|\varphi\| \geqslant |\varphi(I)| \geqslant |\mathrm{Re}(\varphi(I))| = |\mathrm{Re}(\varphi)(I)| \geqslant |\varphi|(I) = \|\varphi\| \tag{3}$$

Thus $|\varphi(I)| = |\mathrm{Re}(\varphi(I))|$. Since $\mathrm{Re}(\varphi(I)) = \mathrm{Re}(\varphi)(I) \geqslant 0$, it follows that $\varphi(I) = \mathrm{Re}(\varphi(I)) \geqslant 0$. It also follows from (3) that $\varphi(I) = \|\varphi\|$. Hence $\varphi \geqslant 0$ ([3], Proposition 1.5.2). Q.E.D

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