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POINT INTERACTIONS AS LIMITS OF SHORT RANGE INTERACTIONS

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1. INTRODUCTION

In the physical literature, in particular in nuclear physics and in solid state physics, there has been a long standing interest in studying models of a quantum mechanical particle moving under the influence of fixed centers, the force between the particle and the centers being a point one (zero range), of different strengths (we speak for short of models with δ-potentials). For such models see e.g., [1]–[6]. The mathematical description of the Hamiltonian for these models is by standard quadratic forms theory in the one dimensional case (the δ-potential being in this case relatively form-bounded with respect to $-\frac{d^2}{dx^2}$, see e.g., [7] (Ch. X.2), [8]). In the two and three-dimensional cases the definition of the Hamiltonian requires more subtle methods. The problem for the case of one center had been discussed in the physical literature starting with [9] (see also [10]), mathematical solutions were then provided by Berezin and Faddeev [11], by using Kreĭn's method of self-adjoint extensions, by Streit and ourselves using the method of Dirichlet forms [12] and by Nelson [34] and Fenstad and ourselves using non standard analysis [13] (in the latter paper the many centers problem is also solved).[1] The non standard analysis version exhibits the Hamiltonian as a smooth perturbation of $-\Delta$ by a potential of infinitesimal support. An explicit computation of the resolvent was also given, by these non standard methods.

The detailed study of the resolvent and of the scattering was pursued further (without non standard analytical methods) in [14], [15], where also applications to models of solid state physics were given; see also [16]. A natural question that arises is whether the so defined Hamiltonians for δ-interactions can be approximated by Hamiltonians with smooth local potentials, i.e., by operators of the form $-\Delta + V \equiv -\Delta + \sum_i V_i(x - x_i)$, with some smooth functions V_i. This

[1] See also [38]. For other references concerning one-center δ-potentials see also e.g., [39].

question of approximation is an interesting one whenever one has Hamiltonians for "singular" potentials. Mostly the approximation is sought because questions about the definition of the scattering matrix are more easily solved mathematically for Hamiltonians given by smooth potentials (see [17]), however in the case of δ-potentials it is rather the opposite point of view the interesting one. Namely for δ-potentials all physical quantities (resolvent, bound states, resonances, scattering) can be computed explicitely, and hence one can use these results, provided one has a sufficient strong approximation theorem, to draw conclusions for the physical quantities associated with the approximating Hamiltonians $-\Delta + V$ (from this point of view the δ-potentials Hamiltonians are looked upon as a idealized case, the realistic ones being those described by Hamiltonians of the above form $-\Delta + V$). Some of the approximation procedures discussed in the physical literature and also the one given by Berezin and Faddeev in [11] are by separable non local potentials. More recently Friedman [35] has studied a class of local approximations and shown strong resolvent convergence. This result was reinterpreted and extended in non standard analytic terms by Nelson [34] and Fenstad and ourselves [13]. Larger classes of approximations and more detailed results have been obtained, again with the help of non standard analytic means, by Alonso y Coria [36] (who also considers the case of convergence to the free Hamiltonian in the case of a multiparticle system interacting by δ-potentials). In the present paper we study approximations of the form $\frac{\lambda(\varepsilon)}{\varepsilon^2} V(x/\varepsilon)$ with V a local potential on $\mathbf{R}^3$ in a general class (containing the one considered by Alonso y Coria) namely integrable and Rollnik, i.e., fulfilling

$$\int_{\mathbf{R}^3} |V(x)|\, \mathrm{d}x < \infty, \quad \int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{|V(x)|\,|V(y)|}{|x-y|^2}\, \mathrm{d}x\, \mathrm{d}y < \infty. \tag{1.1}$$

We put the asymptotic behaviour as $\varepsilon \to 0$ in relation with the presence of zero energy resonances (for a discussion of the presence or absence of zero energy resonances see also [37]). We also give the corresponding results for the many (finitely many, infinitely many) centers problem. Our results also provide the basis for a perturbation theory around the limiting case of δ-interactions with approximating local potentials which is now being developed [19], [40] and which gives a mathematical expression to ideas one often encounters in the physical literature (scattering length approximation, shape independent approximation, effective range expansion, which all are forms of a low energy expansion, see e.g., [18]). Let us also mention that ideas related to the one used in this paper have been recently applied to the discuss of the Efimov effect in three particle systems [20].

Let us now shortly describe the structure of the paper.

In Section 2 we recall the properties of the Hamiltonians associated with potentials of the form (1.1), and we derive technical tools concerning the norm limit of certain basic operators used in the subsequent sections.

These results are then used in Section 3 to study the limit of $H_\varepsilon \equiv -\Delta + \lambda(\varepsilon)\varepsilon^{-2}V\left(\frac{x}{\varepsilon}\right)$ as $\varepsilon \to 0$, in the resolvent sense. We prove in particular that H_ε converges to $-\Delta$ if there is no zero energy resonance for $-\Delta + V$ and, in the case when there is a zero energy resonance and this is either simple or $V \leqslant 0$ [2] the limit is $-\Delta_\alpha$, where $-\Delta_\alpha$ is the Hamiltonian of a δ-potential of strength $\alpha = -\lambda'(0)\left(\sum_{i=1}^{n}|(V, \psi_i)|^2\right)^{-1}$, with ψ_i the resonance functions $(-\Delta + V)\psi_i = 0$, normalized by $\int \psi_i(x)V(x)\psi_j(x)\,dx = -\delta_{ij}$. If $V \leqslant 0$ has at least one zero energy resonance then $-\Delta_\alpha$ has also exactly one resonance at the point $k = -4\pi i\alpha$ if $\lambda'(0) \leqslant 0$ and exactly one negative eigenvalue $-16\pi^2\alpha^2$ if $\lambda'(0) > 0$ (and in this case there is an eigenvalue E_ε of H_ε which converges as $\varepsilon \to 0$ to $-16\pi^2\alpha^2$). Further results are obtained in the case when V has compact support. In particular in this case all negative eigenvalues and resonances of H_ε tend to infinity as $\varepsilon \to 0$ except, in the case of some zero energy resonance for $-\Delta + V$, for one of the negative eigenvalues resp. resonances, according to whether $\lambda'(0) > 0$ or $\lambda'(0) \leqslant 0$, respectively, and these converge to the negative energy eigenvalue resp. resonance for $-\Delta_\alpha$. In Section 4 we extend the results to the case of a discrete (finite or infinite) family of centers in $\mathbf{R}^3$.

2. INTERACTIONS BY SHORT RANGE POTENTIALS

We start by summarizing some known results on Hamiltonians given by potentials V in the Rollnik class defined below.

Let V be a measurable function on $\mathbf{R}^3$ and define

$$\|V\|_{\mathbf{R}}^2 \equiv (4\pi)^{-2}\iint \frac{|V(x)|\,|V(y)|}{|x-y|^2}\,dx\,dy. \tag{2.1}$$

If $\|V\|_{\mathbf{R}} < \infty$ then one calls V a *Rollnik potential* ([21], [22]). Rollnik potentials have been extensively studied especially by H. Rollnik ([21]), A. Grossmann and T.I.Wu ([23]) and B. Simon ([22]). They form a complete vector space $\mathscr{R}$ in the Rollnik norm $\|\ \|_{\mathbf{R}}$ (containing e.g., $L^{3/2}(\mathbf{R}^3)$ and $L^1(\mathbf{R}^3) \cap L^2(\mathbf{R}^3)$). Let us denote by $\mathscr{R}$ the set of all Rollnik potentials. If $V \in \mathscr{R}$ then one knows, see e.g., [7] (Th.X.19, p. 170) that V is infinitesimally small in the sense of quadratic forms in $L^2(\mathbf{R}^3)$ with respect to $-\Delta$, $-\Delta$ being the Laplacian in $L^2(\mathbf{R}^3)$. Thus by the KLMN theorem (see e.g., [7], Th. X. 17, p. 167) there exists a unique self-adjoint operator H with quadratic form domain $Q(H)$ equal to the one of $H_0 \equiv -\Delta$, Δ being the Laplacian on $L^2(\mathbf{R}^3)$, i.e., $Q(H) = D(H_0^{1/2})$, and such that H is the sum

[2] The situation for V not restricted to be $\leqslant 0$ is discussed in [40].

of H_0 and V in the sense of quadratic forms i.e., $(\psi, H\psi) = (\psi, H_0\psi) + (\psi, V\psi)$ for all $\psi \in Q(H)$. H is bounded from below and any domain of essential self-adjointness for H_0 is a form core for H.

Let now $k \in \mathbf{C}$, $k^2 \notin \operatorname{spec}(H) \cup \operatorname{spec}(H_0)$, where spec means the spectrum. It is known (see e.g., [22], Th. II. 34) that if $V \in \mathscr{R}$

$$(H - k^2)^{-1} = G_k - G_k v(1 + uG_k v)^{-1} u G_k\,, \tag{2.2}$$

where $G_k \equiv (H_0 - k^2)^{-1}$, $u \equiv \operatorname{sign} V|V|^{1/2}$, $v \equiv |V|^{1/2}$. Furthermore it is known (see e.g., [22], II.3) that for $\operatorname{Re} k = 0$, $\operatorname{Im} k > 0$ sufficiently big, one has

$$(H - k^2)^{-1} = G_k + \sum_{l=1}^{\infty} (-1)^l (G_k V)^l G_k\,, \tag{2.3}$$

where the series on the right hand side is norm convergent.

REMARK. In fact if $V \in \mathscr{R}$ one has that $H_0 + \lambda V$ is a holomorphic family of type B in the whole λ plane ([24], p. 395).

The kernel $G_k(x - y)$ of $(H_0 - k^2)^{-1}$ is for $\operatorname{Im} k > 0$ given by

$$G_k(x - y) = \frac{e^{ik|x-y|}}{4\pi|x - y|}\,. \tag{2.4}$$

Let from now on $V \in \mathscr{R}$. Then one has for all $\operatorname{Im} k \geqslant 0$

$$\|uG_k v\|_2^2 = (4\pi)^{-2} \iint \frac{|V(x)|\,|V(y)|}{|x - y|^2}\, e^{-2\operatorname{Im} k|x-y|}\, dx\, dy, \tag{2.5}$$

where $\|\ \|_2$ is the Hilbert-Schmidt norm of the kernel $uG_k v$. In particular we have thus

$$\|uG_0 v\|_2^2 = (4\pi)^{-2} \iint \frac{|V(x)|\,|V(y)|}{|x - y|^2}\, dx\, dy.$$

From this it follows that the map $k \to uG_k v$ is analytic from the half-plane $\operatorname{Im} k > 0$ into the space of Hilbert-Schmidt operators with respect to the Hilbert-Schmidt norm and it is continuous from $\operatorname{Im} k \geqslant 0$, $k^2 \notin \operatorname{spec}(H)$ into the same space. Moreover one has, from (2.5) and the Lebesgue dominated convergence theorem, that

$$\|uG_k v\|_2 \to 0 \quad \text{as } |k| \to \infty \text{ in } \operatorname{Im} k > 0,\ k^2 \notin \operatorname{spec}(H).$$

From the same fact we have also that for any $\varepsilon > 0$ there is a $k_\varepsilon \geqslant 0$ such that $\|uG_k v\|_2 \leqslant \varepsilon$ for all $\operatorname{Im} k \geqslant k_\varepsilon$. For $\operatorname{Re} k = 0$ we have $\|uG_k v\|_2 = \|G_k^{1/2} V G_k^{1/2}\|_2$, and this is less or equal ε for $\operatorname{Im} k \geqslant k_\varepsilon$.

This in turn implies

$$|(\varphi, V\varphi)| \leqslant \varepsilon(\varphi, (-\Delta)\varphi) + a_\varepsilon \|\varphi\|^2,$$

for all $\varphi \in D((-\Delta)^{1/2})$, which then shows that $|V|^{1/2}$ is infinitesimally small with respect to $(-\Delta)^{1/2}$.

It is also well known ([25] — [28]), and follows easily from these facts, that if $V \in \mathcal{R}$ then H has at most

$$\min\left((4\pi)^{-2}\iint \frac{V(x)V(y)}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y,\quad (4\pi)^{-2}\iint \frac{V_-(x)V_-(y)}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right) \tag{2.6}$$

negative eigenvalues, where $V_- = \min(V, 0)$.

REMARK. In fact one has that $H_0 + \lambda V$ has at most n negative eigenvalues if $|\lambda| < 1/|\mu_n|$, where $\mu_0 \leqslant \mu_1 \leqslant \ldots$ are the negative eigenvalues of uG_0v. Let us also mention that it is known ([22], Th. III. 13) that the set of positive energy eigenvalues is bounded and nowhere dense.

From the fact that uG_kv is a compact operator for $\mathrm{Im}k \geqslant 0$, $k^2 \notin \mathrm{spec}(H)$ we have from (2.2), by the analytic Fredholm theory (see e.g., [29], [22]), that if $V \in \mathcal{R}$ the singularities in $\mathrm{Im}k > 0$ of $(1 + uG_kv)^{-1}$ are isolated poles of finite order, namely the points (on $\mathrm{Re}k = 0$, H being self-adjoint) where -1 is an eigenvalue of uG_kv, as an operator on $L^2(\mathbf{R}^3)$. The poles in $\mathrm{Im}k > 0$ are in 1-1 correspondence (by $k \to k^2$, $\mathrm{Im}k > 0$) with the strictly negative eigenvalues of H and the multiplicity of the pole corresponds to the multiplicity of the corresponding eigenvalue.

We summarize these results in the following.

PROPOSITION 2.1. *If $V \in \mathcal{R}$ then $H = -\Delta + V$ is defined as a self-adjoint operator given by the sum of $H_0 = -\Delta$ and V in the sense of quadratic forms. H is bounded from below and any domain of essential self-adjointness of H_0 is a form core for H. For k^2 complex such that $k^2 \notin \mathrm{spec}(H) \cup \mathrm{spec}(H_0)$ we have*

$$(H - k^2)^{-1} = G_k - G_kv(1 + uG_kv)^{-1}uG_k\,,$$

where $G_k \equiv (H_0 - k^2)^{-1}$, $u = \mathrm{sign}V|V|^{1/2}$, $v \equiv |V|^{1/2}$, and for $\mathrm{Re}k = 0$, $\mathrm{Im}k > 0$ sufficiently big

$$(H - k^2)^{-1} = G_k + \sum_{l=1}^{\infty} (-1)^l (G_kV)^l G_k\,,$$

where the series on the right hand side is norm convergent.

For $V \in \mathcal{R} \cap L^1(\mathbf{R}^3)$, $k^2 \notin \mathrm{spec}(H)$, $(H - k^2)^{-1}$ is an integral operator of Carleman type i.e., it has a kernel $(H - k^2)^{-1}(x, y)$ such that it is in $L^2(\mathbf{R}^3)$ sepa-

rately in x, y, in fact $(H - k^2)^{-1} - (H_0 - k^2)^{-1}$ *is of trace class. One has*

$$(H-k^2)^{-1}(x, y)=G_k(x-y)-\iint G_k(x-y)v(x_1)(1+uG_kv)^{-1}(x_1, x_2)u(x_2)G_k(x_2-y)\mathrm{d}x_1\mathrm{d}x_2.$$

For $V \in \mathscr{R}$ *we have that* $k \to uG_kv$ *is an analytic map from* $\mathrm{Im}k > 0$ *into the space of Hilbert-Schmidt operators with their natural norms and is a continuous map from* $\mathrm{Im}k \geqslant 0$, $k^2 \notin \mathrm{spec}(H)$, *into the same space.*

The singularities in $\mathrm{Im}k > 0$ *of* $(1 + uG_kv)^{-1}$ *are finitely many isolated poles of finite order on the imaginary axis, the points where* -1 *is an eigenvalue of* uG_kv. *The map* $k \to k^2$ *gives a one-to-one correspondence of the poles in* $\mathrm{Im}k > 0$ *with the strictly negative eigenvalues of H, and the correspondence preserves multiplicities. If* φ *satisfies* $uG_kv\varphi = -\varphi$ *then* ψ *satisfies* $(-\Delta + V)\psi = k^2\psi$ *whenever* $\varphi = -u\psi$. *If* $\mu_0 \leqslant \mu_1 \leqslant \dots$ *are the negative eigenvalues of* uG_0v *then* $-\Delta + \lambda V$ *has at most n negative eigenvalues, if* $|\lambda| < 1/|\mu_n|$, $n = 0, 1, \dots$.

For $\mathrm{Im}k > 0$, $(H - k^2)^{-1}$ *is norm-analytic and* $(H - k^2)^{-1}(x, y)$ *is for* $x \neq y$, *analytic in k except for poles of finite order at the points k where* uG_kv *has the eigenvalue* -1.

REMARK 1. It is also known (see e.g., [22], [31] XI.6, p. 101) that if $V \in \mathscr{R} \cap L^1(\mathbf{R}^3)$ either $(1 + uG_kv)^{-1}$ exists for no k in $\mathrm{Im}k \geqslant 0$ or the set $\mathscr{E}$ of real k such that $(1 + uG_kv)^{-1}$ does not exist as a bounded everywhere defined operator is a bounded closed subset of $\mathbf{R}$ of Lebesgue measure 0. If k^2 is a positive eigenvalue of H then $k^2 \in \mathscr{E}$. If $\|V\|_R < 1$ then $\mathscr{E}$ is void. (In fact in this case H and H_0 are unitarily equivalent.) If V falls off exponentially at infinity in the sense that $e^{\alpha|x|}V(x) \in \mathscr{R}$ for some $\alpha > 0$ then uG_kv can be continued analytically to an Hilbert-Schmidt operator in the whole region $\mathrm{Im}k > -\frac{\alpha}{2}$, hence in particular $\mathrm{Im}k = 0$ is in this case in the analyticity domain and the analytic Fredholm theory yields the discreteness of $\mathscr{E}$. If V has compact support then by (2.4) $k \to uG_kv$ is a Hilbert-Schmidt-valued, analytic map for all $k \in \mathbf{C}$. In this case the only singularities in k of $(1 + uG_kv)^{-1}$ are isolated poles of finite order where -1 is an eigenvalue of uG_kv. Consequently in this case the only singularities of $(H - k^2)^{-1}$ and $(H - k^2)^{-1}(x, y)$ for $x \neq y$ as maps from $k \in \mathbf{C}$ into bounded operators resp. functions are poles of finite order at the points k where uG_kv has the eigenvalue -1. The poles in the half-plane $\mathrm{Im}k < 0$ (the so called "unphysical half plane") are called *resonances* of H. We recall that we have already identified, in Proposition 2.1, the poles in $\mathrm{Im}k > 0$ (the so called "physical half plane") with the negative eigenvalues at k^2 of H.

REMARK 2. For any $V \in \mathscr{R}$ one can also show ([32], XIII, p. 100) that the number of eigenvalues of H in the interval $(-\infty, 0]$ is $\leqslant \left((4\pi)^{-2}\iint \frac{V_-(x)V_-(y)}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right)$. In particular thus the multiplicity of the eigenvalue -1 of uG_0v is finite. In the case when $V \in \mathscr{R}$, $V \geqslant 0$ we see that there are no eigenvalues which are negative or zero.

Throughout this paper we shall use the following.

DEFINITION. Let $V \in \mathscr{R}$. We say that *H has a zero energy resonance* if -1 is an eigenvalue of the operator uG_0v in $L^2(\mathbf{R}^3)$. We say that the zero energy resonance is *simple* if -1 is a simple eigenvalue of uG_0v.

Assume now that $V \in \mathscr{R}$ and that $-\Delta+V$ has a zero energy resonance.[3] By the fact mentioned above, this resonance has a finite multiplicity. Let φ be an eigenfunction to the eigenvalue -1 of uG_0v i.e.,

$$\varphi = -uG_0v\varphi. \tag{2.7}$$

Let $\psi \equiv G_0v\varphi$, then we get from (2.7)

$$\psi = -G_0V\psi \tag{2.8}$$

i.e.,

$$(-\Delta + V)\psi = 0. \tag{2.9}$$

We shall call ψ a resonance function. Note that ψ need not be in $L^2(\mathbf{R}^3)$, in general. Since $u = \operatorname{sign}V \cdot v$ and $v = \operatorname{sign}V \cdot u$ we have

$$vG_0u = \operatorname{sign}V(uG_0v)\operatorname{sign}V, \tag{2.10}$$

so that uG_0v is unitarily equivalent to its adjoint vG_0u. Hence, setting $\tilde{\varphi} \equiv -\operatorname{sign}V \cdot \varphi$, we have

$$\tilde{\varphi} = -vG_0u\tilde{\varphi}. \tag{2.11}$$

If the zero energy resonance is simple, i.e., -1 is a simple eigenvalue of uG_0v, then we have from the spectral theory of compact operators that we may normalize φ in such a way that $(\tilde{\varphi}, \varphi) = -1$, where $(\,,)$ is the scalar product in $L^2(\mathbf{R}^3)$. Then we have, in the norm sense,

$$\lim_{\varepsilon \to 0} \varepsilon(1 + uG_0v + \varepsilon 1)^{-1} = -|\varphi\rangle\langle\tilde{\varphi}| = |\varphi\rangle\langle\tilde{\varphi}|/(\tilde{\varphi}, \varphi), \tag{2.12}$$

where $|\varphi\rangle\langle\tilde{\varphi}|$ denotes the spectral projection corresponding to the simple eigenvalue -1, i.e.,

$$|\varphi\rangle\langle\tilde{\varphi}|f = (\tilde{\varphi}, f)\varphi$$

for all $f \in L^2(\mathbf{R}^3)$.

[3] For a recent discussion of when such a situation can arise see [37].

Moreover the same theory gives that if A is a bounded operator such that $(\tilde{\varphi}, A\varphi) \neq 0$ then

$$\lim_{\varepsilon \to 0} \varepsilon(1 + uG_0v + \varepsilon A)^{-1} = (\tilde{\varphi}, A\varphi)^{-1} |\varphi\rangle \langle\tilde{\varphi}|. \tag{2.13}$$

Let us now assume that we have a zero energy resonance which is not necessarily simple. If $V \leqslant 0$ then $u = -v$, hence $uG_0v = -vG_0v$, hence uG_0v is symmetric in this case. Moreover 1 is an eigenvalue (of finite multiplicity) of vG_0v.

REMARK. The case when $V \geqslant 0$ is trivial since in this case there do not exist any zero energy resonance functions. The situation for V not necessarily of a definite sign is discussed in [40].

Let us now assume that $V \leqslant 0$. Call P the orthogonal projection onto the eigenvalue 1 of vG_0v. Let $\varphi_1, \ldots, \varphi_n$ be an orthonormal base in the corresponding subspace $PL^2(\mathbf{R}^3)$. Then by the spectral theory of symmetric compact operators we get that if A is a bounded operator such that the $n \times n$ matrix $(\varphi_i, A\varphi_j)$, $i, j = 1, \ldots, n$ is non singular then

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon(1 + uG_0v + \varepsilon A)^{-1} &= \lim_{\varepsilon \to 0} \varepsilon(1 - vG_0v + \varepsilon A)^{-1} = \\ &= \sum_{i,j=1}^{n} |\varphi_i\rangle [(\varphi_j, A\varphi_i)]^{-1} \langle\varphi_j|, \end{aligned} \tag{2.14}$$

where $[(\varphi_j, A\varphi_i)]^{-1}$ is the inverse matrix to the matrix $(\varphi_i, A\varphi_j)$.

Thus we have the following:

LEMMA 2.2. *Assume that $V \in \mathcal{R}$ and we are either in the situation of no zero energy resonance or that we have a zero energy resonance which is simple or that we have a potential $V \leqslant 0$. Then if A is a bounded operator we have in the norm sense*

$$\lim_{\varepsilon \to 0} \varepsilon(1 + uG_0v + \varepsilon A + o(\varepsilon))^{-1} = B(A),$$

where $o(\varepsilon)$ is any bounded operator such that $\varepsilon^{-1}\|o(\varepsilon)\| \to 0$ as $\varepsilon \to 0$. The bounded operator $B(A)$ is the zero operator if there is no zero energy resonance; it is equal to

$$B(A) = \sum_{i,j=1}^{n} |\varphi_i\rangle [(\varphi_j, A\varphi_i)]^{-1} \langle\varphi_j|$$

if V has a definite sign, and where φ_i, $i = 1, \ldots, n$ is an orthonormal base for the eigenspace of vG_0v to the eigenvalue 1, and it is assumed that $(\varphi_i, A\varphi_j)$ is a non singular matrix. If there is a simple zero energy resonance then

$$B(A) = |\varphi\rangle (\tilde{\varphi}, A\varphi)^{-1} \langle\tilde{\varphi}|,$$

where $|\varphi\rangle \langle\tilde{\varphi}|$ *denotes the projection onto the eigenspace to the eigenvalue* -1 *of* uG_0v, *and it is assumed that* $(\tilde{\varphi}, A\varphi) \neq 0$.

For simplicity we shall say that $V \in \mathscr{R}$ is *admissible* if either of the cases of Lemma 2.2 occurs, i.e., either V has no zero energy resonance or it has a zero energy resonance but this is simple or $V \leqslant 0$.

In the following we shall assume that $V \in \mathscr{R} \cap L^1(\mathbf{R}^3)$. We shall call *short range potentials* the elements V of $\mathscr{R} \cap L^1(\mathbf{R}^3)$ which satisfy

$$(1 + |x|)^2 V(x) \in L^1(\mathbf{R}^3).$$

If V is a short range potential then we have $u, v \in L^2(\mathbf{R}^3)$ and it is easily verified by expansion that

$$uG_{\varepsilon k}v = uG_0v + \varepsilon \frac{ik}{4\pi} |u\rangle \langle v| + O(\varepsilon^2) \tag{2.15}$$

for all $\mathrm{Im}k \geqslant 0$, in the norm sense. Let now $\lambda(\cdot)$ be a differentiable function on $\mathbf{R}$ such that $\lambda(0) = 1$. Then we have from (2.15)

$$\lambda(\varepsilon)uG_{\varepsilon k}v = uG_0v + \varepsilon\lambda'(0)uG_0v + \frac{ik}{4\pi} |u\rangle \langle v| + O(\varepsilon^2). \tag{2.16}$$

From Lemma 2.2 we then have, for V admissible, with

$$A \equiv \frac{ik}{4\pi} |u\rangle \langle v| + \lambda'(0)uG_0v, \tag{2.17}$$

$$\lim_{\varepsilon \to 0} \varepsilon(1 + \lambda(\varepsilon)uG_{\varepsilon k}v)^{-1} = B(A). \tag{2.18}$$

Suppose now we have a zero energy resonance and let φ be any eigenfunction to the eigenvalue -1 of uG_0v, then we have by the definition $\psi = G_0v\varphi$:

$$-\Delta\psi = v\varphi. \tag{2.19}$$

From this and (2.9) we obtain

$$v\varphi = -V\psi, \tag{2.20}$$

which shows that one has, since $v, \varphi \in L^2(\mathbf{R}^3)$, $V\psi \in L^1(\mathbf{R}^3)$.

Let us now assume that the zero energy resonance is simple. With $\tilde{\varphi}$ defined as in Lemma 2.2 and choosing the normalization $(\varphi, \tilde{\varphi}) = -1$ we have

$$\begin{aligned} -1 &= (\tilde{\varphi}, \varphi) = -(vG_0u\tilde{\varphi}, \varphi) = \\ &= -(G_0u\tilde{\varphi}, v\varphi) = (G_0u\tilde{\varphi}, V\psi) = \\ &= -(G_0V\psi, V\psi) \end{aligned} \tag{2.21}$$

where we used (2.20), (2.11) and

$$v\varphi = u\tilde{\varphi} = -VG_0u\tilde{\varphi} = -V\psi, \tag{2.22}$$

which follows from (2.11) using the definition $\tilde{\varphi} = -\operatorname{sign} V\varphi$.

But using (2.8) we then get that (2.21) is equal to $(\psi, V\psi)$, hence

$$-1 = (\psi, V\psi). \tag{2.23}$$

In the case when the zero energy resonance is not necessarily simple but $V \leqslant 0$, then, with the φ_i as in Lemma 2.2 and setting $\psi_i = G_0v\varphi_i$ (these functions are then again called resonance functions), we have similarly as above that

$$(\psi_i, V\psi_j) = -\delta_{ij}. \tag{2.24}$$

We now see that because of the value (2.17) of A the expression $(\tilde{\varphi}, A\varphi)$ entering $B(A)$ by (2.18) becomes

$$(\tilde{\varphi}, A\varphi) = \left(\tilde{\varphi}, \frac{ik}{4\pi}|u\rangle\langle v|\varphi\right) + (\tilde{\varphi}, \lambda'(0)uG_0v\varphi). \tag{2.25}$$

The first term, using (2.7), (2.11) and $\psi = G_0v\varphi = G_0u\tilde{\varphi}$, becomes

$$\begin{aligned} &\frac{ik}{4\pi}(-vG_0u\tilde{\varphi}, u)(v, -uG_0v\varphi) = \\ &= \frac{ik}{4\pi}(v\psi, u)(v, u\psi) = \frac{ik}{4\pi}(\psi, V)(V, \psi), \end{aligned} \tag{2.26}$$

which is finite since $V\psi \in L^1(\mathbf{R}^3)$.

The second term in (2.25) becomes

$$(\varphi, \lambda'(0)uG_0v\varphi) = -\lambda'(0)(\tilde{\varphi}, \varphi) = -\lambda'(0)(\psi, V\psi) \tag{2.27}$$

where we used (2.7) and (2.21).

Inserting this and (2.26) into (2.25) we get

$$(\tilde{\varphi}, A\varphi) = -\lambda'(0)(\psi, V\psi) + \frac{ik}{4\pi}(\psi, V)(V, \psi). \tag{2.28}$$

This then yields for $B(A)$ in (2.18)

$$B(A) = -\left(\lambda'(0)(\psi, V\psi) - \frac{ik}{4\pi}(\psi, V)(V, \psi)\right)^{-1}|\varphi\rangle\langle\tilde{\varphi}|. \tag{2.29}$$

Similarly in the case when one has a zero energy resonance and $V \leqslant 0$ and ψ_j, $j = 1, \ldots, n$ are the resonance functions with the normalization (2.24) we get

$$B(A) = \sum_{i,j=1}^{n} |\varphi_i\rangle \left[\lambda'(0) + \frac{ik}{4\pi}(\psi_i, V)(V, \psi_j)\right]^{-1} \langle\varphi_j| . \tag{2.30}$$

We have thus proven the

LEMMA 2.3. *Assume that V is a short range potential and let $\lambda(\varepsilon)$ be a differentiable function with $\lambda(0) = 1$. Then we have in the operator norm*

$$\lim_{\varepsilon \to 0} \varepsilon(1 + \lambda(\varepsilon)uG_{\varepsilon k}v)^{-1} = B(k).$$

$B(k)$ is zero if $-\Delta + V$ has no zero energy resonance. If $-\Delta + V$ has a simple zero energy resonance we have that both $V\psi$ and $V|\psi|^2$ are in $L^1(\mathbf{R}^3)$. In this case

$$B(k) = -\left(\lambda'(0)(\psi, V\psi) - \frac{ik}{4\pi}(\psi, V)(V, \psi)\right)^{-1} |\varphi\rangle \langle\tilde{\varphi}|,$$

where $|\varphi\rangle\langle\tilde{\varphi}|$ is the spectral projection to the eigenvalue -1 of uG_0v (and in particular $\psi = G_0v\varphi$). If $-\Delta + V$ has a zero energy resonance which is not necessarily simple but $V \leqslant 0$ then calling φ_i, $i = 1, \ldots, n$ an orthonormal base in the spectral subspace to the eigenvalue -1 of uG_0v and calling $\psi_i = G_0v\varphi_i$ the corresponding resonance functions, then we have that $V|\psi_i|^2$ and $V\psi_i$ are in $L^1(\mathbf{R}^3)$, $(\psi_i, V\psi_j) = -\delta_{ij}$ and

$$B(k) = \sum_{i,j=1}^{n} |\varphi_i\rangle \left[\lambda'(0)\delta_{ij} + \frac{ik}{4\pi}(\psi_i, V)(V, \psi_j)\right]^{-1} \langle\varphi_j| .$$

REMARK 1. In above expression for $B(k)$ we have assumed that the inverse of the matrix $\lambda'(0)\delta_{ij} + \frac{ik}{4\pi}(\psi_i, V)(V, \psi_j)$ exists. If $\lambda'(0) = 0$ and $n > 1$ this assumption is violated and it can be shown ([40]) that in this case the limit $B(k)$ does not exist. The complete discussion is given in [40].

REMARK 2. The condition $V \in L^1(\mathbf{R}^3)$ is used in this lemma to establish $V\psi \in L^1(\mathbf{R}^3)$, which follows from (2.20). Note that the condition that (V, ψ) exists seems to be needed, because this expression appears explicitely in $B(k)$, however the result of Lemma 2.3 in the case of no zero energy resonance holds also under the sole assumption $V \in \mathscr{R}$.

3. POINT INTERACTIONS AS LIMITS OF SHORT RANGE INTERACTIONS

Let V be a short range potential and let U_ε be the unitary dilation group in $L^2(\mathbf{R}^3)$ i.e., for $\varepsilon > 0$

$$(U_\varepsilon\psi)(x) \equiv \varepsilon^{-3/2}\psi\left(\frac{x}{\varepsilon}\right). \tag{3.1}$$

We see that $U_\varepsilon V U_\varepsilon^{-1} = V\left(\frac{x}{\varepsilon}\right)$, $U_\varepsilon \Delta U_\varepsilon^{-1} = \varepsilon^2 \Delta$ and

$$U_\varepsilon G_k U_\varepsilon^{-1} = \varepsilon^{-2} G_{\varepsilon^{-1}k}. \tag{3.2}$$

Let $H_\varepsilon \equiv -\Delta + \lambda V_\varepsilon$, $V_\varepsilon \equiv \varepsilon^{-2} V\left(\frac{x}{\varepsilon}\right)$ and $H = -\Delta + \lambda V$, then we have

$$H_\varepsilon = \varepsilon^{-2} U_\varepsilon H U_\varepsilon^{-1}, \tag{3.3}$$

and for $(\varepsilon k)^2 \notin \operatorname{spec} H$:

$$(H_\varepsilon - k^2)^{-1} = \varepsilon^2 U_\varepsilon (H - (\varepsilon k)^2)^{-1} U_\varepsilon^{-1}. \tag{3.4}$$

Let now $k_0 \geqslant 0$ be such that, by (2.4), $\|uG_k v\|_2 < 1$ for all $\operatorname{Im} k > k_0$. We shall now prove the following

LEMMA 3.1. *Let V be an admissible short range potential. Let λ be a non negative differentiable function $\lambda(\varepsilon)$ of ε, with $\lambda(0) = 1$. Let H_ε be defined by $-\Delta + \lambda(\varepsilon)\varepsilon^{-2} V\left(\frac{x}{\varepsilon}\right)$; then $(H_\varepsilon - k^2)^{-1}$ converges weakly in $L^2(\mathbf{R}^3)$ as $\varepsilon \to 0$, for all k such that $\operatorname{Im} k^2 \neq 0$.*

Proof. From the resolvent equation we have by iteration for $\operatorname{Im} k > 0$

$$(H - k^2)^{-1} = G_k - \lambda G_k V G_k + \lambda^2 G_k V (H - k^2)^{-1} V G_k. \tag{3.5}$$

This together with (3.4) gives

$$(H_\varepsilon - k^2)^{-1} = G_k - \lambda G_k V_\varepsilon G_k + \varepsilon^{-2} \lambda^2 G_k U_\varepsilon V (H - (\varepsilon k)^2)^{-1} V U_\varepsilon^{-1} G_k. \tag{3.6}$$

Denoting by $(H - (\varepsilon k)^2)^{-1}(x, y)$ the kernel of $(H - (\varepsilon k)^2)^{-1}$, we can rewrite (3.6) in the form

$$\begin{aligned}(H_\varepsilon - k^2)(x, y) &= G_k(x - y) - \lambda(\varepsilon)\, \varepsilon \int G_k(x - \varepsilon x_1) V(x_1) G_k(\varepsilon x_1 - y) \mathrm{d}x_1 + \\ &+ \lambda(\varepsilon)^2 \varepsilon \iint G_k(x - \varepsilon x_1) V(x_1) (H - (\varepsilon k)^2)^{-1}(x_1, x_2) V(x_2) G_k(\varepsilon x_2 - y) \mathrm{d}x_1 \mathrm{d}x_2.\end{aligned} \tag{3.7}$$

Now the trace norm of $\int G_k(x - \varepsilon x_1) V(x_1) G(\varepsilon x_1 - y) \mathrm{d}x_1$ is bounded by

$$\int |G_k(x - \varepsilon x_1)|^2 |V(x_1)| \, \mathrm{d}x_1 \mathrm{d}x = \|V\|_1 \|G_k\|_2^2, \tag{3.8}$$

where $\| \;\|_p$ is the L^p-norm, $p = 1,2$. The right hand side of (3.8) is finite independent of ε, by the assumption $\mathrm{Im}k > 0$ and the estimate

$$\|G_k\|_2^2 = \int_{\mathbf{R}^3} |G_k(x)|^2 \mathrm{d}x \leqslant \frac{1}{(4\pi)^2} \int_{\mathbf{R}^3} \frac{\mathrm{e}^{-2\mathrm{Im}k|x|}}{|x|^2} \mathrm{d}x < \infty. \tag{3.9}$$

Hence the norm of the second term in (3.7) is bounded by $\lambda(\varepsilon)\varepsilon\|V\|_1 \|G_k\|_2^2 \to 0$ as $\varepsilon \to 0$. Using (2.2) the third term in (3.7) can be written as

$$\lambda(\varepsilon)^2 \varepsilon \iint G_k(x - \varepsilon x_1)V(x_1)G_{\varepsilon k}(x_1 - x_2)V(x_2)G_k(\varepsilon x_2 - y)\mathrm{d}x_1\mathrm{d}x_2 -$$

$$- \lambda(\varepsilon)^3 \varepsilon \iiiint G_k(x - \varepsilon x_1)V(x_1)G_{\varepsilon k}(x_1 - x_2)v(x_2)(1 + \lambda uG_{\varepsilon k}v)^{-1}(x_2, x_3) \cdot \tag{3.10}$$

$$\cdot u(x_2)G_{\varepsilon k}(x_3 - x_4)V(x_4)G_k(\varepsilon x_4 - y)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 .$$

Now the first term in (3.10) has the form

$$\lambda^2\varepsilon \iint [G_k(x - \varepsilon x_1)v(x_1)][u(x_1)G_{\varepsilon k}(x_1 - x_2)v(x_2)][u(x_2)G_k(\varepsilon x_2 - y)]\mathrm{d}x_1\mathrm{d}x_2 . \tag{3.11}$$

This is of the form $\lambda(\varepsilon)^2 \varepsilon A_\varepsilon B_\varepsilon C_\varepsilon$, where A_ε, B_ε, C_ε are the operators with kernels

$$A_\varepsilon(x, y) \equiv G_k(x - \varepsilon y)v(y),$$

$$B_\varepsilon(x, y) \equiv u(x)G_{\varepsilon k}(x - y)v(y),$$

and

$$C_\varepsilon(x, y) \equiv u(x)G_k(\varepsilon x - y)$$

respectively. The Hilbert-Schmidt norms of A_ε and C_ε are given by (3.8), hence are bounded independently of ε. The Hilbert-Schmidt norm of B_ε is bounded by $\|V\|_{\mathbf{R}}$. Hence (3.11), which was the first term in (3.10), converges to zero in operator norm as $\varepsilon \to 0$. The second term in (3.10) is of the form

$$-\lambda(\varepsilon)^3 A_\varepsilon B_\varepsilon D_\varepsilon B_\varepsilon C_\varepsilon , \tag{3.12}$$

where $D_\varepsilon \equiv \varepsilon(1 + \lambda uG_{\varepsilon k}v)^{-1}$.

From Proposition 2.1 we know the Hilbert-Schmidt continuity of uG_kv for $\mathrm{Im}k \geqslant 0$, $k^2 \notin \mathrm{spec}(H)$, hence $B_\varepsilon \to B_0$ as $\varepsilon \to 0$ in the Hilbert-Schmidt norm. From Lemma 2.3 we have that in the case of an admissible V, D_ε converges in the operator norm sense, as $\varepsilon \to 0$, to an operator $B(k)$. Above, in (3.8), we have already seen that the Hilbert-Schmidt norms of A_ε, C_ε are bounded independently of ε.

On the other hand by

$$(G_k f)(x) = \frac{1}{4\pi} \int \frac{e^{ik|x-y|}}{|x-y|} f(y) dy \tag{3.13}$$

we see that for $\mathrm{Im}k > 0$ we have that G_k maps the space $\mathscr{C}_\infty(\mathbf{R}^3)$ of continuous functions vanishing at infinity into itself. Therefore if $f \in \mathscr{C}_0(\mathbf{R}^3)$, $\mathscr{C}_0(\mathbf{R}^3)$ being the space of continuous function of compact support, we have that $(G_k f)(\varepsilon x)$ converges in sup-norm as $\varepsilon \to 0$ to $(G_k f)(0)$. This implies that $C_\varepsilon f(x)$ converges as $\varepsilon \to 0$ (strongly) to $u(x) \int G_k(y) f(y) dy$. Thus, C_ε being norm bounded uniformly in ε, we have that C_ε converges strongly as a bounded operator on $L^2(\mathbf{R}^3)$, as $\varepsilon \to 0$, to $|u><G_k|$, where $|u><G_k|$ is the operator defined by

$$(|u><G_k|\, g)(x) = u(x)\,(G_k(\cdot), g), \quad \text{for } g \in L^2(\mathbf{R}^3),$$

$(,)$ being the scalar product in $L^2(\mathbf{R}^3)$ (note that $G_k(\cdot) \in L^2(\mathbf{R}^3)$, for $\mathrm{Im}k > 0$, by (3.9)).

In the same way we prove that $A_\varepsilon^* \to |v><G_k|$, strongly as bounded operators in $L^2(\mathbf{R}^3)$, as $\varepsilon \to 0$, where A_ε^* is the adjoint of A_ε. From these results we see that for any $f, g \in L^2(\mathbf{R}^3)$ we have

$$(f, A_\varepsilon B_\varepsilon D_\varepsilon B_\varepsilon C_\varepsilon g) = (A_\varepsilon^* f, B_\varepsilon D_\varepsilon B_\varepsilon C_\varepsilon g) \to (A_0^* f, B_0 D_0 B_0 C_0 g) \tag{3.14}$$

as $\varepsilon \to 0$, where $A_0^* \equiv |v><G_k|$, $B_0 = uG_0 v$, $D_0 = B(k)$, $C_0 = |u><G_k|$. This together with the results concerning (3.11) and (3.12) gives the weak convergence of $(H_\varepsilon - k^2)^{-1}$ as $\varepsilon \to 0$ for $\mathrm{Im}k > k_0$, $\mathrm{Re}k = 0$. On the other hand by (3.4) we have that $(H_\varepsilon - k^2)^{-1}$ is analytic in k for all $\varepsilon > 0$ when $(H - (\varepsilon k)^2)^{-1}$ has these properties. We know by Proposition 2.1 that if e.g., $V \in \mathscr{R}$ hence a fortiori for our V, $(H - k^2)^{-1}$ is analytic in k for $\mathrm{Im}k > 0$ except for finitely many poles of finite order on $\mathrm{Im}k > 0$, $\mathrm{Re}k = 0$. In particular thus $(H_\varepsilon - k^2)^{-1}$ is analytic in k^2 for all k^2 with $k^2 \notin [-k_0^2, \infty)$.

This finishes the proof of the lemma.

REMARK 1. As in Remark 1 of Section 2 the case $\lambda'(0) = 0$, $n > 1$ is excluded by our present assumptions. However the result itself is valid also in this case, as proven in [40].

Let us now look at the proof of the preceding lemma, in order to identify the limit. We saw that the second term in (3.7) vanishes in the limit $\varepsilon \to 0$, the first terms is independent of ε, and the third term converges as $\varepsilon \to 0$ to the limit of (3.12) as $\varepsilon \to 0$, which was computed to be, in the weak sense,

$$|G_k><v|\, uG_0 v\, B(k) uG_0\, v\, |u><G_k|. \tag{3.15}$$

From Lemma 2.3 we have that $B(k) = 0$, hence (3.15) is zero if there is no zero energy resonance; hence in this case we have simply $(H_\varepsilon - k^2)^{-1} \to G_k$. Assume now that V is such that there is a simple zero energy resonance. Then again $B(k)$ is given by the corresponding case in Lemma 2.3 i.e.

$$B(k) = \left(\frac{ik}{4\pi}(\psi, V)(V, \psi) - \lambda'(0)(\psi, V\psi)\right)^{-1} |\varphi\rangle \langle \tilde{\varphi}|. \tag{3.16}$$

By (2.20) we have $v\varphi = -V\psi$ and since, by (2.8), $\psi = -G_0 V\psi$, we have, recalling that $uv = V$:

$$(v, uG_0 v\varphi) = -(V, G_0 V\psi) = (V, \psi). \tag{3.17}$$

In a corresponding way we find

$$(\tilde{\varphi}, uG_0 vu) = (\psi, V). \tag{3.18}$$

From (3.16), (3.17), (3.18) inserted in (3.15) we get

$$\begin{aligned} &|G_k\rangle (V, \psi)\left[\left(\frac{ik}{4\pi}\right)(\psi, V)(V, \psi) - \lambda'(0)(\psi, V\psi)\right]^{-1}(\psi, V)\langle G_k| = \\ &= |G_k\rangle \left(\frac{ik}{4\pi} - \alpha\right)^{-1}\langle G_k|, \end{aligned} \tag{3.19}$$

with

$$\alpha = \lambda'(0)(\psi, V\psi)/|(\psi, V)|^2. \tag{3.20}$$

It was proven in [13] that the sum of (3.19) and G_k is the resolvent of the self-adjoint operator $-\Delta_\alpha$, where $-\Delta_\alpha$ is defined as that self-adjoint extension of $-\Delta$ restricted to $\mathscr{C}_0^2(\mathbf{R}^3 - \{0\})$ that is given by the boundary condition $u'(0) = 4\pi\alpha u(0)$ in the space $L^2(\mathbf{R}^+, r^2 dr)$, for $-d^2/dr^2$ (one decomposes $-\Delta$ and $L^2(\mathbf{R}^3)$ in radial and angular coordinates).

The last case we have to examine is the one when $V \leqslant 0$ and there are zero energy resonances. In this case we have (3.19) in a corresponding way, using Lemma 2.3 but with α replaced by

$$\alpha = -\lambda'(0)\left(\sum_i |(\psi_i, V)|^2\right)^{-1}, \tag{3.21}$$

where we have used the normalization as in Lemma 2.3, $(\psi_i, V\psi_j) = -\delta_{ij}$. Hence we get in all cases that the weak limit of the resolvent $(H_\varepsilon - k^2)^{-1}$ is again a resolvent, namely $(-\Delta_\alpha - k^2)^{-1}$, with $\alpha = 0$ in the case of no zero energy resonance and α given by (3.20) resp. (3.21) in the case where there is a simple energy resonance resp. there is a zero energy resonance and $V \leqslant 0$.

It is a standard result (see e.g., [30], Ch. VIII, p. 284) that if the weak limit of a resolvent is a resolvent then the limit is a strong one. Hence, recalling also the Lemma 2.3, we have proven the following:

THEOREM 3.1. *Let V be a short range potential and let $\lambda(\varepsilon)$ be differentiable with $\lambda(0) = 1$. Then if $-\Delta + V$ has no zero energy resonance then* $\left(-\Delta + \lambda(\varepsilon)\varepsilon^{-2}V\left(\frac{x}{\varepsilon}\right) - k^2\right)^{-1}$ *converges strongly as* $\varepsilon \to 0$ to $(-\Delta - k^2)^{-1}$ *for all k such that* $\text{Im}k^2 \neq 0$. *If $-\Delta+V$ has a zero energy resonance then this resonance is of finite multiplicity and if we call ψ_i, $i = 1, \ldots, n$ the corresponding resonance functions (satisfying $(-\Delta + V)\psi_i = 0$) then we have that $V|\psi_i|^2$ as well as $V\psi_i$ are in $L^1(\mathbf{R}^3)$. Moreover if the resonance is simple* (i.e., $n = 1$) *or $V \leqslant 0$ then the strong limit as $\varepsilon \to 0$ of* $(-\Delta + \lambda(\varepsilon)\varepsilon^{-2}V\left(\frac{x}{\varepsilon}\right) - k^2)^{-1}$ is

$$(-\Delta_\alpha - k^2)^{-1} = G_k - |G_k > \left(\frac{ik}{4\pi} - \alpha\right)^{-1} < G_k|,$$

where $\alpha = -\lambda'(0)\left(\sum_{i=1}^{n} |(V, \psi_i)|^2\right)^{-1}$, *the resonance functions being normalized according to* $(\psi_i, V\psi_j) = -\delta_{ij}$.

From the expression for

$$(-\Delta_\alpha - k^2)^{-1} = G_k - |G_k > \left(\frac{ik}{4\pi} - \alpha\right)^{-1} < G_k| \tag{3.22}$$

of Theorem 3.1 we see that, for $\alpha \neq 0$, $-\Delta_\alpha$ has exactly one resonance (in $\text{Im}k \leqslant 0$) resp. one eigenvalue k^2 (in $\text{Im}k > 0$) at the point $k^2 = -16\pi^2\alpha^2$ for $k = -4\pi i\alpha$. The first case is realized when $\lambda'(0) \leqslant 0$ and the second one when $\lambda'(0) > 0$. In the second case and because of the strong resolvent convergence $\left(-\Delta + \lambda(\varepsilon)\varepsilon^{-2}V\left(\frac{x}{\varepsilon}\right) - z\right)^{-1} \to (-\Delta_\alpha - z)^{-1}$, $z \notin [-k_0^2, \infty)$ we see that an eigenvalue E_ε of $-\Delta + \lambda(\varepsilon)\varepsilon^{-2} V\left(\frac{x}{\varepsilon}\right)$ must converge as $\varepsilon \to 0$ to the eigenvalue $-16\pi^2\alpha^2$ of $-\Delta_\alpha$ (by the principle of non sudden expansion of the limiting operator, see e.g., [24], VIII, 2, Th. 1.14). We formulate this as a

COROLLARY. *In the case when V has at least one zero energy resonance then we have that the limit operator $-\Delta_\alpha$ has exactly one resonance at $k = -4\pi i\alpha$ if $\lambda'(0) \leqslant 0$ and exactly one negative eigenvalue $-16\pi^2\alpha^2$ if $\lambda'(0) > 0$. Thus in the case $\lambda'(0) > 0$*

there exists an eigenvalue E_ε *of* $-\Delta + \lambda(\varepsilon)\varepsilon^{-2} V\left(\frac{x}{\varepsilon}\right)$ *which converges as* $\varepsilon \to 0$ *to the unique eigenvalue* $-16\pi^2\alpha^2$ *of* $-\Delta_\alpha$.

REMARK 2. The same observation as in Remark 1 holds here and in particular we refer to [40] for a discussion of the case when the condition $V \leqslant 0$ is not necessarily fulfilled.

Let us now assume that V has compact support. By what we recalled in Remark 1 after Proposition 2.1 in this case $(-\Delta + V - k^2)^{-1}(x, y)$ is, for $x \neq y$, analytic in the whole complex k-plane with only finite poles at the points k^2 where -1 is an eigenvalue of uG_kv (these are the resonances of $-\Delta + V$ for $\mathrm{Im}k \leqslant 0$ and the negative eigenvalues k^2 for $\mathrm{Im}k > 0$). From the strong convergence of Theorem 3.1 we see that $(H_\varepsilon - k^2)^{-1}(x, y)$ converges for $x \neq y$ pointwise almost everywhere, and hence by continuity everywhere, by subsequences as $\varepsilon \to 0$ to $(-\Delta_\alpha - k^2)^{-1}(x, y)$. Hence in particular the poles in k converge to those of $(-\Delta_\alpha - k^2)^{-1}(x, y)$, thus the eigenvalues and resonances of H_ε converge to those of $-\Delta_\alpha$. However the latter has exactly one resonance (if $\lambda'(0) \leqslant 0$) or one eigenvalue (if $\lambda'(0) > 0$), at $k^2 = -16\pi^2\alpha^2$. In particular if $-\Delta + V$ has no zero energy resonance, i.e., $-\Delta_\alpha$ is to be replaced by $-\Delta$, then the eigenvalues and resonances of H_ε have to tend to the poles of $(-\Delta - k^2)^{-1}(x, y)$, $x \neq y$, i.e., to infinity (since $(-\Delta - k^2)^{-1}(x, y)$, $x \neq y$ is analytic in the whole complex plane). Similarly, if $-\Delta + V$ has a zero energy resonance and we suppose that this is either simple or we have $V \leqslant 0$ then, for $\lambda'(0) > 0$, $-\Delta_\alpha$ has no resonance and exactly one eigenvalue at $k^2 = -16\pi^2\alpha^2$, hence all the resonances and all but one of the eigenvalues of H_ε have to tend to infinity as $\varepsilon \to 0$, and that eigenvalue converges to $k^2 = -16\pi^2\alpha^2$. The case $\lambda'(0) \leqslant 0$ is discussed similarly, and we arrive thus to the following:

THEOREM 3.2. *Let* V, λ *be as in Theorem* 3.1 *and suppose that* V *has compact support. Then if* $-\Delta + V$ *has no zero energy resonance then all the negative eigenvalues and resonances of* $-\Delta + \varepsilon^{-2}\lambda(\varepsilon)V\left(\frac{x}{\varepsilon}\right)$ *tend to infinity as* $\varepsilon \to 0$.

If V *is such that* $-\Delta + V(x)$ *has a zero energy resonance and either* $V \leqslant 0$ *or the zero energy resonance is simple, then if* $\lambda'(0) > 0$ *all the resonances of* $-\Delta + \varepsilon^{-2}\lambda(\varepsilon)V\left(\frac{x}{\varepsilon}\right)$ *tend to infinity as* $\varepsilon \to 0$ *and so do all but exactly one of the eigenvalues, and this one tends to* $k^2 = -16\pi^2\lambda'(0)^2(\sum_i |(V, \psi_i)|^2)^{-2}$, *(where* ψ_i *are the resonance functions such that* $(-\Delta + V)\psi_j = 0$, $(\psi_i, V\psi_j) = -\delta_{ij}$*). On the other hand, if* $\lambda'(0) \leqslant 0$ *then all the eigenvalues tend to infinity together with all but one of the resonances, and this one tends to* $k = -4\pi i\lambda'(0)(\sum_j |(V, \psi_j)|^2)^{-1}$.

REMARK. $-\Delta_\alpha$ is the Hamiltonian for a point interaction at the origin, of strength α (see [13]). The translate $-\Delta_{(x_0,\alpha)}$ of $-\Delta_\alpha$ by the vector $x_0 \in \mathbf{R}^3$ is then the Hamiltonian for a point interaction at x_0 of strength α. By (3.19) we have

$$(3.23)\quad (-\Delta_{(x_0,\alpha)}-k^2)^{-1}(x,y) = G_k(x-y) - \left(\frac{ik}{4\pi} - \alpha\right)^{-1} G_k(x-x_0)G_k(y-x_0).$$

These interactions and superpositions of these were studied in [12], [13], [14], [15]. In particular our results above show that the bound states and resonance spectrum of the zero range interaction on $\mathbf{R}^3$ can be approximated by those of short range potentials. Applications of this observation will be given elsewhere.

We shall now study in more details the case of a potential V of compact support, and these results will also be applied in the next section. Let V, $\lambda(\varepsilon)$ be as in Theorem 3.1. From Proposition 2.1 we have that

$$(3.24)\qquad (-\Delta + V_\varepsilon - k^2)^{-1} = G_k + \sum_{m=1}^{\infty} (-1)^m (G_k V_\varepsilon)^m G_k$$

converges absolutely in norm for Imk sufficiently large. From the resolvent identity we have

$$(3.25)\qquad (-\Delta + V_\varepsilon - k^2)^{-1}\, V_\varepsilon G_k = G_k - (-\Delta + V_\varepsilon - k^2)^{-1}$$

so by Theorem 3.1 we have in the strong operator sense that $(-\Delta+V_\varepsilon-k^2)^{-1}V_\varepsilon G_k$ converges to zero as $\varepsilon \to 0$ if V has no zero energy resonance, or to (3.19) with α given by (3.21), if V has a zero energy resonance and this is simple or $V \leqslant 0$. Applying (3.19) to $(-\Delta - k^2)f$ for $f \in \mathscr{C}_0^2(\mathbf{R}^3)$ we get from (3.25)

$$(3.26)\qquad \lim_{\varepsilon\to 0}((-\Delta + V_\varepsilon - k^2)^{-1}V_\varepsilon f)(x) = \left(\frac{ik}{4\pi} - \alpha\right)^{-1} G_k(x)f(0).$$

Let now B_r be the ball of radius r, i.e., $B_r \equiv \{x \in \mathbf{R}^3 \mid |x| \leqslant r\}$ and let $\mathscr{C}(B_r)$ be the Banach space of continuous functions on B_r, and set $\tilde{B}_r \equiv \mathbf{R}^3 \setminus B_r$. Let $L(\mathscr{C}(B_s), \mathscr{C}_\infty(\tilde{B}_t))$ be the space of bounded linear maps from $\mathscr{C}(B_s)$ to $\mathscr{C}_\infty(\tilde{B}_t)$, for s, $t > 0$, and let $\|\cdot\|_{s,t}$ be the Banach norm in $L(\mathscr{C}(B_s), \mathscr{C}_\infty(\tilde{B}_t))$. We shall see that $\|(-\Delta + V_\varepsilon - k^2)V_\varepsilon\|_{s,t}$ is bounded uniformly in ε for $\varepsilon < \varepsilon_0$ and $k \geqslant k_0$. Suppose now that V has compact support, say supp$V \subset B_{r_0}$, for some $r_0 > 0$. From (3.7) we have

$$(3.27)\qquad \begin{aligned}(-\Delta + V_\varepsilon - k^2)^{-1}V_\varepsilon(x,y) &= \lambda(\varepsilon)\varepsilon \int G_k(x-\varepsilon x_1)V(x_1)\delta(\varepsilon x_1 - y)\mathrm{d}x_1 - \\ &- \lambda(\varepsilon)^2\varepsilon \iint G_k(x-\varepsilon x_1)V(x_1)(H-(\varepsilon k)^2)^{-1}(x_1,x_2)V(x_2)\delta(\varepsilon x_2 - y)\mathrm{d}x_1\mathrm{d}x_2.\end{aligned}$$

Applying the first term in (3.27) to a continuous function f we get

$$\lambda(\varepsilon)\,\varepsilon \int G_k(x - \varepsilon x_1)V(x_1)f(\varepsilon x_1)\mathrm{d}x_1, \tag{3.28}$$

the absolute value of which is bounded for $x \in \tilde{B}_r$, $r > \varepsilon r_0$ by

$$\frac{|\lambda(\varepsilon)|\varepsilon}{4\pi(r - \varepsilon r_0)}\, \mathrm{e}^{-\mathrm{Im}k(r-\varepsilon r_0)}\, \|V\|_1 \sup_{|x| \leqslant \varepsilon r_0} |f(x)|. \tag{3.29}$$

Thus the $\|\cdot\|_{\varepsilon r_0, r}$-norm of the first term in (3.27) is smaller than or equal to the expression (3.29), without the term $\sup\limits_{|x| \leqslant \varepsilon r_0} |f(x)|$.

The second term of (3.27) applied to f yields

$$-\lambda(\varepsilon)^2\varepsilon \iint G_k(x - \varepsilon x_1)V(x_1)(H - (\varepsilon k)^2)^{-1}(x_1, x_2)V(x_2)f(\varepsilon x_2)\mathrm{d}x_1\mathrm{d}x_2\,. \tag{3.30}$$

Using (2.2) this can be rewritten as

$$\begin{aligned} &- \lambda(\varepsilon)^2\varepsilon \iint G_k(x - \varepsilon x_1)V(x_1)G_{\varepsilon k}(x_1 - x_2)V(x_2)f(\varepsilon x_2)\mathrm{d}x_1\mathrm{d}x_2 + \\ &+ \lambda(\varepsilon)^3\varepsilon \iiiint G_k(x - \varepsilon x_1)V(x_1)G_{\varepsilon k}(x_1 - x_2)v(x_2)(1 + \lambda u G_{\varepsilon k} v)^{-1}(x_1, x_2)\cdot \\ &\cdot u(x_3)G_{\varepsilon k}(x_3 - x_4)V(x_4)f(\varepsilon x_4)\mathrm{d}x_1 \ldots \mathrm{d}x_4\,. \end{aligned} \tag{3.31}$$

The first term in (3.31) is of the form $-\lambda(\varepsilon)^2\varepsilon A_\varepsilon B_\varepsilon \tilde{C}_\varepsilon$, where A_ε, B_ε were already defined in the proof of Lemma 3.1 and $\tilde{C}_\varepsilon$ the operator with kernel $\tilde{C}_3(x, y) \equiv \equiv u(x)\delta(\varepsilon x - y)$. The second term in (3.31) is of the form $\lambda(\varepsilon)^3\varepsilon A_\varepsilon B_\varepsilon D_\varepsilon B_\varepsilon \tilde{C}_\varepsilon$, where D_ε was also defined in the proof of Lemma 3.1. Since $\mathrm{supp} V \subset B_{r_0}$ we have that $\tilde{C}_\varepsilon$ maps $\mathscr{C}(B_{\varepsilon r_0})$ into $L^2(\mathbf{R}^3)$ with bounded norm equal to $\|u\|_2$ and converges strongly as $\varepsilon \to 0$ to $u(x)\delta(y)$ as a map from $\mathscr{C}(B_\delta)$, $\varepsilon < r_0^{-1}\delta$, $\delta > 0$ arbitrary, into $L^2(\mathbf{R}^3)$. Moreover B_ε is a uniformly bounded map of $L^2(\mathbf{R}^3)$ into itself, converging strongly as $\varepsilon \to 0$ to B_0 (as we remarked already in the proof of Lemma 3.1). A_ε is a norm bounded map from $L^2(\mathbf{R}^3)$ into $\mathscr{C}_\infty(\tilde{B}_r)$ with norm bounded by

$$\|A_\varepsilon\|_{L^2(\mathbf{R}^3),\, \mathscr{C}_0(\tilde{B}_r)} \leqslant \|v\|_2 \frac{1}{4\pi(r - \varepsilon r_0)}\, \mathrm{e}^{-\mathrm{Im}k(r-\varepsilon r_0)}. \tag{3.32}$$

Moreover A_ε converges strongly as $\varepsilon \to 0$ to $A_0 = |G_k><v|$. Thus the first term

of (3.31) has a $\|\cdot\|_{\varepsilon r_0, r}$-norm which is less or equal

$$c \frac{(\lambda(\varepsilon))^2\varepsilon}{4\pi(r-\varepsilon r_0)} e^{-\mathrm{Im}k(r-\varepsilon r_0)} \|V\|_1, \tag{3.33}$$

for some constant c.

The second term in (3.31) is controlled using the above estimates on A_ε, B_ε, $\tilde{C}_\varepsilon$ and the fact that D_ε is uniformly bounded and converges strongly to D_0 as a map from $L^2(\mathbf{R}^3)$ into itself. We have thus proven the following:

THEOREM 3.3. *Let V be a short range potential with support in a ball B_{r_0} of radius $r_0 > 0$. Let λ, V_ε be as in Theorem 3.1. Let $s, t > 0$ and let $L_{s,t} \equiv L(\mathscr{C}(B_s), \mathscr{C}_\infty(\tilde{B}_t))$ be the space of bounded continuous linear maps from $\mathscr{C}(B_s)$ into $\mathscr{C}_\infty(\tilde{B}_t)$, where $\mathscr{C}(B_s)$ is the space of bounded continuous functions on B_s and $\mathscr{C}_\infty(\tilde{B}_t)$ is the space of bounded continuous functions on $\tilde{B}_t \equiv \mathbf{R}^3 \setminus B_t$ which tend to zero at infinity. If $\|\cdot\|_{s,t}$ is the norm in $L_{s,t}$ then there is a constant c depending only on V such that if $\varepsilon < \min\{r_0^{-1}s, r_0^{-1}t\}$ then for* Im$k > k_0$ *we have*

$$\|(-\Delta + V_\varepsilon - k^2)^{-1} V_\varepsilon\|_{s,t} \leqslant c\, e^{-\mathrm{Im}k(t-\varepsilon r_0)}.$$

If $-\Delta + V$ has no zero energy resonance then $(-\Delta + V_\varepsilon - k^2)^{-1} V_\varepsilon \to 0$ strongly in $L_{s,t}$. If $-\Delta + V$ has a zero energy resonance and either $V \leqslant 0$ or the resonance is simple then

$$(-\Delta + V_\varepsilon - k^2)^{-1} V_\varepsilon(x, y) \to \left(\frac{ik}{4\pi} - \alpha\right)^{-1} G_k(x)\delta(y)$$

strongly in $L_{s,t}$, where $\alpha = -\lambda'(0)\left(\sum_{i=1}^{n} |(V, \psi_i)|^2\right)^{-1}$, ψ_i being the resonance functions, normalized such that $(\psi_i, V\psi_j) = -\delta_{ij}$.

4. SHORT RANGE INTERACTIONS WITH A DISCRETE SET OF CENTERS

Let V_i, $i = 1, 2, \ldots$ a finite or countable set of short range potentials. We suppose first for simplicity, that $-\Delta + V_i$ has, for all i, a zero energy resonance. Let $X \equiv \{x_i\}$ be a discrete subset of $\mathbf{R}^3$. Consider the operators

$$-\Delta + W_{\varepsilon,n}(x) \tag{4.1}$$

where

$$W_{\varepsilon,n}(x) = \sum_{i=1}^{n} V_{i,\varepsilon}(x) \tag{4.2}$$

with

$$V_{i,\varepsilon}(x) \equiv \varepsilon^{-2}\lambda_i(\varepsilon)\, V_i\left(\frac{x-x_i}{\varepsilon}\right) \tag{4.3}$$

and where $\lambda_i(\varepsilon)$ are differentiable functions with $\lambda_i(0) = 1$. Then $W_{\varepsilon,n}$ is a short range potential, hence by Proposition 2.1 we have that

$$(-\Delta + W_{\varepsilon,n} - k^2)^{-1} = G_k + \sum_{m=1}^{\infty} (-1)^m (G_k W_{\varepsilon,n})^m G_k \tag{4.4}$$

is absolutely norm convergent for $\mathrm{Im} k > 0$ large enough. Hence we get, inserting the definition (4.2) and performing a partial resummation

$$\begin{aligned}(-\Delta + W_{\varepsilon,n} - k^2)^{-1} = G_k - \sum_{m=1}^{\infty} \sum_{\substack{i_1 \neq i_2 \\ i_{m-1} \neq i_m}}^{n} & (-1)^m (-\Delta + V_{i_1,\varepsilon} - k^2)^{-1} V_{i_1,\varepsilon} \cdot \\ & \cdot (-\Delta + V_{i_2,\varepsilon} - k^2)^{-1} V_{i_2,\varepsilon} \ldots V_{i_m,\varepsilon} G_k .\end{aligned} \tag{4.5}$$

We want now to study the limit as $\varepsilon \to 0$ of (4.5).

From the resolvent identity we have

$$(-\Delta + V_{j,\varepsilon} - k^2)^{-1} V_{j,\varepsilon} G_k = G_k - (-\Delta + V_{j,\varepsilon} - k^2)^{-1}, \tag{4.6}$$

so by Theorem 3.1 we have, in the strong operator sense, that $\lim_{\varepsilon\to 0} (-\Delta + V_{j,\varepsilon} - k^2)^{-1} V_{j,\varepsilon} G_k$ converges to zero if $V_{j,\varepsilon}$ has no zero energy resonance, or to

$$G_k - (-\Delta_\alpha - k^2)^{-1} = | G_k > \left(\frac{ik}{4\pi} - \alpha\right)^{-1} < G_k | \tag{4.7}$$

where $\alpha = -\lambda'(0)(\sum_i |(V_{j,\varepsilon}, \psi_i)|^2)^{-1}$, if $V_{j,\varepsilon}$ has a zero energy resonance which is either simple or one has $V_{j,\varepsilon} \leqslant 0$.

Let now $f \in \mathscr{C}_0^2(\mathbf{R}^3)$ and apply (4.7) to $(-\Delta - k^2)f$. Then we get

$$\lim_{\varepsilon\to 0}(-\Delta + V_{j,\varepsilon} - k^2)^{-1} V_{j,\varepsilon} f(x) = \left(\frac{ik}{4\pi} - \alpha\right)^{-1} G_k(x) f(0). \tag{4.8}$$

Now to study the limit of (4.5) as $\varepsilon \to 0$ we recall results from Theorem 3.3. Let $s, t > 0$ and let $B_s(x)$, $\tilde{B}_t(x)$ be the translates by x of the ball B_s and of $\tilde{B}_t = \mathbf{R}^3 \setminus B_t$, respectively. Suppose $\mathrm{supp} V_i \subset B_{r_0}$, for some fixed $r_0 > 0$, independent of $i = 1, 2, \ldots$. From Theorem 3.3 it follows with $\varepsilon < \min\{r_0^{-1}s, r_0^{-1}t\}$

$$\|(-\Delta + V_{i,\varepsilon} - k^2)^{-1}V_{i,\varepsilon}\|_{s,t,x_i} \leqslant c\, e^{-\mathrm{Im}k(t-\varepsilon r_0)}, \tag{4.9}$$

where $\|\cdot\|_{s,t,x_i}$ is the norm in $L(\mathscr{C}(B_s(x_i)), \mathscr{C}_\infty(\tilde{B}_t(x_i)))$.

Since $X = \{x_i\}$ is a discrete subset of $\mathbf{R}^3$ we have that if $y \notin X$ then $G_k(\cdot - y) \in \mathscr{C}(B_s(x_i))$ for all x_i, if s is sufficiently small, and in addition we have

$$\sup_{x \in B_s(x_i)} |G_k(x - y)| \leqslant c_i\, e^{-\mathrm{Im}k|x_i - y|}. \tag{4.10}$$

From (4.9) and (4.10) we get that if K is any compact subset of $\mathbf{R}^3$ such that $K \cap X = \emptyset$, then

$$\sup_{x \in K} |[(-\Delta + V_{i_1,\varepsilon} - k^2)^{-1} V_{i_1,\varepsilon}(-\Delta + V_{i_2,\varepsilon} - k^2)^{-1} V_{i_2,\varepsilon} \ldots$$

$$\ldots(-\Delta + V_{i_m,\varepsilon} - k^2)^{-1} V_{i_m,\varepsilon} G_k](x, y)| \leqslant C^m \exp[-\mathrm{Im}k(|x_{i_1} - x_{i_2}| + \tag{4.11}$$

$$+ |x_{i_2} - x_{i_3}| + \ldots + |x_{i_{m-1}} - x_{i_m}| + |x_{i_m} - y|)],$$

for some constant C. From (4.9) and (4.10) we also get that if f is a bounded continuous function of compact support with $\mathrm{supp}\varphi \cap X = \emptyset$, then if $f \geqslant 0$ with $\int f(y)\mathrm{d}y = 1$:

$$\sup_{x \in B_{\varepsilon r_0}(x_{i_1})} |(-\Delta + V_{i_1,\varepsilon} - k^2)^{-1} V_{i_1,\varepsilon} \ldots (-\Delta + V_{i_m,\varepsilon} - k^2)^{-1} V_{i_m} G_k f(x, y)| \leqslant$$

$$\leqslant C' \exp[-\mathrm{Im}k(|x_{i_1} - x_{i_2}| + \ldots + |x_{i_m} - y_0|)] \tag{4.12}$$

where $y_0 \equiv \int y f(y)\,\mathrm{d}y$ and C' is a constant, independent of ε, if $B_{\varepsilon r_0}(x_i) \cap B_{\varepsilon r_0}(x_j) = \emptyset$, $i \neq j$.

Since $\mathrm{supp} V_{i_1,\varepsilon} \subset B_{\varepsilon r_0}(x_i)$ and $(-\Delta + V_{i_1,\varepsilon} - k^2)^{-1} V_{i_1,\varepsilon}$ maps $L^2(\mathbf{R}^3)$ into $L^2(\mathbf{R}^3)$ we have moreover

$$\|(-\Delta + V_{i_1,\varepsilon} - k^2)^{-1} V_{i_1,\varepsilon}(-\Delta + V_{i_2,\varepsilon} - k^2) \ldots (-\Delta + V_{i_m,\varepsilon} - k^2)^{-1} V_{i_m,\varepsilon} G_k f\|_2 \leqslant$$

$$\leqslant \tilde{C}_\varepsilon \exp[-\mathrm{Im}k(|x_{i_1} - x_{i_2}| + \ldots + |x_{i_m} - y_0|)]. \tag{4.13}$$

For fixed $\varepsilon \leqslant \varepsilon_0$ we have from (4.13) and (4.5) that $(-\Delta + W_{\varepsilon,n} - k^2)^{-1}$ converges strongly as $n \to \infty$ to

$$G_k - \sum_{m=1}^{\infty} (-1)^m \sum_{\substack{i_1 \neq i_2 \\ i_{m-1} \neq i_m}} (-\Delta + V_{i_1,\varepsilon} - k^2)^{-1} V_{i_1,\varepsilon} \ldots V_{i_m,\varepsilon} G_k \tag{4.14}$$

for Imk sufficiently large. This being the strong limit of a resolvent it is the resolvent of a self-adjoint operator $-\Delta + W_\varepsilon$.

On the other hand by Theorem 3.3 we get that in the case V_{j_1} has a zero energy resonance and either $V_{j_1} \leqslant 0$ or the resonance is simple, one has as $\varepsilon \to 0$

$$(4.15)\qquad (-\Delta + V_{j_1,\varepsilon} - k^2)^{-1} V_{j_1,\varepsilon}(x, y) \to -\left(\frac{ik}{4\pi} - \alpha_{j_1}\right)^{-1} G_k(x - x_{j_1})\delta(y - x_{j_1})$$

strongly as a map from $\mathscr{C}(B_{\varepsilon_0 r}(x_{j_1}))$ to $\mathscr{C}(B_{\varepsilon_0 r}(x_{j_2}))$ for $j_1 \neq j_2$. This and the uniform estimate (4.13) imply that (4.14) converges strongly as an operator from $\mathscr{C}(K)$ into itself, where K is any compact subset of $\mathbf{R}^3$ such that $K \cap X = \emptyset$. By (4.14) and (4.13) we have that the limit has a kernel given by

$$(4.16)\qquad G_k(x - y) + \sum_{m=1}^{\infty} (-1)^m \sum_{\substack{i_1 \neq i_2 \\ i_{m-1} \neq i_m}} G_k(x - x_{i_1}) \left(\frac{ik}{4\pi} - \alpha_{i_1}\right)^{-1} G_k(x_{i_1} - x_{i_2}) \ldots \left(\frac{ik}{4\pi} - \alpha_{i_m}\right)^{-1} G_k(x_{i_m} - y)$$

for x and y not in X. Summing up the right hand side of (4.16) we get

$$(4.17)\qquad G_k(x - y) + \sum_{i_1, i_2} G_k(x - x_{i_1}) \left[\frac{ik}{4\pi} - \alpha - g_k\right]^{-1}_{i_1, i_2} G_k(x_{i_2} - y),$$

where $\left[\frac{ik}{4\pi} - \alpha - g_k\right]^{-1}_{i_1, i_2}$ is the inverse of the matrix $\frac{ik}{4\pi} - \alpha - g_k$ which is given by

$$(4.18)\qquad \begin{aligned} \left(\frac{ik}{4\pi} - \alpha - g_k\right)_{i_1, i_2} &= \frac{ik}{4\pi} - \alpha_{i_1} \qquad \text{for } i_1 = i_2, \\ \left(\frac{ik}{4\pi} - \alpha - g_k\right)_{i_1, i_2} &= -G_k(x_{i_1} - x_{i_2}) \qquad \text{for } i_1 \neq i_2. \end{aligned}$$

Note that (4.17) is the limit as $\varepsilon \to 0$ of the kernel of (4.14), which is equal to the strong limit as $n \to \infty$ of $(-\Delta + W_{\varepsilon,n} - k^2)^{-1}$. Thus we have that $\lim_{\varepsilon\to 0} \lim_{n\to\infty} (-\Delta + W_{\varepsilon,n} - k^2)^{-1}(x, y)$ is given by (4.17), the convergence in $\lim_{\varepsilon\to 0}$ being the strong one for operators on $\mathscr{C}(K)$.

If $X = X_n = \{x_1, \ldots, x_n\}$ is a finite set the sums in (4.17) are finite, namely $i_1, i_2 = 1, \ldots, n$. In this case (4.17) was shown in [13], [14] to be the resolvent of a self-adjoint operator $-\Delta_{(X_n,\alpha)}$, where $\Delta_{(X_n,\alpha)}$ denotes the Hamiltonian for a sum of point interactions at $x_1, \ldots, x_n$ of respective strength α_i, $i = 1, \ldots, n$. We define α

to be the function on X_n such that $\alpha(x_i)=\alpha_i$. $\Delta_{(X_n,\alpha)}$ has been defined in [13], [14]. If $\mathrm{Re}k = 0$ and $\mathrm{Im}k$ is large enough it is easy to see that

$$(4.19)\quad (-\Delta_{(X_n,\alpha)} - k^2)^{-1}(x, y) = G_k(x - y) - \sum_{i_1,i_2}^{n} G_k(x - x_{i_1})\left[\frac{ik}{4\pi} - \alpha - g_k\right]^{-1}_{i_1,i_2} G_k(x_{i_2} - y)$$

as an operator on $L^2(\mathbf{R}^3)$ is increasing in n.

Let now again X be an arbitrary discrete subset of $\mathbf{R}^3$ and define α on X as the function such that $\alpha(x_i) = \alpha_i$. By the monotonicity we have (see [15]) that $(-\Delta_{(X_n,\alpha)} - k^2)^{-1}$ converges strongly monotonically increasing as $n \to \infty$ in such a way that $(-\Delta_{(X_n,\alpha)} - k^2)^{-1}(x, y)$ converges to (4.17).

It was shown in [15] that this is in fact the resolvent of a self-adjoint operator $-\Delta_{(X,\alpha)}$. Hence $\lim_{\varepsilon\to 0}\lim_{n\to\infty} (-\Delta+W_{\varepsilon,n}-k^2)^{-1}=(-\Delta_{(X,\alpha)}-k^2)^{-1}$ where the $\lim_{\varepsilon\to 0}$ is understood in the weak sense of operators on $L^2(\mathbf{R}^3)$.

However this weak convergence implies strong convergence, the limit being a resolvent.

Observe finally that allowing some of the V_i not to have a zero energy resonance merely amounts replacing X be the subset of those x_j for j such that $-\Delta + V_j$ has a zero energy resonance. We have then the following

THEOREM 4.1. *Let $X = \{x_i\}$ be a discrete subset of* $\mathbf{R}^3$. *Let α be a function on X with $\alpha(x_i) = x_i$. Let*

$$\left[\frac{ik}{4\pi} - \alpha - g_k\right]_{i_1,i_2} \equiv \begin{cases} \dfrac{ik}{4\pi} - \alpha_{i_1}, & i_1 = i_2 \\ G_k(x_{i_1} - x_{i_2}), & i_1 \neq i_2. \end{cases}$$

Then $G_k(x - y) + \sum_{i_1,i_2} G_k(x - x_{i_1})\left[\frac{ik}{4\pi} - \alpha - g_k\right]^{-1}_{i_1,i_2} G_k(x_{i_2} - y)$, *where* $[\ \cdot\ ]^{-1}_{i_1,i_2}$ *is the inverse matrix to* $[\ \cdot\ ]$, *is the resolvent kernel* $(-\Delta_{X,\alpha} - k^2)^{-1}(x, y)$ *of a self-adjoint operator* $-\Delta_{X,\alpha}$. *This operator coincides with* $-\Delta$ *on* $\mathscr{C}^2$ *functions which are zero on X. Let* V_i *be short range potentials of compact support such that for each* $i = 1, 2, \ldots, -\Delta + V_i$ *has a zero energy resonance and either this is simple or* $V_i \leqslant 0$.

Let $\lambda_i(\varepsilon)$ *be a differentiable function with* $\lambda_i(0) = 1$. *For each* i *set* $\alpha_i = -\lambda'(0)[\sum_j |(V_i, \psi_{ij})|^2]^{-1}$, *where* ψ_{ij}, $j = 1, \ldots, m_i$ *are the resonance functions for* $-\Delta + V_i$ i.e., $(-\Delta + V_i)\psi_{ij} = 0$, *normalized such that* $(\psi_{ij}, V\psi_{ij}) = -\delta_{ij}$. *Then*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\left(-\Delta + \sum_{j=1}^{n} \lambda_j(\varepsilon)\varepsilon^{-2}V_j\left(\frac{|x - x_j|}{\varepsilon}\right) - k^2\right)^{-1} = (-\Delta_{(\tilde{X},\alpha)} - k^2)^{-1},$$

where the limit is in the strong operator sense of operators on $L^2(\mathbf{R}^3)$. $\tilde{X}$ *is the subset of X consisting of those* x_j *such that* $-\Delta + V_j$ *has a zero energy resonance.*

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