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QUASISIMILARITY AND HYPONORMAL OPERATORS

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S. Clary proved in [8] that quasisimilar hyponormal operators have equal spectra and he asked whether quasisimilar hyponormal operators have equal essential spectra. The present author studied this question in [26] and proved in [27] that quasisimilar quasinormal operators have equal essential spectra. The purpose of this note is to study further the above mentioned question of Clary and to present some other results that relate to quasisimilarity and hyponormal operators.

If $\mathscr{H}$ is a Hilbert space, let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. (In this note, we shall use the term *operator* to mean an element of $\mathscr{L}(\mathscr{H})$ for some complex Hilbert space $\mathscr{H}$.) If T is an operator and $TT^* \leqslant T^*T$, then T is said to be *hyponormal*. If $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces and $X:\mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear transformation having trivial kernel and dense range, then X is called a *quasiaffinity*. If $T_1 \in \mathscr{L}(\mathscr{H}_1)$ and $T_2 \in \mathscr{L}(\mathscr{H}_2)$ and there exist quasiaffinities $X:\mathscr{H}_1 \to \mathscr{H}_2$ and $Y:\mathscr{H}_2 \to \mathscr{H}_1$ satisfying $XT_1 = T_2X$ and $T_1Y = YT_2$, then T_1 and T_2 are said to be *quasisimilar*. If T is an operator, let $\sigma(T)$ denote the spectrum of T, $\mathscr{K}(T)$ the kernel of T, and $\mathscr{R}(T)$ the range of T. If $T \in \mathscr{L}(\mathscr{H})$ and $\mathscr{H}$ is an infinite dimensional Hilbert space, let $\sigma_e(T)$ denote the essential spectrum of T, i.e., the spectrum of $\widetilde{T}$, where $T \to \widetilde{T}$ is the natural quotient map of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}$. ($\mathscr{C}$ denotes the norm-closed ideal of all compact operators in $\mathscr{L}(\mathscr{H})$.)

As stated earlier, our primary interest is in the class of hyponormal operators. However, many of the results in this note are valid for the more general class of dominant operators. (Recall that an operator T is said to be dominant if $\mathscr{R}(T-\lambda) \subseteq \mathscr{R}((T-\lambda)^*)$ for each λ in $\sigma(T)$. The study of dominant operators was introduced by Stampfli and Wadhwa in [25]. It follows easily from Theorem 1 of [11] that every hyponormal operator is dominant.) Some of the properties of hyponormal operators extend to the class of dominant operators. For example, it is easy to verify that a dominant operator on a finite dimensional Hilbert space is normal. On the other hand there exist dominant operators which behave quite differently from hyponormal operators. For instance, it is known that a compact hyponormal operator is normal [1], [3]. It is also known that a quasinilpotent hyponormal operator is necessarily zero. However, there exist nonnormal compact quasinilpotent dominant operators [25]. It also appears that the question of whether quasisimilar dominant operators have equal spectra is open. Thus we will concentrate here on the class of hyponormal operators. However, we will state and prove the results for the more general case of dominant operators whenever appropriate.

1. NORMAL AND PURE PARTS OF QUASISIMILAR HYPONORMAL OPERATORS

Suppose that T is an operator. Then $T = T_1 \oplus T_2$ where T_1 is normal and T_2 is pure, i.e., if $\mathscr{M}$ is a reducing subspace of T_2 and $T_2|\mathscr{M}$ is normal, then $\mathscr{M} = (0)$. The operator T_1 is called the *normal part* of T and T_2 the *pure part* of T. (Note that either of the operators T_1 or T_2 may be the zero operator on the zero Hilbert space.) J. Conway proved in [9] that the normal parts of quasisimilar subnormal operators are unitarily equivalent. The following theorem generalizes this result to the class of dominant operators. We remark that if T is a dominant operator, $\mathscr{M}$ is an invariant subspace for T, and $T|\mathscr{M}$ is normal, then $\mathscr{M}$ reduces T [25]. Thus if T is a pure dominant operator, then the point spectrum of T is empty. We shall use this fact freely.

THEOREM 1.1. *Suppose that T_1 and T_2 are two quasisimilar dominant operators. Let $T_i = N_i \oplus V_i$ on the Hilbert space $\mathscr{H}_i \oplus \mathscr{K}_i$ where N_i and V_i are the norma part and pure part, respectively, of T_i, $i = 1, 2$. Then N_1 and N_2 are unitarily equivalent and there exist bounded linear transformations $X_0: \mathscr{K}_1 \to \mathscr{K}_2$ and $Y_0: \mathscr{K}_2 \to \mathscr{K}_1$ having dense ranges such that $X_0 V_1 = V_2 X_0$ and $V_1 Y_0 = Y_0 V_2$.*

In order to prove Theorem 1.1 we shall need the following theorem and lemma. The reader should compare Theorem 1.2 with Theorem 3 of [25], Theorem 6 of [26], and Theorem 2 of [27].

THEOREM 1.2. *Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, T_1 is a norma operator in $\mathscr{L}(\mathscr{H}_1)$, T_2 is a dominant operator in $\mathscr{L}(\mathscr{H}_2)$, $X: \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear transformation, and $XT_1 = T_2 X$. If T_2 is pure, then $X = 0$.*

Proof. Let $\mathscr{M} = \mathscr{R}(X)^-$. The subspace $\mathscr{M}$ is invariant under T_2. Let $T_0 = T_2|\mathscr{M}$ and let $X_0: \mathscr{H}_1 \to \mathscr{M}$ be defined by $X_0 z = Xz$ for each z in $\mathscr{H}_1$. Observe that T_0 is dominant, X_0 has dense range, and $X_0 T_1 = T_0 X_0$. Hence, by Theorem 1 of [25], T_0 is normal. Thus Lemma 2 of [25] implies that $\mathscr{M}$ reduces T_2. Therefore, since T_2 is pure, we have $\mathscr{M} = (0)$; thus $X = 0$.

LEMMA 1.1. *Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, N_i is a norma operator in $\mathscr{L}(\mathscr{H}_i)$, $i = 1, 2$, $X: \mathscr{H}_1 \to \mathscr{H}_2$ and $Y: \mathscr{H}_2 \to \mathscr{H}_1$ are bounded linear transformations having trivial kernels, and $XN_1 = N_2 X$ and $N_1 Y = YN_2$. Then N_1 and N_2 are unitarily equivalent.*

Proof. Let $\mathscr{M} = \overline{\mathscr{R}(X)}$ and $\mathscr{N} = \overline{\mathscr{R}(Y)}$. Lemma 4.1 of [12] implies that $\mathscr{M}$ reduces N_2, $\mathscr{N}$ reduces N_1, N_1 is unitarily equivalent to $N_2|\mathscr{M}$, and N_2 is unitarily equivalent to $N_1|\mathcal{N}$. Thus, according to Theorem 1.3 of [15] or Lemma 2.2 of [9], N_1 and N_2 are unitarily equivalent.

Proof of Theorem 1.1. There exist quasiaffinities $X:\mathscr{H}_1 \oplus \mathscr{K}_1 \to \mathscr{H}_2 \oplus \mathscr{K}_2$ and $Y:\mathscr{H}_2 \oplus \mathscr{K}_2 \to \mathscr{H}_1 \oplus \mathscr{K}_1$ such that $XT_1 = T_2X$ and $T_1Y = YT_2$. Let

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \text{ and } \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$$

be the matrices of X and Y, respectively, with respect to $\mathscr{H}_1 \oplus \mathscr{K}_1$ and $\mathscr{H}_2 \oplus \mathscr{K}_2$. A matrix calculation shows that $X_3N_1 = V_2X_3$ and $Y_3N_2 = V_1Y_3$. Thus Theorem 1.2 implies that $X_3 = Y_3 = 0$. It follows that X_1 and X_2 have trivial kernels and a matrix calculation shows that $X_1N_1 = N_2X_1$ and $N_1Y_1 = Y_1N_2$. Hence, by Lemma 1.1, N_1 and N_2 are unitarily equivalent. Observe that X_4 and Y_4 have dense ranges and $X_4V_1 = V_2X_4$ and $V_1Y_4 = Y_4V_2$. Thus the proof is complete.

Suppose that T_1 and T_2 are quasisimilar hyponormal operators. Theorem 1.1 implies that if T_1 is normal, then T_2 is also normal and if T_1 is pure, then T_2 is pure. Moreover, in view of Theorem 1.1, in order to answer the question of whether $\sigma_e(T_1) = \sigma_e(T_2)$, it suffices to answer the question of whether $\sigma_e(V_1) = \sigma_e(V_2)$, where V_1 and V_2 are the pure parts of T_1 and T_2, respectively. We know that V_1 and V_2 are related by the equations $XV_1 = V_2X$ and $V_1Y = YV_2$ where X and Y are bounded linear transformations having dense ranges. (We remark that it is known that the pure parts of quasisimilar hyponormal operators need not be quasisimilar [9],[27].) The fact that X and Y are not necessarily quasiaffinities appears to be of little consequence. For instance, we know that the above stated conditions imply at least that $\sigma(V_1) = \sigma(V_2)$ [8]. Also, if V_1 and V_2 are quasinormal, then $\sigma_e(V_1) = \sigma_e(V_2)$ [27]. Moreover, the above stated conditions on V_1 and V_2 imply that $\dim\mathscr{K}((V_1 - \lambda)^*) = \dim\mathscr{K}((V_2 - \lambda)^*)$ for each complex number λ. (We know that since V_1 and V_2 are pure $\dim\mathscr{K}(V_1 - \lambda) = \dim\mathscr{K}(V_2 - \lambda) = 0$ for each complex number λ.) These observations are important in view of the fact that if T is an operator, then $\sigma_e(T) = \{\lambda \in \mathbf{C}: T-\lambda \text{ is not Fredholm}\}$. (Recall that an operator T is Fredholm if both $\mathscr{K}(T)$ and $\mathscr{K}(T^*)$ are finite dimensional and $\mathscr{R}(T)$ is closed.) Nevertheless, the question of whether $\sigma_e(V_1) = \sigma_e(V_2)$ seems difficult to answer even if V_1 and V_2 are subnormal.

2. HYPONORMALS COMMUTING WITH COMPACTS

The following theorem appears in [26]:

THEOREM A. *Suppose that T_1 and T_2 are hyponormal operators and there exist quasiaffinities X and Y such that $XT_1 = T_2X$ and $T_1Y = YT_2$. If either X or Y is compact, then $\sigma_e(T_1) = \sigma_e(T_2)$.*

The following question also appears in [26]: If T_1 and T_2 satisfy the hypotheses of Theorem A, then are T_1 and T_2 unitarily equivalent or similar? We shall answer this question later in this section. Observe that if T_1 and T_2 satisfy the hypotheses of Theorem A, then T_1 commutes with the compact quasiaffinity YX. Likewise T_2 commutes with the compact quasiaffinity XY. Moreover, Theorem 1.1 and its proof imply that the pure parts of T_1 and T_2 commute with compact operators having dense ranges. These observations inspire the following theorems.

THEOREM 2.1. *Suppose that T is a pure dominant operator, K is a compact operator having dense range, and $KT = TK$. Then $\sigma_e(T) = \sigma(T)$.*

Proof. Consider the equation $T^*K^* = K^*T^*$. Note that $\mathscr{K}(K^*) = (0)$. So by following the proof of Theorem 5.1 of [16] we can conclude that if $T^* - \lambda$ is Fredholm, then $\mathscr{K}(T^* - \lambda) = (0)$. We know that, since T is a pure dominant operator, $\mathscr{K}(T - \lambda) = (0)$ for each complex number λ. Thus if $\lambda \notin \sigma_e(T)$, then $\lambda \notin \sigma(T)$. Therefore $\sigma_e(T) = \sigma(T)$.

THEOREM 2.2. *Suppose that T is a pure dominant operator, K is a compact operator, and $KT = TK$. Then K is quasinilpotent and $\mathscr{K}(K)$ is either the zero or an infinite dimensional subspace.*

Proof. Suppose that λ is a complex number. Then, since $(K - \lambda)T = T(K - \lambda)$, $\mathscr{K}(K - \lambda)$ is an invariant subspace for T. Let $T_0 = T|\mathscr{K}(K - \lambda)$. The operator T_0 is dominant and Lemma 2 of [25] implies that T_0 is pure. Since the point spectrum of T_0 is empty, it follows that $\mathscr{K}(K - \lambda)$ is infinite or zero dimensional. Hence the compact operator K has no nonzero eigenvalues, i.e., $\sigma(K) = \{0\}$. Also, if $\lambda = 0$ is an eigenvalue, then $\mathscr{K}(K)$ is infinite dimensional.

Let K be as in Theorem 2.2. We remark that both the cases, $\mathscr{K}(K)$ is trivial and $\mathscr{K}(K)$ is infinite dimensional, can occur. In the following example K is constructed so that $\mathscr{K}(K)$ is infinite dimensional. In Example 2.2, K is constructed so that $\mathscr{K}(K)$ is trivial.

EXAMPLE 2.1. Let $\mathscr{H} = \mathbf{C}^2$ and let

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } K_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with respect to some orthonormal basis for $\mathscr{H}$. Note that $P\left(\frac{1}{2^n} K_0\right) = \left(\frac{1}{2^{n+1}} K_0\right) P$, $n = 0, 1, 2, \ldots$. Let $\widehat{\mathscr{H}} = \sum_{n=1}^{\infty} \oplus \mathscr{H}_n$ and $\hat{P} = \sum_{n=1}^{\infty} \oplus P_n$ where for each n, $\mathscr{H}_n = \mathscr{H}$ and $P_n = P$. Let $V_{\mathscr{H}}$ denote the unilateral shift on $\widehat{\mathscr{H}}$, i.e., $V_{\mathscr{H}}(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ for each $(x_1, x_2, \ldots)$ in $\widehat{\mathscr{H}}$. Observe that the dilated shift $T = V_{\mathscr{H}}\hat{P}$ is a pure quasinormal operator [5] and if $K = \sum_{n=1}^{\infty} \oplus \frac{1}{2^n} K_0$ on $\widehat{\mathscr{H}}$, then $KT = TK$. The operator K is compact and nilpotent and $\mathscr{K}(K)$ is infinite dimensional.

The following theorem shows that the answer to the question mentioned at the beginning of this section is affirmative if one of the two quasisimilar hyponormal operators is an isometry.

THEOREM 2.3. *Suppose that T_1 is an isometry, T_2 is a dominant operator, and there exist quasiaffinities X and Y such that $XT_1 = T_2X$ and $T_1Y = YT_2$. If either X or Y is compact, then T_1 and T_2 are unitarily equivalent unitary operators.*

Proof. We have $(YX)T_1 = T_1(YX)$ where YX is a compact quasiaffinity. Corollary 6.4 of [12] implies that T_1 is unitary. Hence, since T_1 is normal, Theorem 1.1 implies that T_2 is also normal and unitarily equivalent to T_1. Thus T_2 is also unitary.

In spite of the above theorem, the following example shows that the answer to the above mentioned question is negative.

EXAMPLE 2.2. Let $\{e_n\}_{n=-\infty}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathscr{H}$ and let $\{c_n\}_{n=-\infty}^{\infty}$ and $\{d_n\}_{n=-\infty}^{\infty}$ be bounded sequences of positive numbers such that for each integer n, $c_{n-1} < d_n < c_n$. Let P_1 and P_2 be operators in $\mathscr{L}(\mathscr{H})$ defined by $P_1e_n = c_ne_n$ and $P_2e_n = d_ne_n$ for each integer n. Let $T_1 = V_{\mathscr{H}}\hat{P}_1$ and $T_2 = V_{\mathscr{H}}\hat{P}_2$ on $\hat{\mathscr{H}}$. (Here we are using the notation of Example 2.1.) The operators T_1 and T_2 are pure quasinormal operators. For each positive integer i define X_i and Y_i in $\mathscr{L}(\mathscr{H})$ by

$$X_ie_n = \frac{1}{2^{|n|}}\left(\frac{d_n}{c_n}\right)^i e_n$$

and

$$Y_ie_n = \frac{1}{2^{|n|}}\left(\frac{c_{n-1}}{d_n}\right)^i e_{n-1},$$

$n = \ldots -2, -1, 0, 1, 2, \ldots$. Observe that X_i and Y_i are compact quasiaffinities, $i = 1, 2, \ldots$, that $\|X_i\| \to 0$ and $\|Y_i\| \to 0$, and that $X_iP_1 = P_2X_{i-1}$ and $Y_iP_2 = P_1Y_{i-1}$, $i = 2, 3, \ldots$. Let $X = \sum_{i=1}^{\infty} \oplus X_i$ and $Y = \sum_{i=1}^{\infty} \oplus Y_i$ on $\hat{\mathscr{H}}$. Thus X and Y are compact quasiaffinities, $XT_1 = T_2X$, and $T_1Y = YT_2$. Hence T_1 and T_2 are quasisimilar pure quasinormal operators with the implementing quasiaffinities both compact. The operators T_1 and T_2 are unitarily equivalent to $\sum_{n=-\infty}^{\infty} \oplus c_nS$ and $\sum_{n=-\infty}^{\infty} \oplus d_nS$, respectively, where S denotes the unilateral shift of multiplicity one. Hence $T_1 - d_0$ has closed range and $T_2 - d_0$ does not have closed range. Thus T_1 and T_2 are not similar.

It is possible to extend Theorem A to a class of dominant operators. We shall need some additional terminology. Suppose that $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, where $x \neq 0$. Then $f(\lambda) = (T - \lambda)^{-1}x$ is a vector valued analytic function on $\mathbf{C}\setminus\sigma(T)$, the resolvent set of T. If $\tilde{f}$ is an analytic function whose domain is an open set containing $\mathbf{C}\setminus\sigma(T)$ and which satisfies $(T-\lambda)\tilde{f}(\lambda) = x$ for all λ in the domain of $\tilde{f}$, then $\tilde{f}$ is said to be an analytic extension of f. If any two analytic extensions of f agree on the intersection of their domains, then T is said to have the *single valued extension property*. (We note that dominant operators have the single valued extension property since their normal parts clearly do and their pure parts have empty point spectra.) If T has the single valued extension property, let $\tilde{x}(\lambda)$ denote the maximal analytic extension of $(T-\lambda)^{-1}x$, let $\rho_T(x)$ denote the domain of $\tilde{x}$, and let $\sigma_T(x)=\mathbf{C}\setminus\rho_T(x)$. The set $\rho_T(x)$ is the *local resolvent* of x and $\sigma_T(x)$ is the *local spectrum* of x. The set $\sigma_T(x)$ is a nonempty compact subset of the plane and $\sigma_T(x) \subseteq \sigma(T)$. (For convenience, let $\sigma_T(x) = \emptyset$ if $x = 0$.)

An operator T is said to satisfy Dunford's Condition C if T has the single valued extension property and for each closed subset F of $\mathbf{C}$, the linear manifold $\{x \in \mathscr{H} : \sigma_T(x) \subseteq F\}$ is closed. It is known that hyponormal operators satisfy Dunford's Condition C (cf. [21], [23]). Thus the following theorem extends Theorem A.

THEOREM 2.4. *Suppose that T_1 and T_2 are dominant operators satisfying Dunford's Condition* C *and there exist quasiaffinities X and Y such that $XT_1 = T_2X$ and $T_1Y = YT_2$. If either X or Y is compact, then $\sigma_e(T_1) = \sigma_e(T_2)$ (and $\sigma(T_1) = \sigma(T_2)$).*

For the proof of Theorem 2.4, we shall need the following lemma and theorem. Lemma 2.1 is easy to verify.

LEMMA 2.1. *Suppose that $T_1 \in \mathscr{L}(\mathscr{H}_1)$ and $T_2 \in \mathscr{L}(\mathscr{H}_2)$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces. Then $T_1 \oplus T_2$ satisfies Dunford's Condition* C *if and only if both T_1 and T_2 do.*

J. G. Stampfli proves in [24] that if T_1 and T_2 are quasisimilar operators that satisfy Dunford's Condition C, then $\sigma(T_1) = \sigma(T_2)$. Close scrutiny of his proof reveals that the fact that the intertwining quasiaffinities are injective is not used to establish the above result. Hence we have the following theorem.

THEOREM 2.5 (Stampfli). *Suppose that $T_1 \in \mathscr{L}(\mathscr{H}_1)$ and $T_2 \in \mathscr{L}(\mathscr{H}_2)$, where $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces, and suppose that both T_1 and T_2 satisfy Dunford's Condition* C. *If there exist bounded linear transformations $X:\mathscr{H}_1 \to \mathscr{H}_2$ and $Y:\mathscr{H}_2 \to \mathscr{H}_1$ having dense ranges such that $XT_1 = T_2X$ and $T_1Y = YT_2$, then $\sigma(T_1) = \sigma(T_2)$.*

Proof of Theorem 2.4. We know from Theorem 2.5 that $\sigma(T_1) = \sigma(T_2)$. Let $T_i = N_i \oplus V_i$ be as in Theorem 1.1, $i = 1,2$. Since N_1 and N_2 are unitarily equivalent, the proof is complete if we show that $\sigma_e(V_1) = \sigma_e(V_2)$. Theorem 1.1 implies that there exist bounded linear transformations X_0 and Y_0 having dense ranges such that $X_0V_1 = V_2X_0$ and $V_1Y_0 = Y_0V_2$. So, by Theorem 2.5, $\sigma(V_1) = \sigma(V_2)$. Clearly X_0 or Y_0 is compact (see the proof of Theorem 1.1), V_1 commutes with Y_0X_0, and V_2 commutes with X_0Y_0. Thus, by Theorem 2.1, $\sigma_e(V_i) = \sigma(V_i)$, $i = 1, 2$, and the proof is complete.

We remark that there exist dominant operators satisfying Dunford's Condition C that are not hyponormal. (A nonzero quasinilpotent dominant operator is an example.) However, it appears to be unknown whether every dominant operator satisfies Dunford's Condition C (cf. [20]).

3. BIQUASITRIANGULARITY AND HYPONORMAL OPERATORS

In this section all Hilbert spaces are separable. Also, unless stated otherwise, all Hilbert spaces are infinite dimensional. We shall need the following notation and terminology. If T is an operator, let $\sigma_{le}(T)[\sigma_{re}]$ denote the left [right] essential spectrum of T, i.e., the left [right] spectrum of $\tilde{T}$. Let $\sigma_{ap}(T)$ denote the approximate point spectrum of T. Recall that $\sigma_{ap}(T) = \{\lambda \in \mathbf{C}: T - \lambda$ is not bounded below$\}$. Recall also that a *hole* in $\sigma_e(T)$ is bounded component of $\mathbf{C}\diagdown\sigma_e(T)$ and a *pseudohole* in $\sigma_e(T)$ is a component of $(\operatorname{int}\sigma_e(T))\diagdown\sigma_{le}(T)$ or $(\operatorname{int}\sigma_e(T))\diagdown\sigma_{re}(T)$. Hence a hole or a pseudohole in $\sigma_e(T)$ is an open subset of the complex plane. An operator T is *semi-Fredholm* if either $\mathscr{K}(T)$ or $\mathscr{K}(T^*)$ is finite dimensional and $\mathscr{R}(T)$ is closed. If T is semi-Fredholm, let $i(T) = \dim\mathscr{K}(T) - \dim\mathscr{K}(T^*)$ denote the index of T. If T is an operator and H is a hole or pseudohole in $\sigma_e(T)$, then, for λ in H, $T - \lambda$ is semi-Fredholm and $i(T - \lambda)$ is constant on H. (See, for example, [18] for a discussion of the above.)

It is known that an operator T is quasitriangular if and only if for each complex number λ such that $T - \lambda$ is semi-Fredholm, $i(T - \lambda) \geqslant 0$ [2], [13], [14]. It follows that an operator T is biquasitriangular if and only if for each complex number λ such that $T - \lambda$ is semi-Fredholm, $i(T - \lambda) = 0$. Hence the adjoint of a dominant operator is quasitriangular. Thus a dominant operator is quasitriangular if and only if it is biquasitriangular.

A normal operator T is biquasitriangular and satisfies $\sigma(T) = \sigma_{ap}(T)$. The following theorem shows that this latter condition characterizes those hyponormal (and more generally dominant) operators that are biquasitriangular.

THEOREM 3.1. *Suppose that T is a dominant operator. Then T is biquasitriangular if and only if $\sigma(T) = \sigma_{ap}(T)$.*

Proof. Suppose that T is biquasitriangular. Let $\lambda \in \mathbf{C}\diagdown\sigma_{ap}(T)$. Then $T - \lambda$ is injective and $R(T - \lambda)$ is closed, i.e., $T - \lambda$ is semi-Fredholm. Since T is biquasitriangular, we have $i(T - \lambda) = 0$. Thus $T - \lambda$ is invertible, and hence $\lambda \in \mathbf{C}\diagdown\sigma(T)$. Therefore, $\sigma(T) = \sigma_{ap}(T)$.

Now suppose that $\sigma(T) = \sigma_{ap}(T)$. There exist Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ satisfying $\dim\mathscr{H}_1 \leqslant \aleph_0$, $\dim\mathscr{H}_2 = 0$ or $\dim\mathscr{H}_2 = \aleph_0$, and $\dim\mathscr{H}_1 \oplus \mathscr{H}_2 = \aleph_0$ such that $T = T_1 \oplus T_2$ on $\mathscr{H}_1 \oplus \mathscr{H}_2$, where T_1 is a normal operator in $\mathscr{L}(\mathscr{H}_1)$ and T_2 is a pure dominant operator in $\mathscr{L}(\mathscr{H}_2)$. Suppose that $T - \lambda$ is semi-Fredholm. Then λ belongs to a hole or pseudohole H in $\sigma_e(T)$. Observe that if $\dim\mathscr{H}_1 = \aleph_0$, then $\lambda \notin \sigma_e(T_1)$ and, since T_1 is normal, $\sigma(T_1)\diagdown\sigma_e(T_1)$ consists of at most isolated points. If $1 \leqslant \dim \mathscr{H}_1 < \aleph_0$, then, of course, $\sigma(T_1)$ consists of isolated points. It follows that there exists λ_1 in H such that $T_1 - \lambda_1$ is injective. Note that $T_2 - \lambda_1$ is injective. Hence $T - \lambda_1$ is injective and $\mathscr{R}(T - \lambda_1)$ is closed. Thus, $\lambda_1 \in \mathbf{C} \setminus \sigma_{\mathrm{ap}}(T) = \mathbf{C} \setminus \sigma(T)$. So $T - \lambda_1$ is invertible and $\mathrm{i}(T - \lambda_1) = 0$. It follows that $\mathrm{i}(T - \lambda) = 0$; thus T is biquasitriangular.

COROLLARY 3.1. *Suppose that T is a pure dominant operator. Then T is biquasitriangular if and only if $\sigma(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{le}}(T)$.*

Proof. Since the point spectrum of T is empty, we have $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{le}}(T)$. The proof is now immediate since $\sigma_{\mathrm{le}}(T) \subseteq \sigma_{\mathrm{e}}(T) \subseteq \sigma(T)$.

There are several examples of nonnormal biquasitriangular hyponormal operators in the literature (see, for example, [23] and [26]). The following theorem shows that such operators are easy to construct. (We shall use the convention that if $\mathscr{H}$ is the zero Hilbert space and $T \in \mathscr{L}(\mathscr{H})$, then $\sigma(T) = \emptyset$.)

THEOREM 3.2. *Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are Hilbert spaces satisfying $\dim \mathscr{H}_1 \leqslant \aleph_0$, $\dim \mathscr{H}_2 \leqslant \aleph_0$, and $\dim \mathscr{H}_1 \oplus \mathscr{H}_2 = \aleph_0$. If T_1 is a normal operator in $\mathscr{L}(\mathscr{H}_1)$, $T_2 \in \mathscr{L}(\mathscr{H}_2)$, and $\sigma(T_2) \subseteq \sigma(T_1)$, then $T_1 \oplus T_2$ is biquasitriangular.*

Proof. We may assume that both $\mathscr{H}_1$ and $\mathscr{H}_2$ are nonzero Hilbert spaces. Suppose that $(T_1 \oplus T_2) - \lambda$ is semi-Fredholm. Then λ belongs to a hole or pseudo-hole H in $\sigma_{\mathrm{e}}(T_1 \oplus T_2)$. Since T_1 is normal and H is an open subset of the complex plane, there exists a point λ_1 in H such that $T_1 - \lambda_1$ is invertible. (We have used the fact that if $\dim \mathscr{H}_1 = \aleph_0$, then $\lambda \notin \sigma_{\mathrm{e}}(T_1)$ and, since T_1 is normal, $\sigma(T_1) \setminus \sigma_{\mathrm{e}}(T_1)$ consists of isolated points.) Since $\sigma(T_2) \subseteq \sigma(T_1)$, $T_2 - \lambda_1$ is invertible. Hence $(T_1 \oplus T_2) - \lambda_1$ is invertible and $\mathrm{i}((T_1 \oplus T_2) - \lambda_1) = 0$; thus $\mathrm{i}((T_1 \oplus T_2) - \lambda) = 0$ It follows that $T_1 \oplus T_2$ is biquasitriangular.

It follows from the above theorem that if V is a pure hyponormal operator and N is a normal operator satisfying $\sigma(V) \subseteq \sigma(N)$, then $N \oplus V$ is a nonnormal hyponormal operator that is biquasitriangular. The following is an example of a pure quasinormal operator that is biquasitriangular.

EXAMPLE 3.1. We use the notation of Example 2.1. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathscr{H}$ and let $\{c_n\}_{n=1}^{\infty}$ be a dense sequence in the interval $(0, 1)$. Define P in $\mathscr{L}(\mathscr{H})$ by $Pe_n = c_n e_n$, $n = 1, 2, \ldots$. The operator P is positive definite. Let $T = V_{\mathscr{H}} \hat{P}$. The operator T on $\widehat{\mathscr{H}}$ is a pure quasinormal. Let $\Lambda = \{\lambda \in \mathbf{C}: |\lambda| \leqslant 1\}$. Corollary 2 of [27] implies $\sigma(T) = \sigma_{\mathrm{e}}(T) = \Lambda$. Since T is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus c_n S$, where S denotes the unilateral shift of multiplicity one, $\bigcup_{n=1}^{\infty} \{\lambda \in \mathbf{C}: |\lambda| = c_n\} \subseteq \sigma_{\mathrm{le}}(T)$. Hence $\Lambda \subseteq \sigma_{\mathrm{le}}(T)$. It follows that $\sigma(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{le}}(T) = \sigma_{\mathrm{e}}(T) = \Lambda$. Thus Corollary 3.1 implies that T is biquasitriangular.

Suppose that T_1 and T_2 satisfy the hypotheses of Theorem A (which appears at the beginning of Section 2). Then T_1 commutes with the compact quasiaffinity YX. Theorem 5.1 of [16] implies that if $T_1 - \lambda$ is semi-Fredholm, then $\mathrm{i}(T_1 - \lambda) = 0$ or $\mathrm{i}(T_1 - \lambda) = -\infty$. In view of the above observations and the already mentioned spectral characterization of the biquasitriangular operators, the following question comes to mind: If T_1 and T_2 satisfy the hypotheses of Theorem A, then are T_1 and T_2 necessarily biquasitriangular? The answer is negative. Indeed, neither of the two operators T_1 and T_2 constructed in Example 2.2 is biquasitriangular (because $\sigma(T_i) \neq \sigma_{\mathrm{ap}}(T_i)$, $i = 1, 2$). It was proved in [26] that if T_1 and T_2 are quasisimilar biquasitriangular hyponormal operators, then $\sigma_e(T_1) = \sigma_e(T_2)$. This suggests the following question: If T_1 and T_2 are quasisimilar hyponormal operators and T_1 is biquasitriangular, then must T_2 be biquasitriangular also? The following example shows that the answer is negative.

EXAMPLE 3.2. Again we use the notation of Example 2.1. Let $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ be (one-to-one) sequences of all the rational numbers in the sets (0,1) and $\left(0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, 1\right)$, respectively. Let $\alpha:(0, 1) \to \left(0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, 1\right)$ and $\beta: \left(0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, 1\right) \to (0,1)$ be the one-to-one and onto functions defined as follows:

$$\alpha(x) = \begin{cases} \frac{4}{9}x & \text{if } x \in \left(0, \frac{3}{4}\right) \\ \frac{4}{3}x - \frac{1}{3} & \text{if } x \in \left[\frac{3}{4}, 1\right) \end{cases}$$

and

$$\beta(x) = \begin{cases} \frac{3}{4}x & \text{if } x \in \left(0, \frac{1}{3}\right) \\ \frac{9}{4}x - \frac{5}{4} & \text{if } x \in \left[\frac{2}{3}, 1\right). \end{cases}$$

Observe that $\alpha(x) < x$ for each x in (0, 1) and $\beta(x) < x$ for each x in $\left(0, \frac{1}{3}\right) \cup \left[\frac{2}{3}, 1\right)$. For each positive integer n there exist positive integers j_n and k_n such that $d_{j_n} = \alpha(c_n)$ and $c_{k_n} = \beta(d_n)$. The maps $n \to j_n$ and $n \to k_n$ of the set of positive integers into itself is one-to-one and onto and for each positive integer n, $d_{j_n} < c_n$ and $c_{k_n} < d_n$. Let $\{e_n\}_{n=1}^{\infty}$ be a orthonormal basis for a Hilbert space $\mathscr{H}$. Define P_1 and P_2 in $\mathscr{L}(\mathscr{H})$ by $P_1 e_n = c_n e_n$ and $P_2 e_n = d_n e_n$, $n = 1, 2, \ldots$. Let $T_i = V_{\mathscr{H}} \hat{P}_i$, $i = 1, 2$. The operators T_1 and T_2 in $\mathscr{L}(\hat{\mathscr{H}})$ are pure quasinormals. Define X_i and Y_i in $\mathscr{L}(\mathscr{H})$ by $X_i e_n = \frac{1}{2^n}\left(\frac{d_{j_n}}{c_n}\right)^i e_{j_n}$ and $Y_i e_n = \frac{1}{2^n}\left(\frac{c_{k_n}}{d_n}\right)^i e_{k_n}$, $n = 1, 2, \ldots$, $i = 1, 2, \ldots$.

The operators X_i and Y_i are quasiaffinities, $X_{i+1}P_1 = P_2X_i$, and $Y_{i+1}P_2 = P_1Y_i$, $i = 1, 2, \ldots$. Let $X = \sum_{i=1}^{\infty} \oplus X_i$ and $Y = \sum_{i=1}^{\infty} \oplus Y_i$. The operators X and Y are quasiaffinities in $\mathscr{L}(\hat{\mathscr{H}})$ and an easy calculation shows that $XT_1 = T_2X$ and $T_1Y = YT_2$. So T_1 and T_2 are quasisimilar. As in Example 3.1, T_1 is biquasitriangular. On the other hand, since T_2 is unitarily equivalent to $\sum_{n=1}^{\infty} \oplus d_nS$, $T_2 - \lambda$ is semi-Fredholm and $\mathrm{i}(T - \lambda) = -\infty$ for each complex number λ satisfying $\frac{1}{3} < |\lambda| < \frac{2}{3}$. Hence T_2 is not biquasitriangular. (Note that the operators X and Y are also compact.)

Suppose that T is a pure hyponormal operator. Let m_2 denote planar Lebesgue measure. Putnam proved in [19] that if H is an open subset of the complex plane and $H \cap \sigma(T) \neq \emptyset$, then $m_2(H \cap \sigma(T)) > 0$. In particular, $m_2(\sigma(T)) > 0$. There are many examples of a pure hyponormal operator T such that $m_2(\sigma_e(T)) = 0$. The unilateral shift of multiplicity one is an example. However, as the following theorem shows, the situation is different if T is biquasitriangular.

THEOREM 3.3. *Suppose that $\mathscr{H}_1$ is a Hilbert space satisfying* $\dim\mathscr{H}_1 \leqslant \aleph_0$, $\mathscr{H}_2$ *is an infinite dimensional Hilbert space, T_1 is a normal operator in $\mathscr{L}(\mathscr{H}_1)$, T_2 is a pure hyponormal operator in $\mathscr{L}(\mathscr{H}_2)$, and $T_1 \oplus T_2$ is biquasitriangular. If H is an open subset of the complex plane such that $H \cap \sigma_e(T_2) \neq \emptyset$, then* $m_2(H \cap \sigma_e(T_1 \oplus T_2)) > 0$.

Proof. If $m_2(H \cap \sigma_e(T_2)) > 0$, then we are done. So let us suppose that $m_2(H \cap \sigma_e(T_2)) = 0$. Putnam proved that $m_2(H \cap \sigma(T_2)) > 0$ [19]. Thus there exists a point λ_0 in $H \cap \sigma(T_2)$ such that $\lambda_0 \notin H \cap \sigma_e(T_2)$. There exists a hole K in $\sigma_e(T_2)$ such that $\lambda_0 \in K$. Since the point spectrum of T_2 is empty and since $\lambda_0 \in \sigma(T_2) \setminus \sigma_e(T_2)$, we have $T_2 - \lambda_0$ is semi-Fredholm and $\mathrm{i}(T_2 - \lambda_0) < 0$. Hence $\mathrm{i}(T_2 - \lambda) < 0$ for each λ in K. Note that $\mathscr{H}_1$ is infinite dimensional and $\lambda_0 \in \sigma_e(T_1)$. (If the above statement is not true, then $(T_1 \oplus T_2) - \lambda_0$ is semi-Fredholm and $\mathrm{i}((T_1 \oplus T_2) - \lambda_0) = \mathrm{i}(T_2 - \lambda_0) < 0$; thus $T_1 \oplus T_2$ is not biquasitriangular.) Likewise $K \subseteq \sigma_e(T_1)$. Hence $H \cap K \subseteq H \cap \sigma_e(T_1) \subseteq H \cap \sigma_e(T_1 \oplus T_2)$. Therefore, $m_2(H \cap \sigma_e(T_1 \oplus T_2)) > 0$.

We remark that Theorem 3.3 is not true if we assume only that $H \cap \sigma_e(T_1 \oplus T_2) \neq \emptyset$. For example, let T_1 be a normal operator satisfying $\sigma(T_1) = \sigma_e(T_1) = \{\lambda \in \mathbf{C}: |\lambda| \leqslant 1\} \cup \{\lambda \in \mathbf{C}: 1 \leqslant \lambda \leqslant 2\}$, and let T_2 be the unilateral shift (of multiplicity one). Theorem 3.2 implies that $T_1 \oplus T_2$ is biquasitriangular. Let $H = \{\lambda \in \mathbf{C}: |\lambda - 2| < 1\}$. Then $H \cap \sigma_e(T_1 \oplus T_2) \neq \emptyset$. However, $m_2(H \cap \sigma_e(T_1 \oplus T_2)) = 0$.

Suppose that T is a hyponormal operator on a Hilbert space $\mathscr{H}$. Putnam proved in [19] that $\pi\|T^*T - TT^*\| \leqslant m_2(\sigma(T))$. Thus if $m_2(\sigma(T)) = 0$, then T is normal. Since the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}$ is a $*$-algebra isometrically isomorphic to a C^*-subalgebra of $\mathscr{L}(\mathscr{K})$ for some Hilbert space $\mathscr{K}$, it follows from Putnam's result that $\pi\|\tilde{T}^*\tilde{T} - \tilde{T}\tilde{T}^*\| \leqslant m_2(\sigma(\tilde{T})) = m_2(\sigma_e(T))$ (where $\tilde{T}$ denotes the image of T in the Calkin algebra under the natural quotient map). Hence if $m_2(\sigma_e(T)) = 0$, then the self-commutator $T^*T - TT^*$ is compact, i.e., T is essentially normal. The converse of the above statement is not true. Indeed there exist normal operators whose essential spectra have positive measure. Also Clancey and Morrell constructed in [7] an example of a pure hyponormal operator whose self-commutator has rank one and whose essential spectrum is the closed unit disk. Moreover, there exists a pure subnormal operator T such that $T^*T - TT^*$ is compact and $m_2(\sigma_e(T)) \neq 0$. For example, take T to be a certain Bergman operator or let $T = \sum_{n=1}^{\infty} \oplus A_n$ where A_n is defined in [22]. (The author is grateful to R. G. Douglas for pointing out the former example and to J. G. Stampfli for pointing out the latter.) However, if T is a pure quasinormal operator, then $T^*T - TT^*$ is compact if and only if $m_2(\sigma_e(T)) = 0$ (cf. [27]). The above discussion suggests the following question.

QUESTION 3.1. Suppose that T is a pure subnormal operator and $T^*T - TT^*$ has finite rank. Then is $m_2(\sigma_e(T)) = 0$?

Observe that if T is a pure subnormal operator and $T^*T - TT^*$ is a rank one operator, then $T = \alpha S + \beta$ [17], where α and β are complex numbers and S is the unilateral shift of multiplicity one; thus $m_2(\sigma_e(T)) = 0$. Hence Question 3.1 is settled for the case that T has a rank one self-commutator. The following question was asked by J. Conway in [10].

QUESTION 3.2. Suppose that T is a pure subnormal operator such that $T^*T - TT^*$ has finite rank and suppose that N is the minimal normal extension of T. Then is $m_2(\sigma(N)) = 0$?

The following theorem shows that Questions 3.1 and 3.2 are related.

THEOREM 3.4. *Suppose that T is a subnormal operator on a Hilbert space $\mathscr{H}$ such that $T^*T - TT^*$ is compact and suppose that N is the minimal normal extension of T. Then $\sigma_e(T) \subseteq \sigma(N)$.*

Proof. The operator N is unitarily equivalent to the matrix operator

$$\begin{bmatrix} T & R \\ 0 & S \end{bmatrix}$$

on $\mathscr{H} \oplus \mathscr{K}$. A matrix calculation shows that $RR^* = T^*T - TT^*$. Thus R is compact. It follows that $\sigma_e(N) = \sigma_e(T) \cup \sigma_e(S)$. In particular, $\sigma_e(T) \subseteq \sigma(N)$.

If the answer to Question 3.2 is affirmative and T is a pure subnormal operator such that $T^*T - TT^*$ has finite rank, then Theorem 3.4 implies that $m_2(\sigma_e(T)) = 0$. Thus Questions 3.1 and 3.2 are related.

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