J. OPERATOR THEORY
6(1981), 341–349

THE INVARIANT SUBSPACES OF A VOLTERRA OPERATOR

JOSÉ BARRÍA

Let $(a_1, b_1), \ldots, (a_m, b_m)$ $(m \geqslant 1)$ be disjoint subintervals of $X = [0, c]$ $(c > 0)$ such that $b_1 \leqslant a_2, \ldots, b_{m-1} \leqslant a_m$. Let μ be a measure on X such that μ is the Lebesgue measure on (a_i, b_i) $(1 \leqslant i \leqslant m)$, and μ is purely atomic on $X \setminus \bigcup_{i=1}^{m} (a_i, b_i)$ with a finite number of atoms in $(0, c) \setminus \bigcup_{i=1}^{m} (a_i, b_i)$. Let V_μ be the bounded linear operator on $L_2(X, \mu)$ defined by

$$(V_\mu f)(x) = \int_{[0,x)} f(t)\,\mathrm{d}\mu(t) \quad (f \in L_2(X, \mu)).$$

The purpose of this paper is to determine all the (closed) invariant subspaces of V_μ. For $0 \leqslant a \leqslant c$, let L_a (L'_a, resp.) denote the closed subspace of all functions in $L_2(X, \mu)$ which vanish on $[0, a)$ ($[0, a]$, resp.) a.e. $[\mu]$. Since $V_\mu L_a \subset L'_a \subset L_a$, it follows that the subspaces L_a and L'_a are invariant under V_μ. It is easily seen that $L_a \neq L'_a$ if and only if $\mu(\{a\}) > 0$.

THEOREM A. *The subspaces L_a and L'_a $(0 \leqslant a \leqslant c)$ are the only invariant subspaces of V_μ.*

If σ is the Lebesgue measure on X, then the operator V_σ on $L_2[0, c]$ is the usual Volterra operator and in this case the assertion of the theorem is a well known result ([2], [4], [5]). In particular the lattice of invariant subspaces of V_μ has the order type of $[0, 1]$.

In [1] the theorem was proven for a measure μ which is the Lebesgue measure with a finite number of atoms in $(0, c)$. In this case the lattice of invariant subspaces of V_μ has the order type of the chain $[0, 1] \cup [2, 3] \cup \ldots \cup [2n, 2n + 1]$, where n is the number of atoms of μ.

In [3] P. Rosenthal gave an example of an operator whose lattice of invariant subspaces has the order type of the chain $[0, 1] \cup \{2, 3, \ldots, n\}$ $(n \geqslant 2)$. Theorem A implies that the ordinal sum of a finite number of such chains can be realized as the lattice of invariant subspaces of an operator V_μ.

Let χ_A denote the characteristic function of the set A in X. Let $x_1, x_2, \ldots, x_n$ be the atoms of μ (if it has atoms), so arranged that $0 < x_1 < x_2 < \ldots < x_n < c$. By convention we take $x_0 = 0$.

LEMMA 1. *The adjoint of V_μ is given by*

$$(V_\mu^* f)(x) = \int_{(x,c]} f(t)\, \mathrm{d}\mu(t) \quad (f \in L_2(X, \mu)).$$

Furthermore,

$$\ker V_\mu = \{\lambda u_0 : \lambda \in \mathbf{C},\, u_0 = \chi_{[b_m, c] \cap \{x_n\}}\}$$

and

$$\ker V_\mu^* = \{\lambda v_0 : \lambda \in \mathbf{C},\, v_0 = \chi_{[0, a_1] \cap \{x_1\}}\}.$$

Proof. The operator V_μ is the integral operator whose kernel is the characteristic function of the set $\{(x, y) \in X \times X : y < x\}$. Therefore V_μ^* is the integral operator induced by the conjugate transpose kernel, that is, the characteristic function of $\{(x, y) \in X \times X : x < y\}$. Now the given expression for V_μ^* follows immediately.

Assume that $V_\mu f = 0$. Then $\int_{t_1}^{t_2} f(t)\, \mathrm{d}t = 0$ for all t_1 and t_2 in (a_i, b_i). Therefore $= 0$ a.e. $[\sigma]$ on (a_i, b_i) $(1 \leqslant i \leqslant m)$. Since μ has a finite number of atoms it follows easily that $f = 0$ a.e. $[\mu]$ on $[0, b_m)$. Then $\ker V_\mu = \{0\}$ if it has no atoms in $[b_m, c]$. Otherwise, $f = 0$ a.e. $[\mu]$ on $[0, x_n)$ and $V_\mu u_0 = 0$. Similar reasoning can be used to prove the assertion about $\ker V_\mu^*$. ▨

If f and g are vectors in a Hilbert space $\mathscr{H}$, we denote by $f \otimes g$ the rank one operator on $\mathscr{H}$ defined by $(f \otimes g)(z) = (z, g)f$ for all z in $\mathscr{H}$. Write $\lambda_i = \mu(\{x_i\})$ $(1 \leqslant i \leqslant n)$, $\chi_j = \chi_{(x_j, c]}$, and $\chi^{(j)} = \chi_{\{x_j\}}$ $(0 \leqslant j \leqslant n)$.

Let V_j and T_j be the operators on $L_2(X, \mu)$ defined by

$$V_j = V_\mu - \chi_{j+1} \otimes \chi^{(j+1)} - \ldots - \chi_n \otimes \chi^{(n)}, \tag{1}$$

and

$$T_j = V_{j+1} - \lambda_{j+1} M_{j+1} \tag{2}$$

for $j = 0, 1, \ldots, n-1$, where M_i is the multiplication operator induced by χ_i on $L_2(X, \mu)$ $(0 \leqslant i \leqslant n)$, and $V_n = V_\mu$. As an equivalent definition of V_j we have

$$(V_j f)(x) = \int_{[0, x)} f(t)\, \mathrm{d}\mu_j(t) \quad (f \in L_2(X, \mu)), \tag{3}$$

where μ_j $(0 \leqslant j \leqslant n-1)$ is the measure on $X = [0, c]$ such that μ_j and μ coincide on $X \setminus \{x_{j+1}, \ldots, x_n\}$, and $\mu_j(\{x_{j+1}, \ldots, x_n\}) = 0$.

It is easy to check that V_j and T_j leave invariant the subspaces L_a and L'_a for $0 \leqslant a \leqslant c$.

For $0 \leqslant i \leqslant j \leqslant n$ and $j \geqslant 1$ we define

$$A(i,j) = (T_0 + \lambda_1)(T_1 + \lambda_2) \ldots (T_{i-2} + \lambda_{i-1})\, T_i T_{i+1} \ldots T_{j-1},$$

with the convention that

$$A(0,j) = T_0 T_1 \ldots T_{j-1} \quad (1 \leqslant j \leqslant n),$$

$$A(j,j) = (T_0 + \lambda_1)(T_1 + \lambda_2) \ldots (T_{j-2} + \lambda_{j-1}) \quad (2 \leqslant j \leqslant n),$$

and

$$A(1, 1) = I.$$

By induction it follows easily that

$$\sum_{i=0}^{j} \lambda_i A(i,j) = (T_0 + \lambda_1)(T_1 + \lambda_2) \ldots (T_{j-1} + \lambda_j) \tag{4}$$

for $j = 1, 2, \ldots, n$ (with $\lambda_0 = 1$).

LEMMA 2. *If* $1 \leqslant j \leqslant n$ *and if* $j \leqslant k$, *then*

$$V_j^k = \sum_{i=0}^{j} \lambda_i M_i V_0^{k-j} A(i,j). \tag{5}$$

Proof. First it will be proven that

$$V_j^k = V_{j-1}^{k-1} T_{j-1} + \lambda_j M_j V_{j-1}^{k-1} \quad (1 \leqslant j \leqslant n, k \geqslant 1). \tag{6}$$

The proof of (6) will proceed by induction on k. For $k = 1$ the equality follows from (2). From (1) we have that $V_j = V_{j-1} + \chi_j \otimes \chi^{(j)}$. Therefore

$$V_j^{k+1} = (V_{j-1} + \chi_j \otimes \chi^{(j)}) V_j^k =$$

$$= V_{j-1} V_j^k + (\chi_j \otimes \chi^{(j)}) V_j^k =$$

$$= V_{j-1}^k T_{j-1} + S \qquad \text{(by induction)}$$

where

$$S = \lambda_j V_{j-1} M_j V_{j-1}^{k-1} + (\chi_j \otimes \chi^{(j)}) V_j^k.$$

To complete the proof of (6) we have to show that $S = \lambda_j M_j V_{j-1}^k$. Let $f \in L_2(X, \mu)$. From (3) it follows that $Sf(x) = 0$ for all x in $[0, x_j]$. Since the measures μ_{j-1} and μ_j coincide on $[0, x_j)$, from (3) it follows that $(V_j^k f)(x_j) = (V_{j-1}^k f)(x_j)$, and therefore for x in $(x_j, c]$ we have

$$Sf(x) = \lambda_j \int_{(x_j, x)} (V_{j-1}^{k-1} f)(t) \, d\mu_{j-1}(t) + \lambda_j \int_{[0, x_j)} (V_{j-1}^{k-1} f)(t) \, d\mu_{j-1}(t).$$

Since $\mu_{j-1}(\{x_j\}) = 0$, we conclude that $Sf(x) = \lambda_j (V_{j-1}^k f)(x)$. Therefore $S = \lambda_j M_j V_{j-1}^k$.

Now we are ready to prove (5). Equalities (5) and (6) coincide for $j = 1$. If $2 \leqslant j \leqslant n$, then by induction it follows that

$$V_{j-1}^{k-1} = \sum_{i=0}^{j-1} \lambda_i M_i V_0^{k-} A(i, j-1).$$

If $0 \leqslant i \leqslant j-1$, then $M_j M_i = M_j$ and $A(i, j-1) T_{j-1} = A(i, j)$; therefore

$$M_j V_{j-1}^{k-1} = \sum_{i=0}^{j-1} \lambda_i M_j V_0^{k-j} A(i, j-1) =$$

$$= M_j V_0^{k-j} A(j, j) \qquad \text{(by (4))},$$

and

$$V_{j-1}^{k-1} T_{j-1} = \sum_{i=0}^{j-1} \lambda_i M_i V_0^{k-j} A(i, j).$$

Therefore

$$V_{j-1}^{k-1} T_{j-1} + \lambda_j M_j V_{j-1}^{k-1} = \sum_{i=0}^{j} \lambda_i M_i V_0^{k-j} A(i, j).$$

From (6), the proof of (5) is complete. ▨

The following discussion concerning the usual Volterra operator V_σ on $L_2[0, c]$ is well known and is included here only for the sake of completeness.

By induction it follows easily that

$$(V_\sigma^k f)(x) = \frac{1}{(k-1)!} \int_0^x (x - t)^{k-1} f(t) \, dt$$

for $f \in L_2[0, c]$ and $k \geqslant 1$. It e is the constant function one on $[0, c]$, then $V_\sigma f = e * f$ (the convolution of e and f) for all f in $L_2[0, c]$. Therefore $V_\sigma^k f = e_k * f$ where e_k is the k-fold convolution of e with itself. Let f and g be in $L_2[0, c]$, and write $h(x) =$

$= \overline{g(c-x)}$ for x in $[0, c]$. The following computation gives an expression for the inner product $(V_\sigma^k f, g)$ in the space $L_2[0, c]$.

$$(V_\sigma^k f, g) = \int_0^c (V_\sigma^k f)(t)\, h(c-t)\, \mathrm{d}t =$$

$$= ((V_\sigma^k f) * h)(c) =$$

$$= (e_k * f * h)(c) =$$

$$= V_\sigma^k (f * h)(c).$$

Therefore

$$(V_\sigma^k f, g) = \frac{1}{(k-1)!} \int_0^c t^{k-1} (f * h)(c-t)\, \mathrm{d}t \tag{7}$$

for f and g in $L_2[0, c]$ and $k \geqslant 1$.

We write $d_0 = 0$ and $d_i = (b_1 - a_1) + \ldots + (b_i - a_i)$ if $1 \leqslant i \leqslant m$. For $f \in L_2(X, \mu)$ we define $\tilde{f}$ in $L_2[0, d_m]$ such that $\tilde{f}|(d_{i-1}, d_i)$ is the translation of $f|(a_i, b_i)$. More precisely, $\tilde{f}(t) = f(a_i + t)$ for $t \in (d_{i-1}, d_i)$ and $1 \leqslant i \leqslant m$. Since μ_0 is the Lebesgue measure on (a_i, b_i) $(1 \leqslant i \leqslant m)$ and has no atoms, it follows immediately from (3) that $(V_0 f)^\sim = V_\sigma \tilde{f}$, where V_σ is the usual Volterra operator on $L_2[0, d_m]$. Therefore

$$(V_0^k f)^\sim = V_\sigma^k \tilde{f} \quad (f \in L_2(X, \mu), k \geqslant 1). \tag{8}$$

LEMMA 3. *Let f and g be in $L_2(X, \mu)$. Then the inner product $(V_\mu^k f, g)$ in the space $L_2(X, \mu)$ is given by*

$$(V_\mu^k f, g) = \frac{1}{(k-n-1)!} \int_0^{d_m} t^{k-n-1} \left\{ \sum_{i=0}^{2n} \tilde{f}_i * h_i \right\} (d_m - t)\, \mathrm{d}t$$

for all $k \geqslant n+1$, where

$$f_i = \lambda_i A(i, n) f \quad (0 \leqslant i \leqslant n),$$

$$f_{n+j} = \lambda_j V_0^{n-j} A(j, j) f \quad (1 \leqslant j \leqslant n),$$

$$h_i(t) = \tilde{\chi}_i (d_m - t) \overline{\tilde{g}(d_m - t)} \quad \text{for } t \in [0, d_m] \quad (0 \leqslant i \leqslant n),$$

and

$$h_{n+j}(t) = \overline{g(x_j)}\, \tilde{\chi}_{[0, x_j]}(d_m - t) \quad \textit{for } t \in [0, d_m] \quad (1 \leqslant j \leqslant n).$$

Proof. Denote by $(u, v)_0$ the integral $\int_X u(t)\,\overline{v(t)}\,\mathrm{d}\mu_0(t)$ for u and v in $L_2(X, \mu)$. Assume that $n + 1 \leqslant k$. For $0 \leqslant i \leqslant n$, it follows from (7) and (8) that

$$(\lambda_i M_i V_0^{k-n} A(i, n) f, g)_0 = (V_0^{k-n} f_i, \chi_i g)_0 = (V_\sigma^{k-n} \tilde{f}_i, \widetilde{\chi_i g}) =$$

$$= \frac{1}{(k-n-1)!} \int_0^{d_m} t^{k-n-1}\,(\tilde{f}_i * h_i)(d_m - t)\,\mathrm{d}t.$$

Then, from Lemma 2 it follows that

$$(V_\mu^k f, g)_0 = \frac{1}{(k-n-1)!} \int_0^{d_m} t^{k-n-1} \left\{ \sum_{i=0}^{n} \tilde{f}_i * h_i \right\} (d_m - t)\,\mathrm{d}t.$$

Since the measures μ_{j-1} and μ coincide on $[0, x_j)$, $(V_\mu^k f)(x_j) = (V_{j-1}^k f)(x_j)$. Now from Lemma 2 it follows that

$$(V_\mu^k f)(x_j) = \sum_{i=0}^{j-1} \lambda_i V_0^{k-j+1} A(i, j-1) f(x_j) =$$

$$= V_0^{k-j+1} A(j, j) f(x_j) = \qquad \text{(by (4))}$$

$$= \int_{[0, x_j]} V_0^{k-j} A(j, j) f(t)\,\mathrm{d}\mu_0(t) = \qquad \text{(by (3))}$$

$$= (V_0^{k-j} A(j, j) f, \chi_{[0, x_j]})_0.$$

Hence

$$\lambda_j (V_\mu^k f)(x_j) \overline{g(x_j)} = \lambda_j (V_0^{k-n} V_0^{n-j} A(j, j) f, g(x_j) \chi_{[0, x_j]})_0 =$$

$$= (V_0^{k-n} f_{n+j}, g(x_j) \chi_{[0, x_j]})_0 =$$

$$= (V_\sigma^{k-n} \tilde{f}_{n+j}, g(x_j) \tilde{\chi}_{[0, x_j]}) =$$

$$= \frac{1}{(k-n-1)!} \int_0^{d_m} t^{k-n-1} (\tilde{f}_{n+j} * h_{n+j})(d_m - t)\,\mathrm{d}t,$$

for $j = 1, 2, \ldots, n$.

From the definition of the inner product in $L_2(X, \mu)$ it follows that

$$(V_\mu^k f, g) = (V_\mu^k f, g)_0 + \sum_{j=1}^{n} \lambda_j (V_\mu^k f)(x_j)\, \overline{g(x_j)}.$$

Now the lemma follows immediately from the preceding computations. ▨

The next lemma is the crucial point in the proof of Theorem A. The idea is to use the Titchmarsh Convolution Theorem, just as was done in [2]. The support $S_\mu(f)$ of f with respect to the measure μ is the complement of the largest open set on which $f = 0$ a.e. $[\mu]$.

THEOREM B [6]. *Let f and g be functions in $L_2[0, c]$ such that $f * g = 0$ a.e. $[\sigma]$ on $[0, c]$. If $0 \in S_\sigma(f)$, then $g = 0$ a.e. $[\sigma]$ on $[0, c]$.*

LEMMA 4. *Let f be an element in L'_a with $a_1 \leqslant a < b_1$ and $a \in S_\sigma(f)$. Then the closed subspace M spanned by $f, V_\mu f, V_\mu^2 f, \ldots$ is equal to L'_a.*

Proof. It is clear that $M \subset L'_a$. Assume that $a = a_1$. Next it will be shown that if $g \in L'_a \ominus M$ and $g = 0$ a.e. $[\mu]$ on $[b_m, c]$, then $g = 0$. The proof will proceed by induction on the number of atoms of μ in (a_1, b_m).

If μ has no atoms in (a_1, b_m), then $V_\mu^k f = V_0^k f$ a.e. $[\mu]$ on $[0, b_m]$. Thus, by (8), it follows that $(V_\mu^k f, g) = (V_0^k f, g) = (V_\sigma^k \tilde{f}, \tilde{g})$. Now the conclusion follows from Theorem A for the usual Volterra operator V_σ.

Suppose now that μ has at least one atom in (a_1, b_m). From Lemma 3 it follows that

$$(*) \qquad \sum_{i=0}^{2n} \tilde{f}_i * h_i = 0 \quad \text{a.e. } [\sigma] \text{ on } [0, d_m].$$

Let n' be the largest integer such that $x_{n'} \leqslant a_m$. Let m' be the smallest integer such that $x_{n'} \leqslant a_{m'}$. Let $i \leqslant n'$ and $t \in (0, d_m - d_{m'-1})$. Since $d_{m'-1} < d_m - t < d_m$, it follows that $\tilde{\chi}_{(x_i, c]}(d_m - t) = 1$ and $\tilde{\chi}_{[0, x_i]}(d_m - t) = 0$. Now from the definition of h_i we conclude that

$$h_0(t) = h_1(t) = \ldots = h_{n'}(t) = \overline{\tilde{g}(d_m - t)}$$

and

$$h_{n+1}(t) = h_{n+2}(t) = \ldots = h_{n+n'}(t) = 0$$

a.e. $[\sigma]$ on $(0, d_m - d_{m'-1})$.

Let $n' < i \leqslant n$. Then $a_m < x_i$, and therefore $\tilde{\chi}_{(x_i, c]} = 0$ and $g(x_i) = 0$. Hence $h_i = 0$ and $h_{n+i} = 0$. Now from $(*)$ we have

$$(**) \qquad (\tilde{f}_0 + \tilde{f}_1 + \ldots + \tilde{f}_{n'}) * h_0 = 0 \quad \text{a.e. } [\sigma] \text{ on } (0, d_m - d_{m'-1}).$$

By (4) we have

$$f_0 + f_1 + \dots + f_{n'} = (A(0, n) + \lambda_1 A(1, n) + \dots + \lambda_{n'} A(n', n))f =$$
$$= (T_0 + \lambda_1)(T_1 + \lambda_2) \dots (T_{n'-1} + \lambda_{n'})f.$$

From $f \in L'_a$ and $a = a_1$, it follows that $(T_{i-1} + \lambda_i)f = (V_0 + \lambda_i)f$ a.e. $[\mu]$ on $[a_1, b_1)$. Therefore $a \in S_\sigma((T_{i-1} + \lambda_i)f)$. Since $(T_{i-1} + \lambda_i)f \in L'_a$, the equality above implies that $a \in S_\sigma(f_0 + f_1 + \dots + f_{n'})$. Hence $0 \in S_\sigma(\tilde{f}_0 + \tilde{f}_1 + \dots + \tilde{f}_{n'})$. Now (**) and Theorem B imply that $h_0(t) = \tilde{g}(d_m - t) = 0$ a.e. $[\sigma]$ on $(0, d_m - d_{m'-1})$. Thus from the definition of $\tilde{g}$ we conclude that $g = 0$ a.e. $[\sigma]$ on (a_i, b_i) for $m' \leqslant i \leqslant m$. Now it follows immediately from Lemma 1 that $V_\mu^* g = 0$ a.e. $[\mu]$ on $[x_{n'}, c]$. Write $v = \chi_{(a,c]} V_\mu^* g$. Then $v \in L'_a$ and $v = 0$ a.e. $[\mu]$ on $[x_{n'}, c]$. Let ν be the measure on X such that ν and μ coincide on $X \setminus \{x_{n'}\}$, and $\nu(\{x_{n'}\}) = 0$. Let V_ν be the integral operator induced by ν on $L_2(X, \nu)$; then $(V_\mu^k f)(t) = (V_\nu^k f)(t)$ for t in $[0, x_{n'})$. Therefore

$$(V_\nu^k f, v)_\nu = \int_{[a, x_{n'})} (V_\nu^k f)(t)\overline{v(t)}\, d\nu(t) =$$
$$= \int_{[a, x_{n'})} (V_\mu^k f)(t)\, \overline{(V_\mu^* g)(t)}\, d\mu(t) =$$
$$= (V_\mu^k f, V_\mu^* g) = 0$$

for all integers $k \geqslant 0$. Since the set of atoms of ν is $\{x_1, x_2, \dots, x_n\} \setminus \{x_{n'}\}$, by induction it follows that $v = 0$ a.e. $[\nu]$ on X. Now $v(x_{n'}) = 0$ implies that $V_\mu^* g = 0$ a.e. $[\mu]$ on $(a, c]$. From this, as in the proof of Lemma 1, it follows that $g = 0$ a.e. $[\mu]$ on $(a, c]$. Hence $g = 0$.

To complete the proof of the lemma when $a = a_1$, we take any element g in $L'_a \ominus M$. Since μ is purely atomic on $[b_m, c]$, from Lemma 1 it follows that $V_\mu^{*p} g = 0$ a.e. $[\mu]$ on $[b_m, c]$, where p is the number of atoms of μ in $[b_m, c]$. If $g_1 = \chi_{(a,c]} V_\mu^{*p} g$, then $g_1 \in L'_a \ominus M$ and $g_1 = 0$ a.e. $[\mu]$ on $[b_m, c]$. Therefore $g_1 = 0$. Thus $V_\mu^{*p} g = 0$ a.e. $[\mu]$ on $(a, c]$. As in the proof of Lemma 1 we conclude that $g = 0$ a.e. $[\mu]$ on $(a, c]$. Hence $g = 0$.

The case when $a_1 < a < b_1$ can be reduced to the case just considered because $(V_\mu^k f, g)_\mu = (V_\eta^k f, g)_\eta$ where η is the measure on X defined by $\eta(E) = \mu(E \cap [a, c])$ for every Borel set E in X. ▨

Corollary 1. *Let f be an element of L_a such that $a \in S_\mu(f)$. Let M be the span of all vectors $f, V_\mu f, V_\mu^2 f, \dots$. Then*

(i) $M = L'_a$ *if* $f \cdot \chi_{\{a\}} = 0$,

and

(ii) $M = L_a$ *if* $f \cdot \chi_{\{a\}} \neq 0$.

Proof. (i) If $f \cdot \chi_{\{a\}} = 0$, then $f \in L'_a$. If $b_m \leqslant a$, then L'_a is a finite dimensional subspace and $V_\mu | L'_a$ is similar to a Jordan block, and the result follows easily. Assume that $a < b_m$. Let ν be the measure on X such that ν and μ coincide on $\{x_1, x_2, \ldots, x_n\} \cup \cup (a, c]$ and $\nu((a_i, b_i)) = 0$ for all the subintervals (a_i, b_i) which are contained in $[0, a]$. Then $L'_a = L_2((a, c], \nu)$ and $a \in S_\nu(f)$. Furthermore, $V^k_\mu f = V^k_\nu f$ for all $k \geqslant 0$. An application of Lemma 4 to the measure ν implies that the span of the vectors $f, V_\nu f, V^2_\nu f, \ldots$ is equal to $L_2((a, c], \nu)$. Therefore $M = L'_a$.

(ii) If $f \cdot \chi_{\{a\}} \neq 0$, then a is an atom of μ and $f(a) \neq 0$. If $f_1 = V_\mu f$ then $f_1 \in L'_a$ and $a \in S_\mu(f_1)$. Therefore from (i) it follows that $L'_a \subset M$. Now $f(a) \neq 0$ and $f \in M$ imply that $L_a \subset M$. Hence $M = L_a$. ▨

Proof of Theorem A. For f in $L_2(X, \mu)$ we denote by $a(f)$ the infimum of the μ-support of f.

Let M be an invariant subspace for V_μ. If $a = \inf\{a(f) : f \in M\}$, then $M \subset L_a$. Let $\{f_k\}_{k=1}^\infty$ be a sequence in M such that $a(f_k) \to a \; (k \to \infty)$. From Corollary 1 it follows that $L_{a(f_k)} \subset M$ for all $k \geqslant 1$. Therefore $L'_a \subset M \subset L_a$. Since the codimension of L'_a with respect to L_a is either zero or one, it follows that $M = L'_a$ or $M = L_a$. ▨

REFERENCES

1. Barría, J., *On chains of invariant subspaces*, Indiana University Dissertion, 1974.
2. Kalisch, G. K., A functional analysis proof of Titchmarsh's convolution theorem, *J. Math. Anal. Appl.*, **5**(1962), 176–183.
3. Rosenthal, P., Examples of invariant subspace lattices, *Duke Math. J.*, **37**(1970), 103–112.
4. Sarason, D., A remark on the Volterra operator, *J. Math. Anal. Appl.*, **12**(1965), 244–246.
5. Schreiber, M., Remark on a paper of Kalisch, *J. Math. Anal. Appl.*, **7**(1963), 62–63.
6. Titchmarsh, E. C., The zeros of certain integral functions, *Proc. London Math. Soc.*, **25**(1926), 288–302.

JOSÉ BARRÍA
Instituto Venezolano de Investigaciones Científicas,
Departamento de Matemáticas,
Apartado 1827, Caracas 1010 – A,
Venezuela.

Received September 16, 1980.