J. OPERATOR THEORY
5(1981), 119–126

THE APPROXIMATE POINT SPECTRUM OF A SUBNORMAL OPERATOR

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Let S be a pure subnormal operator on a separable Hilbert space $\mathscr{H}$. By "pure" we mean that no subspace of $\mathscr{H}$ reduces S to a normal operator. Let $K_0 = \sigma(S)$ and let $K_1 = \sigma_a(S)$, the spectrum and approximate point spectrum, respectively, of S. Then we must have:

(1) $\partial K_0 \subset K_1 \subset K_0$.

(2) If Δ is an open disk for which $\Delta \cap K_0 \neq \emptyset$, then

$$C(\Delta^- \cap K_0) \neq R(\Delta^- \cap K_0).$$

(For a compact set K in the plane, $C(K)$ is the space of continuous, complex-valued functions on K with the sup-norm, and $R(K)$ is the closure in $C(K)$ of the set of rational functions with poles off K.)

Condition (1) is a well-known result of Halmos [5] and Bram [2], and (2) is the necessary and sufficient condition found by Clancey and Putnam [3] in order that the compact set K_0 be the spectrum of a pure subnormal operator.

In this paper we investigate which pairs (K_0, K_1) of compact sets arise in this way as the spectrum and approximate point spectrum of a pure subnormal operator. In general the necessary conditions (1) and (2) are sufficient. We will require, however, that the subnormal operator S have additional properties. This yields results with implications, for example, to the study of L^2-approximation by rational functions.

An operator $A:\mathscr{H} \to \mathscr{H}$ is *effectually rationally cyclic* (cf. [1]) if there exists a vector $x_0 \in \mathscr{H}$ such that the set

$$\{y \in \mathscr{H} : p(A)y = q(A)x_0 \quad \text{for some polynomials } p \text{ and } q \text{ with } p(A) \neq 0\}$$

is dense in $\mathscr{H}$. Berger and Shaw [1] have shown that an effectually rationally cyclic hyponormal operator S is essentially normal ($S^*S - SS^*$ is compact). If K is a compact subset of the plane, then a hole of K is a bounded component of the complement. The polynomial convex hull of K, denoted $K^\wedge$, is the union of K and all of its holes.

The following results are proved in this paper.

THEOREM 1. *Let K_0 and K_1 be compact subsets of the plane which satisfy* (1) *and* (2). *Then there exists a pure subnormal operator S such that $\sigma(S) = K_0$ and $\sigma_a(S) = K_1$. If K_1 has no interior, then we may require that S be effectually rationally cyclic.*

COROLLARY. *A compact subset K_1 of the plane is the approximate point spectrum of a pure subnormal operator if and only if $K_1^\wedge$ is the closure of its interior.*

THEOREM 2. (i) *Suppose S is a pure, cyclic subnormal operator. Then the components of $\sigma(S)\setminus\sigma_a(S)$ are simply connected.*

(ii) *Suppose S is a pure, rationally cyclic subnormal operator. Then each component of $\sigma_a(S)$ intersects the boundary of $\sigma(S)$.*

THEOREM 3. *Let K_0 and K_1 be compact subsets of the plane satisfying* (1). *In addition suppose that $(K_0\setminus K_1)^- = K_0$.*

(i) *If the components of $K_0\setminus K_1$ are simply connected, then there is a pure, cyclic subnormal operator S such that $\sigma(S) = K_0$ and $\sigma_a(S) = K_1$.*

(ii) *If each component of K_1 intersects the boundary of K_0, then there is a pure, rationally cyclic subnormal operator S such that $\sigma(S) = K_0$ and $\sigma_a(S) = K_1$.*

Two remarks: First, in part (i) of Theorem 2, note that the components of $\sigma(S)\setminus\sigma_a(S)$ are simply connected if and only if the intersection of each component of $\sigma_a(S)$ with $\sigma(S)$ is connected. A similar remark applies to Theorem 3(i). Second, the following lemma shows that the additional condition of Theorem 3 is not unduly restrictive.

LEMMA 1. *Let K_0 and K_1 be compact subsets of the plane satisfying* (1) *and* (2). *Suppose K_1 has no interior, and suppose the diameters of the holes of K_0 are bounded away from zero. Then $(K_0\setminus K_1)^- = K_0$.*

Proof. Let $U = K_0\setminus K_1$ and fix $z \in K_1$. Let Δ be an open disk containing z. If $\Delta \cap K_0$ has interior, then $\Delta \cap U \neq \emptyset$, because K_1 has no interior. If $\Delta \cap K_0$ has interior for every open disk Δ containing z, then $z \in U^-$.

Suppose, on the other hand, that there is an open disk Δ containing z for which $\Delta \cap K_0$ has no interior. Since the diameters of the holes of K_0 are bounded away from zero, we may assume that Δ is so small that Δ does not contain any hole of K_0. It follows that $\Delta^- \cap K_0$ has connected complement. But then a theorem of Lavrentiev (cf. [4, II.8.7]) shows that $C(\Delta^- \cap K_0) = R(\Delta^- \cap K_0)$, a contradiction.

LEMMA 2. (i) *If K is the closure of its interior, then K satisfies* (2).

(ii) *Suppose $K=K^\wedge$. Then K satisfies* (2) *if and only if K is the closure of its interior.*

Proof. (i) Suppose K is the closure of its interior. Then any open disc intersecting K must intersect the interior of K. It follows immediately that K satisfies (2).

(ii) Suppose $K = K^\wedge$. Let U be the interior of K and suppose there exists $z \in K \setminus U^-$. Let Δ be an open disc centered at z for which $\Delta^- \cap U^- = \emptyset$. Then $\Delta^- \cap K$ has no interior. Consequently $\Delta^- \cap K$ has connected complement because any hole of $\Delta^- \cap K$ would be a hole of $K = K^\wedge$. Therefore by the theorem of Lavrentiev mentioned above, $R(\Delta^- \cap K) = C(\Delta^- \cap K)$. Consequently, K does not satisfy (2). The other direction follows directly from (i).

Before proceeding, we establish some additional notation. Let μ be a measure. By "measure" is meant a finite, positive Borel measure on $\mathbf{C}$ with compact support, denoted sptμ. Let K be a compact set containing sptμ, and denote by $R^2(K, \mu)$ the closure in $L^2(\mu)$ of $R(K)$. The norm of an element f of $L^2(\mu)$ is denoted by $\|f\|_\mu$ and the inner product of f and g by $(f, g)_\mu$. (The subscript μ will be dropped when there is no danger of confusion.)

A point $\omega \in \mathbf{C}$ is a bounded point evaluation (b.p.e.) for $R^2(K, \mu)$ if there exists a constant $c > 0$ such that $|f(\omega)| \leqslant c\|f\|$ for every $f \in R(K)$. Let $\Omega = \Omega(K, \mu)$ be the set of all b.p.e.'s for $R^2(K, \mu)$. If $\omega \in \Omega$, then there exists $k_\omega \in R^2(K, \mu)$ such that $f(\omega) = (f, k_\omega)$ for every $f \in R(K)$. Thus we may assume that $f(\omega)$ is well-defined for every $f \in R^2(K, \mu)$ by assuming $f(\omega) = (f, k_\omega)$. If $\omega \in \operatorname{int}\Omega$ (the interior of Ω), then ω is an analytic b.p.e. (a.b.p.e.) if every $f \in R^2(K, \mu)$ is analytic at ω. Let $\Omega_a = \Omega_a(K, \mu)$ be the set of all a.b.p.e.'s for $R^2(K, \mu)$. We will show below that Ω_a is open.

If S is "multiplication by z" on $R^2(K, \mu)$, then S is, up to unitary equivalence, the most general rationally cyclic subnormal operator. Furthermore, $\sigma_a(S) \subset \text{spt}\mu \subset \sigma(S) \subset K$. Also, $\omega \in \Omega$ if and only if $\bar{\omega}$ is an eigenvalue of S^* with eigenvector k_ω. Suppose $K = (\text{spt}\mu)^\wedge$. In this case S is cyclic and is, up to unitary equivalence, the most general cyclic subnormal operator.

The following proposition is a slightly different version of a result of Trent [9, Theorem 1.1].

PROPOSITION. *If S is "multiplication by z" on $R^2(K, \mu)$, then $\Omega_a = \sigma(S) \setminus \sigma_a(S)$.*

Proof. We first show that Ω_a is open. To this end we will show that if $\omega_0 \in \Omega_a$, then $\|k_\omega\|$ is bounded in a neighborhood of ω_0. If not, there exist $\omega_n \in \Omega$ with $\omega_n \to \omega_0 \in \Omega_a$, while $\|k_{\omega_n}\| \to \infty$. Let $n_1 = 1$ and let $\lambda_1 = \omega_1$. Choose $f_1 \in R(K)$ such that $\|f_1\| = 9^{-1}$ and $f_1(\lambda_1) \geqslant 2^{-1} \cdot 9^{-1} \|k_{\lambda_1}\|$. Suppose $n_1, \ldots, n_{j-1}$ and $f_1, \ldots, f_{j-1}$ have been chosen. Choose n_j such that if $\lambda_j = \omega_{n_j}$, then $\|k_{\lambda_j}\| \geqslant 8 \cdot 9^j \cdot \max\{\|f_1 + \ldots + f_{j-1}\|_\infty, j\}$. (The infinity norm is the sup-norm on $C(K)$.) Choose f_j such that $\|f_j\| = 9^{-j}$ and $f_j(\lambda_j) \geqslant 2^{-1} \cdot 9^{-j} \|k_{\lambda_j}\|$. Let $f = \sum f_j$. One may verify that $|f(\lambda_j)| \geqslant 2j$, and hence f is not analytic at ω_0, a contradiction.

Therefore, $\|k_\omega\|$ is bounded in an open neighborhood U of ω_0. Now, if $g_n \in R(K)$ and $g_n \to g$ in $R^2(K, \mu)$, then $g_n \to g$ uniformly in U, and hence g is analytic in U. Thus $U \subset \Omega_a$. Furthermore, it follows that there is a (possibly smaller) open neighborhood V of ω_0 such that $\int_V \|k_\omega\|^2 \,\mathrm{d}\mu(\omega) = d < 1$. This is immediate if $\mu\{\omega_0\} = 0$. If $\mu\{\omega_0\} > 0$, we must show that $\|k_{\omega_0}\|^2\mu\{\omega_0\} < 1$. Let h be the characteristic function of $\{\omega_0\}$. Then k_{ω_0} is the projection into $R^2(K, \mu)$ of $h' = (\mu\{\omega_0\})^{-1}h$. Since k_{ω_0} is analytic at ω_0, we have $k_{\omega_0} \neq h'$. Therefore,

$$\|k_{\omega_0}\|^2 \ \mu\{\omega_0\} < \|h'\|^2 \ \mu\{\omega_0\} = 1.$$

Consequently,

$$\int_{\mathbf{C}\setminus V} |f|^2 \,\mathrm{d}\mu \geqslant (1-d)\|f\|^2 \qquad \text{for } f \in R^2(K, \mu),$$

and hence $\omega_0 \notin \sigma_a(S)$. Since k_{ω_0} is in the kernel of $(S-\omega_0)^*$, we have $\omega_0 \in \sigma(S)$.

Conversely, suppose $\sigma(S)\setminus\sigma_a(S)$ is nonempty. If $\omega \in \sigma(S)\setminus\sigma_a(S)$, then $\ker(S-\omega)^*$ has dimension one. (Trivially, $\ker(S-\omega)^*$ has dimension at most one.) Therefore, for $\omega_0 \in \sigma(S)\setminus\sigma_a(S)$, there is an open neighborhood U of ω_0 and a conjugate analytic mapping of U into $R^2(K, \mu)$ which maps ω into a nonzero vector f_ω in $\ker(S-\omega)^*$ (cf. [8, Theorem 1]). Define $\varphi(\omega)$ by $\varphi(\omega)(f_\omega, 1) = 1$. Then $\varphi(\omega)$ is conjugate analytic and $k_\omega = \varphi(\omega) f_\omega$. Therefore, k_ω is also conjugate analytic, and the proof is complete.

The following examples illustrate the constructions used in the proof of Theorem 1.

EXAMPLE A. Suppose K_0 is the closure of its interior. Let μ be the restriction of Lebesgue area measure to K_0. If S is "multiplication by z" on $R^2(K_0, \mu)$, then $\sigma(S) = K_0$ and $\sigma_a(S) = \partial K_0$.

EXAMPLE B. Let $\{r_n\}$ be a countable dense subset of the interval $(0, 1)$. Let $D_n = \{|z| \leqslant r_n\}$ and let μ_n be Lebesgue linear measure on ∂D_n. If S_n is "multiplication by z" on $R^2(D_n, \mu_n)$ and if $S = \sum \oplus S_n$, then $\sigma(S) = \sigma_a(S) = \{|z| \leqslant 1\}$.

EXAMPLE C. Define μ on the unit disk by $\mathrm{d}\mu(re^{i\theta}) = e^{-1/r}\,\mathrm{d}r\mathrm{d}\theta$. Let $\mathscr{H}$ be the span in $L^2(\mu)$ of $\{z^n : n = 0, \pm 1, \pm 2, \ldots\}$. If S is "multiplication by z" on $\mathscr{H}$, then $\sigma(S) = \{|z| \leqslant 1\}$. Since S has dense range, $0 \in \sigma_a(S)$. On the other hand if U is an open subset of the unit disk not containing zero, then the elements of $\mathscr{H}$ are analytic on U. Therefore $\sigma_a(S) = \partial\sigma(S) \cup \{0\}$.

Proof of Theorem 1. Let $U = K_0\setminus K_1$ and let $E_1 = K_1 \cap \operatorname{int}\{U^-\}$. Note first of all that

$$K_1 = \partial K_0 \cup E_1 \cup (\operatorname{int} K_1)^-.$$

Indeed, the union on the right hand side is clearly a subset of K_1. Suppose $z \in K_1$ but $z \notin \partial K_0 \cup (\mathrm{int}K_1)^-$. Then there is an open disc Δ centered at z such that $\Delta \subset \mathrm{int}K_0$ and $\Delta \cap \mathrm{int}K_1 = \emptyset$. Let $w \in \Delta$. Since $w \in \mathrm{int}K_0$ and $w \notin \mathrm{int}K_1$, there exists a sequence of points $w_n \in K_0$ such that $w_n \notin K_1$ and $w_n \to w$. In other words, $w \in U^-$. Thus $\Delta \subset U^-$ and, consequently, $z \in E_1$.

Let $E_2 = K_0 \setminus U^-$ and let $\{z_n\}$ be a countable, dense subset of E_2. (If E_2 is empty, we may skip this step.) For each n and each $k \geq 1$ let D_{nk} be a closed disc centered at z_n of radius at most $1/k$ which does not intersect U^-. Let ν_{nk} be a measure of total variation 1 carried by $D_{nk} \cap K_0$ such that $\int f \mathrm{d}\nu_{nk} = 0$ for every $f \in R(D_{nk} \cap K_0)$. (We are here using ideas from the proof in [3].) Let τ be the restriction of Lebesgue area measure to U and define h on U by

$$h(z) = \exp\{-1/\mathrm{dist}(z, E_1)\}.$$

(If E_1 is empty, let $h = 1$.) Define the measure μ on K_0 by

$$\mathrm{d}\mu(z) = h(z)\mathrm{d}\tau(z) + \sum_n \sum_k 2^{-n-k}\mathrm{d}|\nu_{nk}|(z).$$

Let $\mathscr{H}_0 = \{r: r \text{ is a rational function with no poles on } K_0 \setminus E_1\}$. By our construction $\mathscr{H}_0 \subset L^2(\mu)$. Let $\mathscr{H}$ be the closure in $L^2(\mu)$ of $\mathscr{H}_0$ and let S be "multiplication by z" on $\mathscr{H}$. Note that S is effectually rationally cyclic.

We first show that S is pure. Suppose E is a measurable set such that $\mathscr{H}$ contains all elements of $L^2(\mu)$ which vanish almost everywhere off E. Clearly $\tau(E) = 0$ since all elements of $\mathscr{H}$ are analytic in U. Let f be an element of $L^2(\mu)$ which is carried by E. Then there exist $r_j \in \mathscr{H}_0$ such that $r_j \to f$ in $L^2(\mu)$. But then $0 = \int r_j \mathrm{d}\nu_{nk} \to \int f \mathrm{d}\nu_{nk}$. It follows that $|\nu_{nk}|(E) = 0$ and, consequently, that $\mu(E) = 0$. Therefore S is pure.

By the construction it is clear that $\mathrm{spt}\mu = K_0$. Therefore, $\sigma(S) = K_0$. Also $U \cap \sigma_a(S) = \emptyset$. Let $z \in E_1$. Since the range of $S - z$ includes $\mathscr{H}_0$ and since $z \in \sigma(S)$, we must have $z \in \sigma_a(S)$. Thus $\partial K_0 \cup E_1 \subset \sigma_a(S) \subset K_1$. As noted above $K_1 = \partial K_0 \cup E_1 \cup (\mathrm{int}K_1)^-$. If $\mathrm{int}K_1 = \emptyset$, we are finished.

Suppose $\mathrm{int}K_1 \neq \emptyset$. Write $\mathrm{int}K_1 = \bigcup_n \Delta_n$, where Δ_n is an open disc. As in Example B, there is a pure subnormal operator S_n such that $\sigma(S_n) = \sigma_a(S_n) = \Delta_n^-$. The operator $S \oplus \sum \oplus S_n$ satisfies the requirements of the theorem.

Proof of Corollary. Suppose K_1 is the approximate point spectrum of a pure subnormal operator with spectrum K_0. Then K_0 satisfies (2), and therefore so does $K_1^\wedge = K_0^\wedge$. By Lemma 2(ii), $K_1^\wedge$ is the closure of its interior. For the converse let $K_0 = K_1^\wedge$ and apply Lemma 2(i) and Theorem 1.

Proof of Theorem 2. (i) There exists a measure μ such that if $K = (\mathrm{spt}\mu)^\wedge$, then S is unitarily equivalent to "multiplication by z" on $R^2(K, \mu)$. Since $\|k_\omega\|$ is locally bounded on $\sigma(S)\setminus\sigma_a(S)$, we have $\|k_\omega\|$ bounded on compact subsets of $\sigma(S)\setminus\sigma_a(S)$. Let Γ be a simple closed Jordan curve in $\sigma(S)\setminus\sigma_a(S)$, and let O be the interior of Γ. Then $\|k_\omega\|$ is bounded on Γ. By the maximum modulus theorem, for $z \in O$ and $f \in R(K)$, we have

$$|f(z)| \leqslant \max\{|f(\omega)| : \omega \in \Gamma\} \leqslant \|f\| \max\{\|k_\omega\| : \omega \in \Gamma\}.$$

Therefore, arguing as in the proof of the Proposition, $O \subset \Omega_a = \sigma(S)\setminus\sigma_a(S)$. Hence the components of $\sigma(S)\setminus\sigma_a(S)$ are simply connected.

(ii) Suppose E is a component of $\sigma_a(S)$ which does not intersect $\partial\sigma(S)$. Then there exists a Jordan curve Γ in $\sigma(S)\setminus\sigma_a(S)$ with interior O such that $E \subset O \subset \sigma(S)$. But the argument above shows that $O \subset \sigma(S)\setminus\sigma_a(S)$, a contradiction.

Proof of Theorem 3. Since $K_0\setminus K_1 \subset \mathrm{int} K_0$ and $(K_0\setminus K_1)^- = K_0$, we have $K_0 = (\mathrm{int} K_0)^-$. By Lemma 2(i), K_0 satisfies (2).

(i) Suppose that the components of $K_0\setminus K_1$ are simply connected. Let $K = K_0^\wedge$, and let V_i, $i \in I$, be the holes of K_0. Choose $z_i \in V_i$ and an open disc Δ_i containing z_i whose closure is contained in V_i. Let Δ_i' be another open disk whose closure is contained in V_i and is disjoint from Δ_i^-. Let ν be Lebesgue area measure restricted to $U = (K_0\setminus K_1) \cup (\bigcup_{i \in I} \Delta_i')$. Let f be an analytic function in $\mathbf{C}\setminus\{z_i : i \in I\}^-$ which is bounded off $\bigcup_{i \in I} \Delta_i$ and has a simple pole at each z_i. Then f is analytic and bounded on U. By the hypotheses on K_0 and K_1, the set $\mathbf{C}\setminus U$ is connected. Hence we may use the construction in [6] to conclude that there is a measure τ, mutually absolutely continuous with respect to ν, for which $f \in R^2(K, \tau)$ and $\Omega_a(K, \tau) \supset U$. Let τ' be the restriction of τ to $K_0\setminus K_1$. Since on compact subsets of $K_0\setminus K_1$ the measure τ' is greater than a constant multiple of area measure, $\Omega_a(K, \tau') \supset K_0\setminus K_1$. Suppose $\Omega_a(K, \tau') \cap \cap V_i \neq \emptyset$. From condition (1) and the Proposition it follows that $V_i \subset \Omega_a(K, \tau')$. Choose $f_n \in R(K)$ such that $f_n \to f$ in $R^2(K, \tau)$. Then, in particular, $f_n \to f$ pointwise on Δ_i'. But as an element of $R^2(K, \tau')$, the function f must be analytic in V_i, which is impossible. Therefore $\Omega_a(K, \tau') \subset K_0$. (The reason for using area measure in Δ_i' is this: Suppose $g_n \in R(K)$ and $g_n \to f$ in $R^2(K, \tau')$. Then the hypotheses do not preclude, for example, the possibility that g_n converges pointwise to zero in V_i.)

Let h be an analytic function in U which cannot be continued analytically across any point of K_1. Again applying the results in [6], we may conclude that there is a measure μ, mutually absolutely continuous with respect to τ', such that $h \in R^2(K,\mu)$ and

$$K_0\setminus K_1 \subset \Omega_a(K, \mu) \subset \Omega_a(K, \tau').$$

Let S be "multiplication by z" on $R^2(K, \mu)$. Then $\sigma_a(S) \subset K_0 = \mathrm{spt}\mu \subset \sigma(S)$. Combining the above results with the Proposition, we have

$$K_0\setminus K_1 \subset \sigma(S)\setminus\sigma_a(S) = \Omega_a(K, \mu) \subset K_0.$$

Therefore $\sigma(S) = K_0$ and $\sigma_a(S) \subset K_1$. Since $h \in R^2(K, \mu)$, no point of K_1 can be an a.b.p.e.. Hence $\sigma_a(S) = K_1$. Finally, since $\mu(\sigma_a(S)) = 0$, we may conclude that S is pure. Indeed, if S is not pure, then $S = S_0 \oplus S_1$, where S_0 is normal. Now S_0 is (unitarily equivalent to) "multiplication by z" on $L^2(\mu|_E)$, where E is a Borel subset of $\sigma(S_0) = \sigma_a(S_0) \subset K_1$. In particular, $\mu(E) > 0$, a contradiction.

(ii) In the above paragraph let τ' be Lebesgue area measure restricted to $K_0 \setminus K_1$ and let $K = K_0$. We would like to follow the above argument, but the result used from [6] requires that the complement of $K_0 \setminus K_1$ have a finite number of components. In the proof of the Proposition in [6], this condition is used to show that the set E_n (defined in [6]) has at most a finite number of holes. But the definition of E_n and the requirement that each component of U' intersects K' guarantee that each hole of E_n contains an open disc of radius $1/n$. Therefore, the finiteness condition is superfluous, and (ii) is proved by the above argument.

We conclude with an observation and a question. For the subnormal operators considered in this paper, we have a nearly complete description of the possibilities for the various parts of the spectrum. In particular, suppose S is a pure, cyclic subnormal operator for which $(\sigma(S) \setminus \sigma_a(S))^- = \sigma(S)$. We have found the possibilities for $\sigma(S)$ and $\sigma_a(S)$. Since the point spectrum of a pure subnormal operator is always empty, the only part remaining is $\sigma_c(S) = \{z : S - z$ does not have dense range$\}$. As noted above, $\sigma_c(S) = \Omega(\sigma(S), \mu)$, where S is unitarily equivalent to "multiplication by z" on $R^2(\sigma(S), \mu)$. In light of the Proposition, we must find the possibilities for $\Omega \setminus \Omega_a = \Omega \cap \sigma_a(S)$. To the author's knowledge, no examples are known (in the cyclic case) for which $\Omega_a \neq \Omega$. On the other hand, an example of Olin [7] shows that in the rationally cyclic case one can have $\Omega \cap \partial\sigma(S) \neq \emptyset$. In general, if S is pure, about all one can say is that any peak point of $R(\sigma(S))$ cannot be a b.p.e.. This suggests the following problem: Suppose S is a pure, cyclic subnormal operator for which $\sigma(S)$ is the closed unit disc. Can $\sigma_a(S)$ and $\sigma_c(S)$ have a nonempty intersection?

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Received December 14, 1979; revised February 26 and April 29, 1980.