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K-THEORY FOR CERTAIN C^*-ALGEBRAS. II

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In a remarkable recent paper [7] M. Pimsner and D. Voiculescu established an exact sequence for the K-groups (and Ext-groups) of crossed product C^*-algebras, that permits to compute these groups for many important C^*-algebras. The method they used is similar to the one used before in [4] to treat the special case of the algebras $\mathcal{O}_n$. In the present paper we follow more closely the ideas of [4] to give a very simple proof of a result that is somewhat more general than the one of Pimsner and Voiculescu.

We consider twisted tensor products by the C^*-algebras $\mathcal{O}_n$. For $n \geqslant 2$, $\mathcal{O}_n$ is the C^*-algebra generated by n isometries $S_1, \ldots, S_n$ such that $\sum S_i S_i^* = 1$, cf. [3]. For $n = 1$ we define $\mathcal{O}_1$ as the C^*-algebra generated by a single unitary S_1 whose spectrum is the whole unit circle.

If $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$, $\mathscr{H}$ some Hilbert space, is a unital C^*-algebra (see 2.2 for the non-unital case) and $\mathscr{U} = (U_1, \ldots, U_n)$ is a family of pairwise commuting unitaries in $\mathscr{L}(\mathscr{H})$, implementing automorphisms $\alpha_i = \mathrm{Ad}\, U_i$ of $\mathscr{A}$, the twisted tensor product $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ is the C^*-subalgebra of $\mathscr{L}(\mathscr{H}) \otimes \mathcal{O}_n$ (note that $\mathcal{O}_n$ is nuclear) generated by $\mathscr{A} \otimes 1$ together with $U_1 \otimes S_1, \ldots, U_n \otimes S_n$. (Essentially repeating the proof given in [3] it can be shown that the algebra $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ does not depend on the choice of the unitaries U_i implementing the automorphisms α_i.) For the case $n = 1$ this reduces to the ordinary crossed product of $\mathscr{A}$ by the single automorphism α_1. While the crossed product is in some sense a universal C^*-algebra for which the automorphisms induced by α_1 on the K-groups becomes trivial, the algebra $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ with $U_1 = \ldots = U_n$ is the universal algebra for which n times these automorphisms become trivial.

We obtain an exact sequence for the K-groups of $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ that generalizes the exact sequence of [7]. The proof we give uses a homotopy argument from [4] to reduce the problem to a trivial exercise in homological algebra.

It should be pointed out that the additional generality gained in considering products by $\mathcal{O}_n$ instead of just ordinary crossed products is not very far reaching. In fact, as in [3] the C^*-algebra $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ can itself be represented as a crossed product of a certain C^*-algebra by a single endomorphism and, investing some more work, one can apply the exact sequence for crossed products to get, for $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$, essentially the same exact sequence that we obtain here. On the other hand, the proof we give here, is much more natural and gives some additional information that would not be obtained the other way. A point we want to make, is that the twisted tensor product by $\mathcal{O}_n$ is a natural generalization of the ordinary crossed product and that both cases can be treated in a parallel way.

Section 2 contains some remarks. In particular, we show in 2.5 that the result on the range of the trace on K_0 of the irrational rotation algebras, obtained in [6], is an easy consequence of the exact sequence for the K-groups of a crossed product.

1. THE MAIN RESULT

As in [4], [7] we will get the desired results applying the exact sequence of K-theory to a standard extension $\mathscr{E}$ of $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$. We consider in the following a fixed unital C^*-algebra $\mathscr{A}$ and a family $\mathscr{U} = (U_1, \ldots, U_n)$ as above and define $\mathscr{E}$ as the C^*-subalgebra of $\mathscr{L}(\mathscr{H}) \otimes \mathcal{O}_{n+1}$ generated by $\mathscr{A} \otimes 1$ together with $U_1 \otimes S_1, \ldots$ $\ldots, U_n \otimes S_n$. If $n = 1$, this is just the "Toeplitz extension" of [7].

Whenever $S'_1, \ldots, S'_n$ are isometries with pairwise orthogonal ranges such that $1 - \sum_{i=1}^{n} S'_i S'^*_i$ is a non-trivial projection, the map $aU_i \otimes S_i \mapsto aU_i \otimes S'_i$ $a \in \mathscr{A}$; $i = 1, \ldots, n$) extends to an isomorphism of $\mathscr{E}$ onto the C^*-algebra generated by $\mathscr{A} \otimes 1$ and $U_1 \otimes S'_1, \ldots, U_n \otimes S'_n$ in $\mathscr{L}(\mathscr{H}) \otimes C^*(S'_1, \ldots, S'_n)$. For $n = 1$ this follows from the well-known uniqueness of the C^*-algebra generated by a single non-unitary isometry [2], while for $n \geqslant 2$ this is a consequence of [3, 1.12], cf. also [4, 3.1]. On $\mathcal{O}_{n+1}$ one has the canonical endomorphism Φ_{n+1} defined by $\Phi_{n+1}(x) = \sum_{i=1}^{n+1} S_i x S_i^*$ $(x \in \mathcal{O}_{n+1})$. The map $aU_i \otimes S_i \mapsto aU_i \otimes \Phi_{n+1}(S_i)$ $(a \in \mathscr{A}; i = 1, \ldots, n)$ defines an isomorphism φ of $\mathscr{E}$ onto some C^*-subalgebra $\varphi(\mathscr{E})$ of $\mathscr{L}(\mathscr{H}) \otimes \mathcal{O}_{n+1}$. We denote by $\hat{\mathscr{E}}$ the C^*-algebra generated by $\mathscr{E}$ together with $\varphi(\mathscr{E})$. Equivalently, $\hat{\mathscr{E}}$ is the C^*-subalgebra of $\mathscr{L}(\mathscr{H}) \otimes \mathcal{O}_{n+1}$ generated by $\mathscr{E}$ together with $\beta(\mathscr{E})$, where the homomorphism $\beta : \mathscr{E} \to \hat{\mathscr{E}}$ is defined by $\beta(x) = (1 \otimes S_{n+1}) x (1 \otimes S_{n+1})^*$. Note that $\beta(\mathscr{A} \otimes 1) = \mathscr{A} \otimes S_{n+1} S^*_{n+1} \subset \mathscr{E}$ since $S_{n+1} S^*_{n+1} = 1 - \sum_{i=1}^{n} S_i S_i^*$.

We write j for the inclusion of $\mathscr{E}$ in $\hat{\mathscr{E}}$ and k for the inclusion of $\mathscr{A}$ as $\mathscr{A} \otimes 1$ in $\mathscr{E}$.

PROPOSITION 1.1. *The subalgebras* $\beta(\mathscr{A} \otimes 1), \beta(\mathscr{E})$ *generate closed ideals* $\mathscr{J}, \hat{\mathscr{J}}$ *in* $\mathscr{E}, \hat{\mathscr{E}}$ *isomorphic to* $\mathscr{K} \otimes \mathscr{A}, \mathscr{K} \otimes \mathscr{E}$, *respectively. The quotients* $\mathscr{E}/\mathscr{J}$ *and*

$\hat{\mathscr{E}}/\hat{\mathscr{J}}$ are both isomorphic to $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ and the diagram

$$\begin{array}{ccccc} \mathscr{K} \otimes \mathscr{A} = \mathscr{J} & \xrightarrow{i} & \mathscr{E} & \xrightarrow{q} & \mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n \\ \downarrow{\scriptstyle \mathrm{id} \otimes k} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathscr{K} \otimes \mathscr{E} = \hat{\mathscr{J}} & \xrightarrow{\hat{i}} & \hat{\mathscr{E}} & \xrightarrow{\hat{q}} & \mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n \end{array}$$

is commutative (as usual $\mathscr{K}$ denotes the compact operators and i, $\hat{i}$, q, $\hat{q}$ the inclusion and quotient maps). Moreover, the diagrams

$$\begin{array}{ccc} K_\#(\mathscr{J}) & \xrightarrow{i_*} & K_\#(\mathscr{E}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}} \\ K_\#(\mathscr{A}) & \xrightarrow{\beta_*} & K_\#(\mathscr{E}) \end{array} \qquad \begin{array}{ccc} K_\#(\hat{\mathscr{J}}) & \xrightarrow{\hat{i}_*} & K_\#(\hat{\mathscr{E}}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \mathrm{id}} \\ K_\#(\mathscr{E}) & \xrightarrow{\beta_*} & K_\#(\hat{\mathscr{E}}) \end{array}$$

are commutative.

Proof. We only give the proof for the case $n = 1$. The proof for the case $n \geqslant 2$ follows word by word that in [4, 3.2]. Let $(e_{ij})_{i,j \in \mathbf{N}}$ be a system of matrix units for $\mathscr{K}$. The closed ideal $\mathscr{J}\,(\hat{\mathscr{J}})$ generated by $\beta(\mathscr{A})\,(\beta(\mathscr{E}))$ in $\mathscr{E}\,(\hat{\mathscr{E}})$ obviously coincides with the closure of the set of all linear combinations of elements of the form $(U_1 \otimes S_1)^i \beta(x)(U_1 \otimes S_1)^{*j}$ with $x \in \mathscr{A}$ $(x \in \mathscr{E})$ and $i, j \in \mathbf{N}$. The map $(U_1 \otimes S_1)^i \beta(x)(U_1 \otimes S_1)^{*j} \mapsto e_{ij} \otimes \beta(x)$ extends to an isomorphism of $\mathscr{J}\,(\hat{\mathscr{J}})$ onto $\mathscr{K} \otimes \beta(\mathscr{A}) \cong \mathscr{K} \otimes \mathscr{A}$ $(\mathscr{K} \otimes \beta(\mathscr{E}) \cong \mathscr{K} \otimes \mathscr{E})$.

In particular (letting $\mathscr{A} = \mathbf{C}$), $C^*(S_1) \subset \mathcal{O}_2$ contains a closed ideal $\mathscr{J}' \cong \mathscr{K}$ and $C^*(S_1)/\mathscr{J}' = \mathcal{O}_1$. Since $\mathscr{L}(\mathscr{H}) \otimes \mathscr{J}' \cap \mathscr{E} = \mathscr{J}\,(\mathscr{L}(\mathscr{H}) \otimes \mathscr{J}' \cap \hat{\mathscr{E}} = \hat{\mathscr{J}})$, the quotient $\mathscr{E}/\mathscr{J}$ $(\hat{\mathscr{E}}/\hat{\mathscr{J}})$ is canonically isomorphic with the image of $\mathscr{E}$ in $\mathscr{L}(\mathscr{H}) \otimes \mathcal{O}_1$. This image is the C^*-subalgebra of $\mathscr{L}(\mathscr{H}) \otimes \mathcal{O}_1$ generated by $\mathscr{A} \otimes 1$ and $U_1 \otimes S$ where S is the unitary generating $\mathcal{O}_1$, thus $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_1$. The rest of the proposition is then clear. Q.E.D.

PROPOSITION 1.2. *The homomorphism $k_*: K_\#(\mathscr{A}) \to K_\#(\mathscr{E})$ induced by the inclusion $k: \mathscr{A} \to \mathscr{E}$ is injective.*

Proof. We follow here the proof in [7, 2.1 b)]. It suffices to prove the assertion for $k_*: K_1(\mathscr{A}) \to K_1(\mathscr{E})$. The corresponding result for K_0 follows upon replacing $\mathscr{A}$ by $C(\mathbf{T}) \otimes \mathscr{A}$ and $U_1, \ldots, U_n$ by $1 \otimes U_1, \ldots, 1 \otimes U_n$, using the fact that the injective homomorphism

$$(\mathrm{id} \otimes k)_*: K_1(C(\mathbf{T}) \otimes \mathscr{A}) = K_1(\mathscr{A}) \oplus K_0(\mathscr{A}) \to$$

$$\to K_1(C(\mathbf{T}) \otimes \mathscr{E}) = K_1(\mathscr{E}) \oplus K_0(\mathscr{E})$$

which then replaces k_* equals $k_* \oplus k_*$.

Similarly, replacing $\mathscr{A}$ by $M_k \otimes \mathscr{A}$ and $U_1, \ldots, U_n$ by $1 \otimes U_1, \ldots, 1 \otimes U_n$ we see that we need not consider matrix algebras over $\mathscr{A}$ and $\mathscr{E}$, but can work with $\mathscr{A}$ and $\mathscr{E}$ themselves. Thus we have to show that any two unitaries $v_0 \otimes 1$ and $v_1 \otimes 1$ in $\mathscr{A} \otimes 1$ that are homotopic in $\mathscr{E}$ correspond to the same class $[v_0] = [v_1]$ in $K_1(\mathscr{A})$. Let w_t ($t \in [0, 1]$) be a continuous path of unitaries in $\mathscr{E}$ such that $w_0 = v_0 \otimes 1$, $w_1 = v_1 \otimes 1$. Set $w_t' = w_t(\theta(w_t)^* + 1 \otimes S_{n+1}S_{n+1}^*)$ where θ is the endomorphism of $\mathscr{E}$ defined by $\theta(a \otimes 1) = a \otimes (1 - S_{n+1}S_{n+1}^*)$ ($a \in \mathscr{A}$) and $\theta(U_i \otimes S_i) = U_i \otimes S_i(1 - S_{n+1}S_{n+1}^*)$ ($i = 1, \ldots, n$). Since $q \circ \theta = q$, all the w_t' are in $\tilde{\mathscr{J}}$, the C^*-algebra generated by $\mathscr{J}$ together with the unit of $\mathscr{E}$. Moreover, w_t' is a continuous path of unitaries such that

$$w_0' = v_0 \otimes S_{n+1}S_{n+1}^* + 1 \otimes (1 - S_{n+1}S_{n+1}^*)$$

and

$$w_1' = v_1 \otimes S_{n+1}S_{n+1}^* + 1 \otimes (1 - S_{n+1}S_{n+1}^*).$$

Thus $[v_0] = [w_0'] = [w_1'] = [v_1]$ in $K_1(\mathscr{J}) = K_1(\mathscr{K} \otimes \mathscr{A}) = K_1(\mathscr{A})$. Q.E.D.

PROPOSITION 1.3. *The homomorphism $\varphi : \mathscr{E} \to \hat{\mathscr{E}}$ is homotopic to j in the topology of pointwise norm convergence in $\hat{\mathscr{E}}$.*

Proof. The restriction of φ to $\mathscr{A} \otimes 1 \subset \mathscr{E}$ coincides with j while $\varphi(U_i \otimes S_i) = (1 \otimes W)(U_i \otimes S_i)$ ($i = 1, \ldots, n$) where $1 \otimes W = 1 \otimes \left(\sum_{i,j=1}^{n+1} S_i S_j S_i^* S_j^* \right)$ is a selfadjoint unitary in $\mathscr{L}(\mathscr{H}) \otimes \mathscr{O}_{n+1}$. For $i, j \neq n+1$, the summand $1 \otimes S_i S_j S_i^* S_j^*$ is in $\hat{\mathscr{E}}$. Also

$$1 \otimes S_{n+1} S_j S_{n+1}^* S_j^* = \beta(U_j \otimes S_j)(U_j \otimes S_j)^*$$

and

$$1 \otimes S_i S_{n+1} S_i^* S_{n+1}^* = (U_i \otimes S_i)\beta(U_i \otimes S_i)^*$$

are in $\hat{\mathscr{E}}$. Since, finally, also

$$1 \otimes S_{n+1}^2 S_{n+1}^{*2} = \beta(1 \otimes S_{n+1}S_{n+1}^*) \in \hat{\mathscr{E}}$$

one sees that $1 \otimes W \in \hat{\mathscr{E}}$.

Let $W = E - F$ be the spectral decomposition of W and $W_t = E + e^{\pi i t}F$ ($t \in [0, 1]$). Then $1 \otimes W_t \in \hat{\mathscr{E}}$ and we can define homomorphisms $\varphi_t : \mathscr{E} \to \hat{\mathscr{E}}$ by $\varphi_t(a \otimes 1) = a \otimes 1$ ($a \in \mathscr{A}$) and

$$\varphi_t(U_i \otimes S_i) = (1 \otimes W_t)(U_i \otimes S_i) = U_i \otimes W_t S_i \quad (i = 1, \ldots, n).$$

Then φ_t is a continuous path of homomorphisms from $\mathscr{E}$ to $\hat{\mathscr{E}}$ connecting j to φ. Q.E.D.

PROPOSITION 1.4. *The homomorphism* $\beta_*:K_\#(\mathscr{E}) \to K_\#(\hat{\mathscr{E}})$ *induced by* $\beta:\mathscr{E} \to \hat{\mathscr{E}}$ *is equal to* $j_* - \sum_{i=1}^{n} \hat{\alpha}_{i*}^{-1} j_*$ *where* $\hat{\alpha}_i$ *is the automorphism* $\mathrm{Ad}(U_i \otimes 1)$ *of* $\hat{\mathscr{E}}$.

Proof. As in the proof of 1.2 we need not consider matrix algebras over $\mathscr{E}$, $\hat{\mathscr{E}}$.

For every projection p (every unitary u) in $\mathscr{E}$, $\varphi(p)\,(\varphi(u))$ is the orthogonal sum of the projections $\beta(p)$ and $(1 \otimes S_i)p(1 \otimes S_i)^*$ $(i = 1, \ldots, n)$ (the product of the unitaries $\beta(u) + (1 - \beta(1))$ and $(1 \otimes S_i)u(1 \otimes S_i)^* + (1 - 1 \otimes S_i S_i^*)$). Since $[(U_i \otimes S_i)p(U_i \otimes S_i)^*] = [p]$ in $K_0(\hat{\mathscr{E}})$ and $[(1 \otimes S_i)p(1 \otimes S_i^*)] = [\hat{\alpha}_i^{-1}((U_i \otimes S_i)p(U_i \otimes S_i)^*)] = \hat{\alpha}_{i*}^{-1}[p]$ (similarly for unitaries) it follows that $\varphi_* = \beta_* + \sum_{i=1}^{n} \hat{\alpha}_{i*}^{-1} j_*$. But $\varphi_* = j_*$ by 1.3. Q.E.D.

THEOREM 1.5. *The sequence*

$$\begin{array}{ccccc}
K_0(\mathscr{A}) & \xrightarrow{\mathrm{id}-\Sigma\alpha_{i*}^{-1}} & K_0(\mathscr{A}) & \xrightarrow{l_*} & K_0(\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n) \\
\uparrow & & & & \downarrow \\
K_1(\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n) & \xleftarrow{l_*} & K_1(\mathscr{A}) & \xleftarrow{\mathrm{id}-\Sigma\alpha_{i*}^{-1}} & K_1(\mathscr{A})
\end{array}$$

is exact. (Here l_* is the homomorphism induced by the natural inclusion of $\mathscr{A} \cong \mathscr{A} \otimes 1$ in $\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ and the vertical arrows will be made explicit in the proof.)

Proof. If we write down the exact sequences for the K-groups derived from the short exact sequences $\mathscr{J} \rightarrowtail \mathscr{E} \twoheadrightarrow \mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ and $\hat{\mathscr{J}} \rightarrowtail \hat{\mathscr{E}} \twoheadrightarrow \mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n$ and take into account Proposition 1.1 we get the following commutative diagram

$$\begin{array}{ccccccccccccc}
K_1(\mathscr{A}) & \xrightarrow{\beta_*} & K_1(\mathscr{E}) & \xrightarrow{q_*} & K_1(\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n) & \xrightarrow{\delta} & K_0(\mathscr{A}) & \xrightarrow{\beta_*} & K_0(\mathscr{E}) & \xrightarrow{q_*} & K_0(\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n) & \xrightarrow{\delta} & \\
\downarrow k_* & & \downarrow j_* & & \downarrow \mathrm{id} & & \downarrow k_* & & \downarrow j_* & & \downarrow \mathrm{id} & & \\
K_1(\mathscr{E}) & \xrightarrow{\beta_*} & K_1(\hat{\mathscr{E}}) & \xrightarrow{\hat{q}_*} & K_1(\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n) & \xrightarrow{\hat{\delta}} & K_0(\mathscr{E}) & \xrightarrow{\beta_*} & K_0(\hat{\mathscr{E}}) & \xrightarrow{\hat{q}_*} & K_0(\mathscr{A} \times_{\mathscr{U}} \mathcal{O}_n) & \xrightarrow{\hat{\delta}} &
\end{array}$$

where the rows are exact. Here $K_\#(\mathscr{J})$ and $K_\#(\hat{\mathscr{J}})$ are identified with $K_\#(\mathscr{A})$ and $K_\#(\mathscr{E})$, respectively. One has $\beta_*(K_\#(\mathscr{E})) \subset j_*(K_\#(\mathscr{E}))$ since $\beta_* = (\mathrm{id} - \sum \hat{\alpha}_{i*}^{-1})j_*$ by 1.4, and since $\hat{\alpha}_i^{-1}$ leaves $\mathscr{E} \subset \hat{\mathscr{E}}$ globally invariant.

Moreover, by 1.2, $k_*:K_s(\mathscr{A}) \to K_s(\mathscr{E})$ $(s = 0, 1)$ is injective. This implies that $\mathrm{Ker}\,\hat{\delta} = \hat{q}_*(K_t(\hat{\mathscr{E}})) \subset \mathrm{Ker}\,\delta = q_*(K_t(\mathscr{E}))$ $(t = 1, 0)$, whence $\hat{q}_*(K_t(\hat{\mathscr{E}})) = q_*(K_t(\mathscr{E}))$ since the converse inclusion is obvious. Since, as remarked above, $\beta_*(K_t(\mathscr{E})) \subset j_*(K_t(\mathscr{E}))$ this shows that $j_*:K_t(\mathscr{E}) \to K_t(\hat{\mathscr{E}})$ is surjective. It now follows from the fifth lemma (more precisely the fourth lemma) that the vertical arrows $k_*:K_\#(\mathscr{A}) \to K_\#(\mathscr{E})$ and $j_*:K_\#(\mathscr{E}) \to K_\#(\hat{\mathscr{E}})$ are isomorphisms.

If we identify $K_{\#}(\mathscr{E})$ and $K_{\#}(\hat{\mathscr{E}})$ with $K_{\#}(\mathscr{A})$ via the isomorphisms k_* and j_*k_*, the homomorphism β_* becomes $\mathrm{id} - \sum_{i=1}^{n} \alpha_{i*}^{-1}$ by 1.4. Q.E.D.

2. SOME REMARKS

2.1. Let $\mathscr{A}$ be as above and $U_1 = \ldots = U_n = 1$. Then $\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n$ is just the tensor product $\mathscr{A} \otimes \mathscr{O}_n$ and applying 1.5 one has the exact sequence

$$\begin{array}{ccccc} K_0(\mathscr{A}) & \xrightarrow{n-1} & K_0(\mathscr{A}) & \longrightarrow & K_0(\mathscr{A} \otimes \mathscr{O}_n) \\ \uparrow & & & & \downarrow \\ K_1(\mathscr{A} \otimes \mathscr{O}_n) & \longleftarrow & K_1(\mathscr{A}) & \xleftarrow{n-1} & K_1(\mathscr{A}) \end{array}$$

($n-1$ denotes multiplication by $n-1$). In particular, $K_s(\mathscr{A} \otimes \mathscr{O}_n) = K_s(\mathscr{A})/(n-1)K_s(\mathscr{A}) = K_s(\mathscr{A}) \otimes_{\mathbf{Z}} \mathbf{Z}_{n-1}$ for $s = 0, 1$ and $n \geqslant 2$ if $K_t(\mathscr{A})$ is torsion-free for $t = 1, 0$. For this cf. [1, 2.2], [5, 6.18].

2.2. Let $\mathscr{A}$ and $\mathscr{U} = (U_1, \ldots, U_n)$ in $\mathscr{L}(\mathscr{H})$ be as above but not unital. The product $\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n$ is then defined as the C^*-algebra generated by all linear combinations of elements of the form $(aU_i) \otimes S_i$ in $\mathscr{L}(\mathscr{H}) \otimes \mathscr{O}_n$ where $a \in \mathscr{A}$. Let $\tilde{\mathscr{A}}$ be the C^*-algebra generated by $\mathscr{A}$ together with the unit of $\mathscr{L}(\mathscr{H})$. Then $\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n$ is a closed ideal in $\tilde{\mathscr{A}} \times_{\mathscr{U}} \mathscr{O}_n$ and we have a split exact sequence

$$0 \to \mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n \to \tilde{\mathscr{A}} \times_{\mathscr{U}} \mathscr{O}_n \to \mathscr{O}_n \to 0.$$

Thus

$$K_0(\tilde{\mathscr{A}} \times_{\mathscr{U}} \mathscr{O}_n) = K_0(\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n) \oplus K_0(\mathscr{O}_n) = K_0(\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n) \oplus \mathbf{Z}_{n-1}$$

$$K_1(\tilde{\mathscr{A}} \times_{\mathscr{U}} \mathscr{O}_n) = K_1(\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n) \oplus K_1(\mathscr{O}_n) = K_1(\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n) \oplus 0, \ (n \geqslant 2).$$

We apply now the exact sequence 1.5 to $\tilde{\mathscr{A}} \times_{\mathscr{U}} \mathscr{O}_n$ where $\tilde{\alpha}_i = \mathrm{Ad}\, U_i$ are the natural extensions of α_i to automorphisms of $\tilde{\mathscr{A}}$ and get

$$\begin{array}{ccccc} K_0(\mathscr{A}) \oplus \mathbf{Z} & \xrightarrow{\mathrm{id} - \Sigma \tilde{\alpha}_{i*}^{-1}} & K_0(\mathscr{A}) \oplus \mathbf{Z} & \longrightarrow & K_0(\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n) \oplus \mathbf{Z}_{n-1} \\ \uparrow & & & & \downarrow \\ K_1(\mathscr{A} \times_{\mathscr{U}} \mathscr{O}_n) \oplus 0 & \longleftarrow & K_1(\mathscr{A}) \oplus 0 & \xleftarrow{\mathrm{id} - \Sigma \tilde{\alpha}_{i*}^{-1}} & K_1(\mathscr{A}) \oplus 0. \end{array}$$

Since

$$\mathrm{id} - \sum \tilde{\alpha}_{i*}^{-1} = (\mathrm{id} - \sum \alpha_{i*}^{-1}) \oplus (n-1)\, \mathrm{id},$$

it follows that the exact sequence of 1.5 generalizes without modification to the non-unital case (the same argument works for $n = 1$).

2.3. Consider again the endomorphism Φ_n of $\mathcal{O}_n$ defined by $\Phi_n(x) = \sum_{i=1}^{n} S_i x S_i^*$. The endomorphism id $\otimes \Phi_n$ of $\mathcal{L}(\mathcal{H}) \otimes \mathcal{O}_n$ leaves $\mathcal{A} \times_{\mathcal{U}} \mathcal{O}_n$ globally invariant thus defines an endomorphism ψ of $\mathcal{A} \otimes_{\mathcal{U}} \mathcal{O}_n$. Repeating the argument in 1.3, cf. also [4, 2.2], one sees that ψ is homotopic to the identity automorphism on $\mathcal{A} \times_{\mathcal{U}} \mathcal{O}_n$. On the other hand, $\psi_* = \sum \alpha_{i*}^{-1}$ on $K_\#(\mathcal{A} \times_{\mathcal{U}} \mathcal{O}_n)$ where α_i are the automorphisms $\mathrm{Ad}U_i \otimes 1$ of $\mathcal{A} \times_{\mathcal{U}} \mathcal{O}_n$, cf. 1.4. Thus $\sum \alpha_{i*}^{-1} = \mathrm{id}$ on $K_\#(\mathcal{A} \times_{\mathcal{U}} \mathcal{O}_n)$.

2.4. Let $S_1, S_2, \ldots$ be a sequence of isometries with pairwise orthogonal ranges and $\mathcal{O}_\infty$ the C^*-algebra generated by these isometries, cf. [3]. Then $\mathcal{O}_\infty$ is the inductive limit of the subalgebras $\mathcal{E}_n$ generated by the isometries $S_1, \ldots, S_n$. If $\mathcal{A}$ is any C^*-algebra, $\mathcal{A} \otimes \mathcal{E}_n$ is the standard extension of $\mathcal{A} \otimes \mathcal{O}_n$ and by the proof of 1.5 the embedding k_n of $\mathcal{A}$ in $\mathcal{A} \otimes \mathcal{E}_n$ induces an isomorphism $k_{n*}: K_\#(\mathcal{A}) \to K_\#(\mathcal{A} \otimes \mathcal{E}_n)$. The diagram

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{E}_n & \hookrightarrow & \mathcal{A} \otimes \mathcal{E}_{n+1} \\ \uparrow k_n & & \uparrow k_{n+1} \\ \mathcal{A} & \xrightarrow{\mathrm{id}} & \mathcal{A} \end{array}$$

is commutative. Since $K_\#(\mathcal{A} \otimes \mathcal{O}_\infty)$ is the inductive limit of the groups $K_\#(\mathcal{A} \otimes \mathcal{E}_n)$ with the homomorphisms induced by the inclusions of $\mathcal{A} \otimes \mathcal{E}_n$ in $\mathcal{A} \otimes \mathcal{E}_{n+1}$, we see that $K_\#(\mathcal{A} \otimes \mathcal{O}_\infty) = K_\#(\mathcal{A})$ for every $\mathcal{A}$. The same remark holds for twisted tensor products by $\mathcal{O}_\infty$. If $\mathcal{A}$ is any simple C^*-algebra, $\mathcal{A} \otimes \mathcal{O}_\infty$ is a simple C^*-algebra, which is infinite in the sense of [3] and has the same K-theory as $\mathcal{A}$.

2.5. For every irrational $\theta \in (0,1)$ let A_θ be the unique C^*-algebra generated by two unitaries u and v such that $uv = e^{2\pi i\theta} vu$. Using the exact sequence in 1.5 one can give a simple proof for the result of Pimsner and Voiculescu in [6] that the range of the homomorphism $\tau_*: K_0(A_\theta) \to \mathbf{R}$ induced by the unique trace τ on A_θ is contained in $\mathbf{Z} + \theta\mathbf{Z}$. Let f be the characteristic function of the interval $\exp(i[0, 2\pi\theta])$ in the unit circle $\mathbf{T}$ and $\mathcal{D}$ the smallest C^*-algebra of functions on $\mathbf{T}$ (with the supremum norm) containing f and invariant under the rotation by the angle $2\pi\theta$. Then $\mathcal{C}(\mathbf{T}) \subset \mathcal{D}$ and we have a commutative diagram

$$\begin{array}{ccc} A_\theta = \mathcal{C}(\mathbf{T}) \times_\alpha \mathbf{Z} & \xrightarrow{\tau} & \mathbf{R} \\ \cap & & \downarrow \mathrm{id} \\ \mathcal{D} \times_\alpha \mathbf{Z} & \xrightarrow{\tau'} & \mathbf{R} \end{array}$$

where $\mathcal{C}(\mathbf{T}) \times_\alpha \mathbf{Z}$ and $\mathcal{D} \times_\alpha \mathbf{Z}$ are the crossed products of $\mathcal{C}(\mathbf{T})$ and $\mathcal{D}$ by the automorphism α induced by the rotation by $2\pi\theta$, respectively, and τ, τ' are the trace states on the crossed products that extend the Lebesgue measure on $\mathcal{C}(\mathbf{T})$ and $\mathcal{D}$. Since $K_1(\mathcal{D}) = 0$ it follows from 1.5 that $K_0(\mathcal{D} \times_\alpha \mathbf{Z}) = l_*(K_0(\mathcal{D}))$ where $l: \mathcal{D} \to \mathcal{D} \times_\alpha \mathbf{Z}$ is the natural inclusion, and it is immediate that $\tau'_*(l_*(K_0(\mathcal{D})) = \mathbf{Z} + \theta\mathbf{Z}$.

2.6. Let $\mathscr{A}$ be a C^*-algebra such that $K_\#(\mathscr{A}) = K_0(\mathscr{A}) \oplus K_1(\mathscr{A})$ is torsion-free. Let N be the class of C^*-algebras $\mathscr{B}$ for which the Künneth formula

$$K_\#(\mathscr{A} \otimes \mathscr{B}) = K_\#(\mathscr{A}) \otimes K_\#(\mathscr{B})$$

holds. Using standard homological algebra one sees that N is closed under taking inductive limits, extensions (if the ideal and the quotient are in N), quotients (if the algebra and its ideal are in N), and ideals (if the algebra and the quotient are in N). The exact sequence 1.5 shows that N is also closed under taking crossed products by a single automorphism. Thus N presumably contains all nuclear C^*-algebras. (Note that N clearly contains all finite-dimensional C^*-algebras.)

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