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SUBALGEBRAS OF REFLEXIVE ALGEBRAS

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1. INTRODUCTION

An algebra $\mathscr{A}$ of operators on a Hilbert space is *reflexive* if the only operators that leave invariant all of the invariant subspaces of $\mathscr{A}$ are the operators belonging to $\mathscr{A}$. In other words if

$$\mathrm{Lat}\,\mathscr{A} = \{P : P = P^* = P^2 \text{ and } (1 - P)AP = 0 \text{ for every } A \text{ in } \mathscr{A}\}$$

and

$$\mathrm{Alg}(\mathrm{Lat}\mathscr{A}) = \{T : \mathrm{Lat}\mathscr{A} \subset \mathrm{Lat}T\},$$

then $\mathscr{A}$ is reflexive precisely when $\mathscr{A} = \mathrm{Alg}(\mathrm{Lat}\mathscr{A})$. An operator T is *reflexive* if the weakly closed unital algebra $\mathscr{A}(T)$ generated by T is reflexive. In [23] D. Sarason proved that all commutative weakly closed unital algebras of normal or of analytic Toeplitz operators are reflexive. The reflexive operators on a finite-dimensional space were characterized by J. Deddens and P. A. Fillmore [8], and all isometries were shown to be reflexive by Deddens [6]. K. Uchiyama [27] has shown that a $C_{\cdot 0}$ contraction with unequal defect indices is reflexive, and P. Y. Wu [29] has shown that finite defect C_{11} contractions are reflexive. Recently, R. Olin and J. Thomson [19] have shown that all subnormal operators are reflexive.

Before Olin and Thomson's work, W. Wogen [28] extended Deddens' result [6] by showing that all quasinormal operators (T and T^*T commute) are reflexive, thereby answering a question of Deddens [7]. Moreover, Wogen [28] proved that if T is quasinormal, then every unital weakly closed subalgebra of $\mathscr{A}(T)$ is reflexive. Call an algebra $\mathscr{A}$ *super-reflexive* if every unital weakly closed subalgebra of $\mathscr{A}$ is reflexive, and call an operator T *super-reflexive* if $\mathscr{A}(T)$ is. Then Sarason's results [23] can be reformulated as the unilateral shift and every commutative von Neumann algebra are super-reflexive.

In this paper we study some properties of algebras that, in the presence of reflexivity, imply super-reflexivity. Our main results show that some of these properties are preserved under direct integrals. Using a result of E. Azoff, C. K. Fong, and F. Gilfeather [3] that says that a direct integral of reflexive algebras is reflexive, we are able to prove that, under certain conditions, a direct integral of super-reflexive algebras is super-reflexive. This leads to conditions under which a direct integral of reflexive operators is reflexive. It is not known whether the direct sum of two reflexive operators must be reflexive. It is also unknown whether the reflexivity of an operator T implies that of T^2.

Along with various applications of our results on direct integrals we include some other results on reflexivity. We show that the characterization by Deddens and Fillmore [8] of reflexivity for operators on a finite-dimensional space can be extended to arbitrary algebraic operators on an arbitrary Banach space. Finally, we consider analogues of our results on super-reflexivity for algebras $\mathscr{A}$ such that either $\mathscr{A} = \mathscr{A}'' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{A})$ or $\mathscr{A} = \mathscr{A}' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{A})$, where $\mathscr{A}'$ denotes the commutant of $\mathscr{A}$. Rosenthal [20] and Sarason [24] have asked whether the latter relation holds for all singly generated algebras; Azoff [2] has shown that even the former relation does not hold in general.

Throughout, let H denote a Hilbert space, and let $B(H)$ denote the set of (bounded linear) operators on H. If $T \in B(H)$, $\mathscr{S} \subset B(H)$, and n is a positive integer or ∞, then $H^{(n)}$ denotes a direct sum of n copies of H, $T^{(n)}$ denotes a direct sum of n copies of T acting on $H^{(n)}$ and $\mathscr{S}^{(n)} = \{S^{(n)} : S \in \mathscr{S}\}$. If $\{H_n\}$ is a sequence of pairwise orthogonal subspaces of H such that $H = H_1 \oplus H_2 \oplus \ldots$, and if $\mathscr{S}_n \subset B(H_n)$ for $n = 1, 2, \ldots$, then

$$\mathscr{S}_1 \oplus \mathscr{S}_2 \oplus \ldots = \{S_1 \oplus S_2 \oplus \ldots \in B(H) : S_n \in \mathscr{S}_n \text{ for } n = 1, 2, \ldots\}.$$

Note, for example, that $\mathscr{S} \oplus \mathscr{S}$ is not the same as $\mathscr{S}^{(2)}$. If $\mathscr{S} \subset B(H)$, then $\mathscr{A}(\mathscr{S})$ denotes the weakly closed algebra generated by $\mathscr{S}$ and 1, and $\mathscr{S}' = \{T \in B(H) : ST = TS$ for every S in $\mathscr{S}\}$ is the commutant of $\mathscr{S}$. If $\mathscr{S} \subset B(H)$ and $f \in H$, then $\mathrm{Cyc}(\mathscr{S}, f)$ denotes the smallest subspace of H that contains f and is invariant for every S in $\mathscr{S}$. We write $\mathrm{Cyc}(S,f)$ if $\mathscr{S} = \{S\}$, and $S|\mathrm{Cyc}(S,f)$ is called a *cyclic part* of S. Also $\mathscr{S}^* = \{S^* : S \in \mathscr{S}\}$. The set of complex numbers is denoted $\mathbf{C}$.

2. PROPERTIES C, D, D(r)

In [28] Wogen defined an operator T to have *property* C if, for each positive integer n, each cyclic part of $T^{(n)}$ is unitarily equivalent to a cyclic part of T. More generally, a subset $\mathscr{S}$ of $B(H)$ has *property* C if, for each positive integer n and each f in $H^{(n)}$, there is a vector g in H and a unitary operator U: $\mathrm{Cyc}(\mathscr{S}^{(n)}, f) \to \mathrm{Cyc}(\mathscr{S}, g)$ such that

$$U^*[S|\mathrm{Cyc}(\mathscr{S}, g)]U = S^{(n)}|\mathrm{Cyc}(\mathscr{S}^{(n)}, f)$$

for every S in $\mathscr{S}$. It is obvious that a subset $\mathscr{S}$ of $B(H)$ has property C if and only if $\mathscr{A}(\mathscr{S})$ does. Here are some other elementary facts concerning property C.

PROPOSITION 2.1. (1) *A direct sum of unital algebras has property* C *if and only if each summand has property* C,

(2) *if* $\mathcal{S}_1 \subset \mathcal{S}_2 \subset B(H)$ *and* $\mathcal{S}_2$ *has property* C, *then so does* $\mathcal{S}_1$,

(3) *a direct sum of operators with property* C *has property* C,

(4) *if* $\mathcal{S} \subset B(H)$, *then* $\mathcal{S}$ *has property* C *precisely when for each* f *and* g *in* H *there exists* h *in* H *such that*

$$\|Af\|^2 + \|Ag\|^2 = \|Ah\|^2$$

for every A *in* $\mathcal{A}(\mathcal{S})$,

(5) *if* $\mathcal{S} \subset B(H)$ *and there is a unitary operator* $U : H^{(2)} \to H$ *such that* $U^*SU = S^{(2)}$ *for every* S *in* $\mathcal{S}$, *then* $\mathcal{S}$ *has property* C,

(6) *the unilateral shift operator has property* C,

(7) *the algebra of scalar multiples of the identity operator has property* C.

Proof. Statement (7) is obvious, and the proofs of statements (4) and (2) are straightforward. Statement (3) follows immediately from (1) and (2), and statement (5) follows immediately from (4). Statement (6) is in [23]. We give a proof of statement (1). Suppose $\mathcal{A}_i \subset B(H_i)$ for each i in some index set I, let H be the direct sum of the H_i's, and let $\mathcal{A}$ be the direct sum of the weakly closed unital algebras $\mathcal{A}_i (i \in I)$. First suppose each $\mathcal{A}_i$ has property C and $f \in H^{(2)}$. Write f as the direct sum of vectors $f_i (i \in I)$ where each f_i is in $H_i^{(2)}$. For each i in I, property C (via (4)) implies the existence of h_i in H_i such that

$$\|A_i f_i\|^2 + \|A_i g_i\|^2 = \|A_i h_i\|^2$$

for all A_i in $\mathcal{A}_i$. It follows that if h is the direct sum of the h_i's and if $A \in \mathcal{A}$, then $\|Af\|^2 + \|Ag\|^2 = \|Ah\|^2$, and consequently (4) implies $\mathcal{A}$ has property C. Conversely, suppose $\mathcal{A}$ has property C, and suppose $j \in I$ and $f, g \in H_j \subset H$. Since $\mathcal{A}$ has property C, there exists an h in H satisfying the preceding equation for all A in $\mathcal{A}$. Since $1 \in \mathcal{A}_i$ for every i, it follows that $h \in H_j$, and thus $\mathcal{A}_j$ has property C.

REMARKS 2.2. (a) A 2×2 complex matrix has property C if and only if it is normal.

(b) For each positive integer n, let J_n denote the $n \times n$ nilpotent Jordan block, and let $\{n_k\}$ be an unbounded sequence of positive integers. Let T be the direct sum of the J_{n_k}'s. If $1 \leqslant m \leqslant n$, then J_m is unitarily equivalent to a cyclic part of J_n. It follows that $T^{(\infty)}$ is unitarily equivalent to the restriction of T to some invariant subspace. It easily follows that T has property C.

We next consider another property, which, for lack of imagination, we call property D. Suppose $\mathcal{A}$ is a unital subalgebra of $B(H)$ that is closed in the weak operator topology. Let $\mathcal{B}$ be the set of linear functionals φ on $\mathcal{A}$ of the form

$$\varphi(A) = (Ae_1, f_1) + (Ae_2, f_2) + \ldots + (Ae_n, f_n) \quad \text{where } e_1, f_1, \ldots, e_n, f_n \in H.$$

The weak operator topology on $\mathscr{A}$ is the weak topology on $\mathscr{A}$ induced by the functionals in $\mathscr{B}$. A basic result from the duality theory of topological vector spaces [21, Th. 3.10] asserts that every linear functional on $\mathscr{A}$ that is continuous in the weak operator topology is an element of $\mathscr{B}$. The algebra $\mathscr{A}$ has *property* D if every functional φ in $\mathscr{B}$ can be written in the form $\varphi(A) = (Af, g)$ for vectors f, g in H. The functional φ does not completely determine the vectors f, g; e.g., if λ is a non-zero complex number, then we can replace the pair f, g by λf, $(1/\lambda)g$. In particular, we can always choose f and g so that $\|f\| = \|g\|$. It is clear that if $\varphi(A) = (Af, g)$ for every A in $\mathscr{A}$, then $\|\varphi\| \leqslant \|f\| \|g\|$. Suppose $r \geqslant 1$. The algebra $\mathscr{A}$ has *property* D(r), if, for each φ in $\mathscr{B}$ and each $s > r$, there are vectors f, g in H such that $\varphi(A) = (Af, g)$ for every A in $\mathscr{A}$ and $\|f\| \|g\| \leqslant s\|\varphi\|$. A set $\mathscr{S} \subset B(H)$ has *property* D (resp. D(r)) if $\mathscr{A}(\mathscr{S})$ does.

Since the sets of strongly and weakly continuous linear functionals on $B(H)$ coincide (see e.g. [1, 1.2.E]), the definition of property D would remain the same if "weakly" were replaced by "strongly". Another consequence of this fact is that a strongly closed algebra (or even convex set) of operators is weakly closed (see also [20, Th.7.1]).

LEMMA 2.3. *If M is a weakly closed subspace of $B(H)$ and φ is a weakly continuous linear functional of norm 1 on M, then for every $\varepsilon > 0$ there is a weakly continuous extension of φ to $B(H)$ having norm less than $1 + \varepsilon$.*

Proof. This is an immediate consequence of the following lemma.

LEMMA 2.4. *Let Y be a Banach space and let $\mathscr{T}$ be a topology on Y that is smaller than the norm topology and that makes Y into a locally convex topological vector space in which the (norm) unit ball is $\mathscr{T}$-compact. If M is a $\mathscr{T}$-closed subspace of Y and if φ is a $\mathscr{T}$-continuous linear functional of norm 1 on M, then, for every $\varepsilon > 0$, there is a $\mathscr{T}$-continuous linear extension of φ to Y having norm less than $1 + \varepsilon$.*

Proof. Choose m in M of norm 1 such that $\varphi(m) > 1/(1 + \varepsilon)$, and let B be the closed ball of radius $1/(1 + \varepsilon)$ centered at m. The ball B is $\mathscr{T}$-compact, and if $A = \ker\varphi$, then A is $\mathscr{T}$-closed and we claim that A does not intersect B. For if $x \in B \cap M$, then

$$|\varphi(x)| \geqslant \varphi(m) - |\varphi(x - m)| > 1/(1 + \varepsilon) - \|x - m\| \geqslant 0 .$$

By the Hahn-Banach theorem [21, Th. 3.4(6)], there is a constant γ and a $\mathscr{T}$-continuous linear functional ψ on Y such that $\mathrm{Re}\psi(x) < \gamma < \mathrm{Re}\psi(y)$ for all x in A and y in B. Since A is a subspace, it follows that $\psi|A = 0$, i.e., $\ker\varphi \supset \ker\psi$. Thus $\gamma > 0$ and $\psi(m) \neq 0$. If $\Phi = (\varphi(m)/\psi(m))\psi$, then Φ is $\mathscr{T}$-continuous, $\ker\Phi \supset \ker\varphi$, $\Phi(m) = \varphi(m)$, and consequently Φ is an extension of φ.

It remains to show that $\|\Phi\| \leqslant 1+\varepsilon$. Let u be a unit vector in Y. Then there is an α in $\mathbf{C}$ and a y in $\ker\Phi$ such that $u = \alpha m + y$. Since B does not intersect $\ker\Phi$, it follows that the distance from m to $\ker\Phi$ exceeds $1/(1+\varepsilon)$. Hence $|\alpha| < 1+\varepsilon$. The norm inequality now follows from

$$|\Phi(u)| = |\alpha|\varphi(m) < 1+\varepsilon\,.$$

The following basic proposition shows some relationships between properties C, D, D(r) and super-reflexivity.

PROPOSITION 2.5. (1) *A reflexive algebra with property* D *is super-reflexive.*

(2) *An algebra with property* C *has property* D(1).

(3) *If* $\mathscr{S}_1 \subset \mathscr{S}_2 \subset B(H)$ *and* $\mathscr{S}_2$ *has property* D *(resp.* D(r)*), then so does* $\mathscr{S}_1$.

(4) *If* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *are unital weakly closed algebras,* $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \mathrm{Alg}(\mathrm{Lat}\mathscr{A}_1)$, *and if* $\mathscr{A}_2$ *has property* D, *then* $\mathscr{A}_1 = \mathscr{A}_2$.

(5) *If* $\mathscr{A}$ *is a unital weakly closed algebra and* $\mathrm{Alg}(\mathrm{Lat}\mathscr{A})$ *has property* D, *then* $\mathscr{A}$ *is super-reflexive.*

(6) *A direct sum of unital algebras has property* D(r) *if and only if each summand has property* D(r).

Proof. Statement (3) follows from Lemma 2.3, and statement (5) follows immediately from (4) and (1). The proof of statement (6) is similar to the proof of Proposition 2.1 (1). To prove (4) suppose $\mathscr{A}_1 \neq \mathscr{A}_2$. It follows from the Hahn-Banach theorem and the fact that $\mathscr{A}_2$ has property D that there are vectors f, g in H such that $(Af, g) = 0$ for every A in $\mathscr{A}_1$, but, for some T in $\mathscr{A}_2$, $(Tf, g) \neq 0$. This implies $g \perp \mathrm{Cyc}(\mathscr{A}_1, f)$. Since $T \in \mathrm{Alg}(\mathrm{Lat}\mathscr{A}_1)$, it follows that $g \perp T(\mathrm{Cyc}(\mathscr{A}_1, f))$. In particular $(Tf, g) = 0$, a contradiction.

We now prove (1). Suppose $\mathscr{A}_0$ is a unital weakly closed subalgebra of $\mathscr{A}$. Clearly, $\mathrm{Alg}(\mathrm{Lat}\mathscr{A}_0) \subset \mathrm{Alg}(\mathrm{Lat}\mathscr{A}) = \mathscr{A}$. It follows from (3) that $\mathrm{Alg}(\mathrm{Lat}\mathscr{A}_0)$ has property D, and, by (4), $\mathscr{A}_0 = \mathrm{Alg}(\mathrm{Lat}\mathscr{A}_0)$. Thus $\mathscr{A}_0$ is reflexive. This proves (1).

To prove (2) suppose $\mathscr{A}$ has property C and φ is a weakly continuous functional of norm one on $\mathscr{A}$. Let $s > 1$. Extend φ to a functional on $B(H)$ that is weakly continuous and of norm less than s using Lemma 2.3. Then there exist $f_1', \ldots, f_m', g_1', \ldots, g_m'$ such that $\varphi(A) = \sum_{j=1}^{m} (Af_j', g_j')$ for all A in $B(H)$. Let T be the finite rank operator defined by $Th = \sum_{j=1}^{m} (h, g_j')f_j'$. If "tr" denotes the trace, then it follows easily that $\varphi(A) = \mathrm{tr}AT$, and it is well known that the norm of φ is just the trace norm of T. Let T have polar decomposition $T = UP$, where U is a partial isometry and P is positive, and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for the range of P consisting of eigenvectors of P. Thus $Pe_j = t_je_j$ for $j = 1, 2, \ldots, n$ and $\|\varphi\| = \sum_{j=1}^{m} t_j$. Let $f_j = t_j^{1/2}Ue_j$ and $g_j = t_j^{1/2}e_j$ for $j = 1, 2, \ldots, n$. Then for A in

$B(H)$, we have $\varphi(A) = \sum_{j=1}^{m} (Af_j, g_j)$ and

$$\|\varphi\| = \left(\sum_{j=1}^{m} \|f_j\|^2\right)^{1/2} \left(\sum_{j=1}^{m} \|g_j\|^2\right)^{1/2} < s.$$

Let $M = \mathrm{Cyc}(\mathscr{A}^{(n)}; (f_1, \ldots, f_n))$. Since $\mathscr{A}$ has property C, there is a vector f in H and a unitary operator $V : M \to \mathrm{Cyc}(\mathscr{A}, f)$ such that $A^{(n)}|M = V^* A V$ for all A in $\mathscr{A}$. Clearly we may assume $V(f_1, \ldots, f_n) = f$. Let g be the image under V of the projection of $(g_1, \ldots, g_n)$ onto M. It follows from the preceding paragraph that for A in $\mathscr{A}$, $\varphi(A) = (A^{(n)}(f_1, \ldots, f_n), (g_1, \ldots, g_n)) = (Af, g)$ and $\|f\|\|g\| < s$. This completes the proof.

REMARKS 2.6. (a) The ideas in the proof of the preceding theorem were used by Sarason, whose proof in [24] that every unital weakly closed commutative algebra of normal operators is reflexive was based on an idea of Goodman [9] that shows that every commutative von Neumann algebra has property D. It was first formally stated by W. Wogen [28] that a reflexive algebra with property C is super-reflexive. Recently, R. Olin and J. Thomson [19] proved that every subnormal operator has property D(r) for some fixed r.

(b) Proposition 2.1(6) and Proposition 2.5(2) imply that the algebra of analytic Toeplitz operators has property D(1). A proof of this can also be based on duality theory (see [19]). Rosenoer has shown that every compression of the unilateral shift to the orthogonal complement of an invariant subspace has property D.

3. DIRECT INTEGRALS

Before we consider more examples of operators or algebras with one or more of the properties of the preceding section, we wish to discuss the relationships between these properties and direct integrals. Throughout this section we assume that H is separable.

Suppose $(X, \mathscr{M}, \mu)$ is a complete sigma-finite measure space. Define $L^2(\mu, H)$ as the set of all measurable functions $f : X \to H$ such that $\int_X \|f(x)\|^2 \, d\mu(x)$ is finite, and define $L^\infty(\mu, B(H))$ as the set of all measurable (relative to the weak operator topology) and essentially (norm) bounded functions $\varphi : X \to B(H)$, where two functions are identified if they agree almost everywhere. The space $L^2(\mu, H)$ becomes a Hilbert space if we define an inner product by $(f, g) = \int_X (f(x), g(x)) d\mu(x)$. Each φ in $L^\infty(\mu, B(H))$ induces an operator M_φ on $L^2(\mu, H)$ defined by $(M_\varphi f)(x) = \varphi(x) f(x)$.

Suppose that $\mathscr{S}_x \subset B(H)$ for every x in X. The *direct integral* of the $\mathscr{S}_x$'s over X, denoted by $\int_X^{\oplus} \mathscr{S}_x \mathrm{d}\mu(x)$, is the set

$$\{M_\varphi : \varphi \in L^\infty(\mu, B(H)), \varphi(x) \in \mathscr{S}_x \text{ a.e.}\}.$$

The most important result of this section shows that if every $\mathscr{S}_x$ has property D(r), then so does the direct integral of the $\mathscr{S}_x$'s.

REMARKS 3.1. (a) It is possible to have $\mathscr{S}_x \neq \emptyset$ for every x in X, but $\int_X^{\oplus} \mathscr{S}_x \, \mathrm{d}\mu(x) = \emptyset$.

(b) There are more general direct integrals than the ones we have defined here, but the more general ones are isomorphic to direct sums of the type we have just defined. Thus the preceding results on direct sums make the results of this section valid for the more general types of direct integrals.

As with many theorems about direct integrals, one of the key ideas used here is von Neumann's principle of measurable selection. The version of this result we use is much stronger than the one von Neumann originally proved [18]. Let Y be a complete separable metric space, and let $(X, \mathscr{M}, \mu)$ be a complete sigma-finite measure space as above. The following is in [12] and [22].

THEOREM 3.2. *Give $X \times Y$ the product Borel structure, let E be a measurable subset of $X \times Y$, and define E^Y to be $\{x \in X : (x, y) \in E$ for some y in $Y\}$. Then $E^Y \in \mathscr{M}$, and there is a measurable function from E^Y into Y whose graph is included in E.*

The following is a consequence of Theorem 3.2 (see [12]).

THEOREM 3.3. *Let E be a subset of $X \times Y$ and define E_x to be $\{y \in Y : (x, y) \in E\}$ for each x in X. Suppose each E_x is closed. Then E is measurable if and only if there is a sequence of measurable functions φ_n on E^Y such that $\{\varphi_n(x) : n = 1, 2, \ldots\}$ is dense in E_x for every x.*

Suppose, for each x in X, that $\mathscr{S}_x$ is a strongly closed subset of the closed unit ball of $B(H)$. The family $\{\mathscr{S}_x\}_{x \in X}$ is *measurable* [3] if and only if there are functions $\varphi_1, \varphi_2, \ldots$ in $L^\infty(\mu, B(H))$ such that $\varphi_1(x), \varphi_2(x), \ldots$ is strongly dense in $\mathscr{S}_x$ for almost every x in X. If Y is taken to be the unit ball of $B(H)$ with the strong operator topology, then Y is a complete separable metric space (see [1] or [25]). Theorem 3.3 then implies that measurability for $\{\mathscr{S}_x\}_{x \in X}$ amounts to the existence of a null set N in X such that if $E = \{(x, A) : x \notin N$ and $A \in \mathscr{S}_x\}$, then E is measurable in $X \times Y$. A family $\{\mathscr{A}_x\}_{x \in X}$ of weakly closed unital algebras is *measurable* if $\{A \in \mathscr{A}_x : \|A\| \leqslant 1\}_{x \in X}$

is measurable. A family $\{\mathscr{A}_x\}$ of weakly closed algebras is *attainable* [3] if there are functions $\varphi_1, \varphi_2, \ldots$ in $L^\infty(\mu, B(H))$ such that $\mathscr{A}_x = \mathscr{A}(\varphi_1(x), \varphi_2(x), \ldots)$ for almost every x in X. It is shown in [3] that attainable families of unital weakly closed algebras are in fact measurable, that direct integrals of such families are weakly closed, and the direct integral of an attainable family of reflexive algebras is reflexive. They use a stronger measure-theoretic hypothesis than ours, but the principle of measurable choice quoted above makes their results valid in our context. Our notion of direct integral of unital algebras does not quite coincide with that of [3], since ours always contain the "diagonal algebra".

The measurability of families of algebras plays a central role in the theory of direct integrals of algebras. Since many of the properties of algebras that concern us here are inherited by subalgebras, we can often omit the assumption of measurability. The key idea is contained in the following lemma, and an illustrative example is contained in the corollary.

LEMMA 3.4. *If $L^2(\mu, H)$ is separable and $\{\mathscr{A}_x\}_{x\in X}$ is a family of weakly closed unital subalgebras of $B(H)$, then there is a measurable family $\{\mathscr{B}_x\}$ of unital weakly closed subalgebras of $B(H)$ such that $\mathscr{B}_x \subset \mathscr{A}_x$ for almost every x in X and*

$$\int_X^{\oplus} \mathscr{B}_x \mathrm{d}\mu(x) = \int_X^{\oplus} \mathscr{A}_x \mathrm{d}\mu(x).$$

Proof. Since $L^2(\mu, H)$ is separable, we can choose a sequence $\{\varphi_n\}$ in $L^\infty(\mu, B(H))$ so that $\{M_{\varphi_n}\}$ is weakly dense in the direct integral of the $\mathscr{A}_x$'s. For each x in X, let $\mathscr{B}_x = \mathscr{A}(\varphi_1(x), \varphi_2(x), \ldots)$. Clearly, $\mathscr{B}_x \subset \mathscr{A}_x$ a.e.; thus the direct integral of the $\mathscr{B}_x$'s is contained in the direct integral of the $\mathscr{A}_x$'s. The reverse inclusion follows from the fact that the direct integral of the $\mathscr{B}_x$'s is weakly closed and contains $\{M_{\varphi_n}\}$.

COROLLARY 3.5. *If $\{\mathscr{A}_x\}_{x\in X}$ is a (not necessarily measurable) family of super-reflexive algebras in $B(H)$, then $\int_X^{\oplus} \mathscr{A}_x \mathrm{d}\mu(x)$ is reflexive.*

We are concerned with situations in which the direct integral of super-reflexive algebras is super-reflexive. The following two theorems are therefore of prime interest. Note the absence of any measurability assumptions in the corollaries.

THEOREM 3.6. *Suppose $r \geqslant 1$ and $\{\mathscr{A}_x\}_{x\in X}$ is a measurable family of unital weakly closed algebras in $B(H)$ and that almost every $\mathscr{A}_x$ has property* D(r). *Then $\int_X^{\oplus} \mathscr{A}_x \mathrm{d}\mu(x)$ has property* D(r).

Proof. Choose $\varphi_1, \varphi_2, \ldots$ in $L^\infty(\mu, B(H))$ so that $\{\varphi_1(x), \varphi_2(x), \ldots\}$ is weakly dense in the unit ball of $\mathscr{A}_x$ for almost every x in X. Let $\mathscr{A}$ be the direct integral of the $\mathscr{A}_x$'s, and let ω be a weakly continuous linear functional on $\mathscr{A}$. Then there are vectors $f_1, g_1, f_2, g_2 \ldots$ in $L^2(\mu, H)$ such that $\omega(A) = \sum_{i=1}^{n} (Af_i, g_i)$ for every A in $\mathscr{A}$. For each x in X, define a weakly continuous linear functional ω_x on $\mathscr{A}_x$ by

$$\omega_x(A) = \sum_{i=1}^{n} (Af_i(x), g_i(x)).$$

It follows from the choice of the φ_k's that the mapping $x \to \|\omega_x\| = \sup\{|\omega_x(\varphi_k(x))| : k \geqslant 1\}$ is measurable.

We next show that $\|\omega\| = \int_X \|\omega_x\| \mathrm{d}\mu(x)$. Suppose $A = M_\varphi \in \mathscr{A}$ where $\varphi \in L^\infty(\mu, B(H))$. Then

$$|\omega(A)| = \left|\int_X \omega_x(\varphi(x))\, \mathrm{d}\mu(x)\right| \leqslant \int_X \|\omega_x\|\, \|\varphi(x)\|\, \mathrm{d}\mu(x) \leqslant$$

$$\leqslant \left(\int_X \|\omega_x\|\, \mathrm{d}\mu(x)\right) \|A\|.$$

Thus $\|\omega\| \leqslant \int_X \|\omega_x\|\, \mathrm{d}\mu(x)$. Conversely let Y be the closed unit ball of $B(H)$, and consider the sets

$$E_1 = \{(x, A) \in X \times Y : A \in A_x\}$$

and

$$E_2 = \left\{(x, A) \in X \times Y : \sum_{i=1}^{n} (Af_i(x), g_i(x) = \|\omega_x\|\right\}.$$

The measurability of $\{\mathscr{A}_x\}$ together with Theorem 3.3 implies the existence of a null set N in X such that $E_1 \setminus (N \times Y)$ is measurable in $X \times Y$. The measurability of each of the functions $(x, A) \to (Af_i(x), g_i(x))$ and the measurability of the mapping $x \to \|\omega_x\|$ imply E_2 is also measurable in $X \times Y$. Thus if $E = E_1 \cap E_2$, then $E \setminus (N \times Y)$ is measurable. Since the unit ball of $\mathscr{A}_x$ is weakly compact and since ω_x is weakly continuous, $\|\omega_x\| = \omega_x(A)$ for some A in the unit ball of $\mathscr{A}_x$. Thus the projection of $E \setminus (N \times Y)$ on X is $X \setminus N$, and by Theorem 3.2, there is a measurable function φ on $X \setminus N$ such that $\omega_x(\varphi(x)) = \|\omega_x\|$. Hence $M_\varphi \in \mathscr{A}$, $\|M_\varphi\| \leqslant 1$, and $\omega(M_\varphi) = \int_X \|\omega_x\|\, \mathrm{d}\mu(x)$, which establishes $\|\omega\| = \int_X \|\omega_x\|\, \mathrm{d}\mu(x)$.

To see that $\mathscr{A}$ has property D(r), take $s > r$ and consider the subset F of $X \times H \times H$ defined by

$$F = \{(x, u, v) : \|u\|^2 = \|v\|^2 \leqslant s\|\omega_x\| \text{ and } \omega_x(\varphi_k(x)) = (\varphi_k(x)u, v) \text{ for all } k\}.$$

Since $\mathscr{A}_x$ has property D(r) a.e., there exists a null set N_1 such that $F \setminus (N_1 \times H \times H)$ is measurable and its projection on X is $X \setminus N_1$. By Theorem 3.2, there exist a.e. defined functions f and g such that $(x, f(x), g(x)) \in F$ a.e. . Thus

$$\int_X \|f\|^2 \,\mathrm{d}\mu = \int_X \|g\|^2 \mathrm{d}\mu \leqslant \int_X s\|\omega_x\| \,\mathrm{d}\mu(x) = s\|\omega\|,$$

so f and g are in $L^2(\mu, H)$, and $\|f\|\|g\| \leqslant s\|\omega\|$. Also, if $M_\varphi \in \mathscr{A}$, then

$$\omega(M_\varphi) = \int_X \omega_x(\varphi(x)) \,\mathrm{d}\mu(x) = \int_X (\varphi(x)f(x), g(x)) = (M_\varphi f, g).$$

Hence $\mathscr{A}$ has property D(r).

COROLLARY 3.7. *Suppose $L^2(\mu, H)$ is separable, $r \geqslant 1$, and, for almost every x in X, $\mathscr{S}_x \subset B(H)$ and $\mathscr{S}_x$ has property* D(r). *Then* $\int_X^{\oplus} \mathscr{S}_x \mathrm{d}\mu(x)$ *has property* D(r).

Proof. Imitate the proof of Corollary 3.4.

THEOREM 3.8. *If $r \geqslant 1$ and $\{\mathscr{A}_x\}_{x \in X}$ is a measurable family of unital weakly closed algebras such that, for almost every x in X, $\mathscr{A}_x$ is reflexive and has property* D(r), *then* $\int_X^{\oplus} \mathscr{A}_x \mathrm{d}\mu(x)$ *is super-reflexive.*

Proof. The algebra $\int_X^{\oplus} \mathscr{A}_x \mathrm{d}\mu(x)$ is reflexive by [3]. It has property D by Theorem 3.6 and is therefore super-reflexive by Proposition 2.5(1).

THEOREM 3.9. *If $r \geqslant 1$, $L^2(\mu, H)$ is separable, for almost every x in X, $\mathscr{A}_x$ is a reflexive subalgebra of $B(H)$ with property* D(r), *then* $\int_X^{\oplus} \mathscr{A}_x \mathrm{d}\mu(x)$ *is super-reflexive.*

COROLLARY 3.10. (Sarason [23]). *Every commutative von Neumann algebra is super-reflexive.*

COROLLARY 3.11. (Wogen [28]). *Every quasinormal operator is super-reflexive.*

COROLLARY 3.12. *If* $L^2(\mu, H)$ *is separable, and, for almost every* x *in* X, $\mathscr{A}_x$ *is a unital weakly closed algebra of analytic Toeplitz operators, then* $\int_X^{\oplus} \mathscr{A}_x \mathrm{d}\mu(x)$ *is super-reflexive and has property* D(1).

REMARK 3.13. Using Proposition 2.1(4) and the techniques used in proving Theorem 3.6, one can also show that a direct integral of a measurable family of unital weakly closed algebras with property C has property C.

4. WEAK AND ULTRAWEAK TOPOLOGIES

R. Olin and J. Thomson [19] have shown that whenever T is a subnormal operator the weak operator topology on $\mathscr{A}(T)$ coincides with the ultraweak operator topology. We will show how property D occasionally provides a technique for showing that these topologies agree. Recall that the ultraweak (σ-weak, weak*) topology on $B(H)$ is the weak* topology when $B(H)$ is regarded as the dual space of the Banach space of trace class operators. A useful way of viewing the ultraweak topology is obtained by noting that a net $\{T_n\}$ in $B(H)$ converges ultraweakly to an operator T if and only if the net $\{T_n^{(\infty)}\}$ converges weakly to $T^{(\infty)}$.

The theory of direct integrals of ultraweakly closed operator algebras proceeds much like the theory for weakly closed algebras. Call a family $\{\mathscr{A}_x\}_{x \in X}$ of unital ultraweakly closed algebras *attainable* if there are functions $\varphi_1, \varphi_2, \ldots$ in $L^\infty(\mu, B(H))$ such that $\mathscr{A}_x$ is the unital weakly closed algebra generated by $\{\varphi_1(x), \varphi_2(x), \ldots\}$ for almost every x. This is equivalent to requiring $\{\mathscr{A}_x^{(\infty)}\}$ to be attainable in the sense of the preceding section. It follows from [3] that $\int_X^{\oplus} \mathscr{A}_x^{(\infty)} \,\mathrm{d}\mu\,(x)$ is weakly closed, and hence we see that the direct integral of an attainable family of unital ultraweakly closed algebras is ultraweakly closed.

We say that an ultraweakly closed unital algebra $\mathscr{A}$ has *property* D_σ if every ultraweakly continuous linear functional φ on $\mathscr{A}$ has the form $\varphi(A) = (Af, g)$ for some f, g in H. If in addition $r \geqslant 1$ and, for every $s > r$, f and g can be chosen so that $\varphi(A) = (Af, g)$ for every A in $\mathscr{A}$ and $\|f\|\,\|g\| \leqslant s\,\|\varphi\|$, then we say that $\mathscr{A}$ has property $\mathrm{D}_\sigma(r)$.

The following proposition shows some of the relevant facts concerning property D_σ and property $\mathrm{D}_\sigma(r)$.

PROPOSITION 4.1. (1) *If $\mathscr{A}$ is an ultraweakly closed unital algebra with property* D_σ *(resp.* $D_\sigma(r)$, $r \geq 1$), *then every ultraweakly closed unital subalgebra of $\mathscr{A}$ has property* D_σ *(resp.* $D_\sigma(r)$).

(2) *A direct sum (resp. direct integral) of a family (resp. measurable family) of unital ultraweakly closed algebras with property* $D_\sigma(r)$, $r \geq 1$, *has property* $D_\sigma(r)$.

(3) *If $\mathscr{A}$ is a weakly closed algebra with property* D_σ, *then the weak and ultraweak topologies coincide on $\mathscr{A}$.*

(4) *If a direct sum of an infinite collection of unital ultraweakly closed algebras has property* D_σ, *then it has property* $D_\sigma(r)$ *for some* $r \geq 1$.

Proof. (1). Imitate the proof of Proposition 2.5 (3). Note that the analogue of Lemma 2.3 follows from Lemma 2.4 and the fact that the unit ball of $B(H)$ is ultraweakly compact.

(2). Imitate the proof of Theorem 3.6.

(3). The weak and ultraweak topologies are, by hypothesis, generated by the same family of linear functionals.

(4). Suppose $\mathscr{A}$ is the direct sum of the $\mathscr{A}_n$'s and $\mathscr{A}$ has property D_σ. By (2), we need show only that each $\mathscr{A}_n$ has property $D_\sigma(r)$ for some r. Assume the contrary. Then there is an increasing sequence $\{k_n\}$ such that $\mathscr{A}_{k_n}$ does not have property $D_\sigma(2^n)$ for $n = 1, 2, \ldots$. Hence for each positive integer n, there is an ultraweakly continuous linear functional φ_n on $\mathscr{A}_{k_n}$ such that $\|\varphi_n\| \leq 1/2^n$ and $\|f\| \|g\| \geq 1$ whenever $f, g \in H$ and $\varphi_n(A) = (Af, g)$ for every A in $\mathscr{A}_{k_n}$. Define a linear functional φ on $\mathscr{A}$ by $\varphi(A_1 \oplus A_2 \oplus \ldots) = \sum_n \varphi_n(A_{k_n})$. Clearly φ is ultraweakly continuous. If $f = f_1 \oplus f_2 \oplus \ldots$ and $g = g_1 \oplus g_2 \oplus \ldots$ and $\varphi(A) = (Af, g)$ for every A in $\mathscr{A}$, then $\varphi_n(A) = (Af_{k_n}, g_{k_n})$ for every A in $\mathscr{A}_{k_n}$ and every positive integer. Thus $\|f_{k_n}\| \|g_{k_n}\| \geq 1$ for every n, a contradiction.

COROLLARY 4.2. *Suppose $r \geq 1$, n is a positive integer, and $\{\mathscr{A}_x\}_{x \in X}$ is a measurable family of weakly closed algebras such that $\mathscr{A}_x^{(n)}$ has property* $D_\sigma(r)$ *for almost every x in X. Then the weak and ultraweak topologies agree on* $\int_X^{\oplus} \mathscr{A}_x \, d\mu(x)$.

COROLLARY 4.3. *For each positive integer n the weak and ultraweak topologies agree on the algebra of all decomposable operators on* $L^2(\mu, \mathbf{C}^{(n)})$.

An operator T is *n-normal* if T is unitarily equivalent to an $n \times n$ operator matrix with commuting normal entries; equivalently, if T is unitarily equivalent to a decomposable operator on $L^2(\mu, \mathbf{C}^{(n)})$ for some sigma-finite measure space (X, μ).

COROLLARY 4.4. *If n is a positive integer and T is an n-normal operator, then the weak and ultraweak topologies agree on $\mathscr{A}(T)$.*

5. EXAMPLES AND APPLICATIONS

Suppose $\mathscr{A}$ is a norm closed unital subalgebra of $B(H)$, $f \in H$, and $\|f\| = 1$. Define mapping $\rho_f : \mathscr{A} \to \mathrm{Cyc}(\mathscr{A}, f)$ by $\rho_f(A) = Af$ for every A in $\mathscr{A}$. If ρ_f is one-to-one, then f is called a *separating* vector for $\mathscr{A}$, and if ρ_f is onto, then f is a *relatively strictly cyclic vector* for $\mathscr{A}$. The algebra $\mathscr{A}$ is *strictly cyclic* if $\mathscr{A}$ has a relatively strictly cyclic unit vector f with $\mathrm{Cyc}(\mathscr{A}, f) = H$ (the vector f is called a *strictly cyclic vector* for $\mathscr{A}$). Strictly cyclic algebras were introduced by A. Lambert [13, 14, 15].

More generally, suppose M is an invariant subspace of $\mathscr{A}$ and define $\rho_M : \mathscr{A} \to \mathscr{A} \mid M$ by $\rho_M(A) = A \mid M$ for every A in $\mathscr{A}$. Call M *separating* if ρ_M is one--to-one. If M is separating and $\mathscr{A} \mid M$ is norm closed, then ρ_M has a bounded inverse $\rho_M^{-1} : \mathscr{A} \mid M \to \mathscr{A}$. It does not, however, follow that ρ_M^{-1} is weakly continuous even if $\mathscr{A} \mid M$ is weakly closed. To see that, consider the example where $\mathscr{A}$ is $B(H)^{(\infty)}$ on $H \oplus H \oplus \ldots$ and $M = H \oplus 0 \oplus \ldots$. Since a net $\{A_n^{(\infty)}\}$ in $\mathscr{A}$ converges weakly to $A^{(\infty)}$ if and only if $A_n \to A$ ultraweakly, weak continuity of ρ_M^{-1} would imply that the weak and ultraweak topologies agree on $B(H)$, which is obviously false (they have different continuous linear functionals). On the other hand, if $\mathscr{A}$ and $\mathscr{A} \mid M$ are both ultraweakly closed and M is separating, then ρ_M^{-1} is ultraweakly continuous. This follows from the fact that a weak* continuous linear mapping between the duals of two Banach spaces that is one-to-one and onto has a weak* continuous inverse (because the weak* continuity implies that the mapping is the dual of a bounded linear operator between the original Banach spaces, so the open mapping theorem applies). The following theorem shows how these ideas relate to property D.

THEOREM 5.1. *Suppose $\mathscr{A}$ is a unital norm closed subalgebra of $B(H)$.*

(1) *If $\mathscr{A}$ has a relatively strictly cyclic separating unit vector f, then $\mathscr{A}$ is weakly closed and $\mathscr{A}$ has property $\mathrm{D}_\sigma(\|\rho_f^{-1}\|)$.*

(2) *If $\mathscr{A}$ has a separating invariant subspace M such that $\mathscr{A}$ and $\mathscr{A} \mid M$ are weakly closed and if $\mathscr{A} \mid M$ has property* D *(resp. $\mathrm{D}(r)$), then $\mathscr{A}$ has property* D *(resp. $\mathrm{D}(r\|\rho_M^{-1}\|)$).*

(3) *If $\mathscr{A}$ has a separating invariant subspace M such that $\mathscr{A}$ and $\mathscr{A}|M$ are ultraweakly closed, and if $\mathscr{A}|M$ has property D_σ (resp. $\mathrm{D}_\sigma(r)$), then $\mathscr{A}$ has property D_σ (resp. $\mathrm{D}_\sigma(r\|\rho_M^{-1}\|)$).*

Proof. (1) The open mapping theorem implies that ρ_f^{-1} is continuous; thus $\mathscr{A}$ is strongly (hence weakly) closed. Suppose ρ is an ultraweakly continuous linear functional on $\mathscr{A}$. It follows from the open mapping theorem that $\rho \circ \rho_f^{-1}$ is a continuous linear functional on $\mathrm{Cyc}(\mathscr{A}, f)$. Hence there is a vector g in $\mathrm{Cyc}(\mathscr{A}, f)$ such that $\rho \circ \rho_f^{-1}(h) = (h, g)$ for every h in $\mathrm{Cyc}(\mathscr{A}, f)$. Hence $\rho(A) = (Af, g)$ for every A in $\mathscr{A}$, and

$$\|f\| \|g\| = \|g\| = \|\rho \circ \rho_f^{-1}\| \leqslant \|\rho\| \|\rho_f^{-1}\|.$$

Thus $\mathscr{A}$ has property $\mathrm{D}_\sigma(\|\rho_f^{-1}\|)$.

The proofs of (2), (3) follow in a similar fashion.

COROLLARY 5.2. *If $\mathscr{A}$ is a unital weakly closed strictly cyclic commutative subalgebra of $B(H)$, then $\mathscr{A}$ has property $\mathrm{D}_\sigma(r)$ for some $r \geqslant 1$.*

Proof. By [15, Section 2], any strictly cyclic vector for $\mathscr{A}$ is separating.

COROLLARY 5.3. *If T is an algebraic operator, then $\mathscr{A}(T)$ has property $\mathrm{D}_\sigma(r)$ for some $r \geqslant 1$.*

COROLLARY 5.4. *If $\mathscr{A}$ is reflexive and $\mathscr{A}$ has a strictly cyclic separating vector, then $\mathscr{A}$ is super-reflexive.*

COROLLARY 5.5. *If $\mathscr{A}$ is a unital weakly (resp. ultraweakly) closed subalgebra of $B(H)$ with property $\mathrm{D}(r)$ (resp. $\mathrm{D}_\sigma(r)$) for some $r \geqslant 1$, and if M is a Hilbert space and $\rho : \mathscr{A} \to B(M)$ is a unital weakly (resp. ultraweakly) continuous algebra homomorphism, then the graph of ρ, i.e., $\{A \oplus \rho(A) : A \in \mathscr{A}\} \subset B(H \oplus M)$, has property $\mathrm{D}(\|\rho\| r)$ (resp. $\mathrm{D}_\sigma(\|\rho\| r)$).*

REMARKS 5.6. (a) The fact that the vector f be both strictly cyclic and separating is important in the preceding theorem. For example, $B(H)$ has a strictly cyclic vector but no separating vectors and $B(H)$ does not have property D (e.g., $B(H)$ is reflexive but not super-reflexive).

(b) In [10] it was shown that the commutant of a quasianalytic unilateral weighted shift (see [26, p. 103]) intersects the commutant of a non-zero compact operator only in the scalars; therefore a quasianalytic shift does not satisfy the hypothesis of Lomonosov's theorem. However, such shifts are reflexive [26, Proposition 37] and strictly cyclic by definition. Thus, by Corollary 4.4, every quasianalytic shift is super-reflexive.

(c) Note that if $\mathscr{A}$ satisfies the hypotheses of the preceding theorem, then $\mathscr{A}^*$ satisfies the conclusions.

There are some situations in which it is possible to use Theorem 5.1 to show that, for certain classes of operators, there is an $r \geqslant 1$ such that every operator in the class has property $\mathrm{D}(r)$. The technique involves finding, for each operator T in the class, a relatively strictly cyclic separating unit vector f for $\mathscr{A}(T)$ with $\|\rho_f^{-1}\| \leqslant r$. Obviously, this technique requires the existence of a lot of relatively strictly cyclic separating vectors, so the technique has limited application. It does work well for 2×2 matrices and (slightly more generally) for operators satisfying a polynomial equation of degree 2. However, the technique breaks down in the 3×3 case and the analogous results do not even hold in the 4×4 case (see Theorem 5.9).

PROPOSITION 5.7. *If $T \in B(H)$ and T satisfies a polynomial equation of degree not exceeding 2, then T has property* D($\sqrt{10}$).

Proof. If the degree of the minimal polynomial of T is less than 2, then T is a scalar and has property D(1). We next consider the case when $\sigma(T) = \{\lambda\}$ is a singleton. In this case $\mathscr{A}(T) = \mathscr{A}(T - \lambda)$, so we may assume $T^2 = 0$. Then, by [11, Theorem 1] T has an operator matrix of the from $\begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$ relative to $H = \ker T \oplus (\ker T)^\perp$. Suppose $0 < r < 1$, and choose a unit vector g in $(\ker T)^\perp$ so that $\|Cg\| \geqslant r\|C\| = r\|T\|$. Let $f = 0 \oplus g$. Then f is a relatively strictly cyclic separating vector for $\mathscr{A}(T)$. Furthermore, $\rho_f^{-1}(f) = 1$ and if $h = (Cg/\|Cg\|) \oplus 0$, then $\rho_f^{-1}(h) = T/\|Cg\|$. Since $\{f, h\}$ is an orthonormal basis for $\mathrm{Cyc}(\mathscr{A}(T), f)$, we can estimate $\|\rho_f^{-1}\|$ as follows. If $a, b \in \mathbf{C}$, then

$$\|\rho_f^{-1}(af + bh) = \|a1 + (bT)/\|Cg\|\,\| \leqslant |a| + |b|\,\|T\|/\|Cg\| \leqslant$$

$$\leqslant |a| + |b|/r \leqslant \|af + bh\|(1 + 1/r^2)^{1/2}.$$

Since $0 < r < 1$ was arbitrary, it follows from Theorem 5.1(1) that T has property D($\sqrt{2}$).

Next suppose $\sigma(T)$ has two points. We can assume that $\sigma(T) = \{0, 1\}$ (since $\mathscr{A}(T)$ is generated by an operator with this property). In this case T has an operator matrix of the form $\begin{bmatrix} 1 & A \\ 0 & 0 \end{bmatrix}$ relative to $H = \ker(T - 1) \oplus (\ker(T - 1))^\perp$. We can assume that $A \neq 0$ (otherwise T is normal and has property D(1)). Suppose $0 < r < 1$ and choose a unit vector g in $(\ker(T - 1))^\perp$ so that $\|Ag\| \geqslant r\|A\|$. Let $f = (1/\sqrt{2})$ $(Ag/\|Ag\| \oplus g)$. Then f is a relatively strictly cyclic separating vector for $\mathscr{A}(T)$. Let $u = (Ag/\|Ag\|) \oplus 0$ and $v = 0 \oplus g$. Then $\{u, v\}$ is an orthonormal basis for $\mathrm{Cyc}(\mathscr{A}(T), f)$. A simple computation shows that

$$\|\rho_f^{-1}(u)\| = \sqrt{2}\,\|T\|/(1 + \|Ag\|) \leqslant \sqrt{2}/r$$

and

$$\|\rho_f^{-1}(v)\| = \|\sqrt{2} - \rho_f^{-1}(u)\| \leqslant \sqrt{2} + 2/r\,.$$

Estimating the norm of ρ_f^{-1} as above, we obtain

$$\|\rho_f^{-1}\| \leqslant \sqrt{2}\,[1/r^2 + (1 + 1/r)^2]^{1/2}.$$

Since $0 < r < 1$ was arbitrary, we conclude that T has property D($\sqrt{10}$).

Since a 2-normal operator is a direct integral of 2×2 matrices, Theorem 3.6 implies the following corollary.

COROLLARY 5.8. *Every 2-normal operator has property* $D(\sqrt{10})$. *More generally, if* (X, μ) *is a sigma-finite measure space, then any commutative subset of* $\{M_\varphi : \varphi \in L^\infty(\mu, B(\mathbf{C}^{(2)}))\}$ *has property* $D(\sqrt{10})$.

REMARK. If n is a positive integer, let

$$T_n = \begin{bmatrix} 1 & n^2 & n \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

If, for each positive integer n, f_n is a relatively strictly cyclic separating unit vector in $\mathbf{C}^{(3)}$ for T_n, then $\|\rho_{f_n}^{-1}\| \to \infty$ as $n \to \infty$. Moreover, for each n, there is a relatively strictly cyclic separating unit vector g_n for T_n^* such that $\sup_n \|\rho_{g_n}^{-1}\| < \infty$.

THEOREM 5.9. *There is no positive number* $r \geqslant 1$ *such that every* 4×4 *matrix has property* $D(r)$. *There is no* $r \geqslant 1$ *such that every operator* T *with* $T^4 = 0$ *has property* $D(r)$. *There is a 4-normal operator that does not have property* D.

Proof. Suppose n is a positive integer and let

$$T_n = \begin{bmatrix} 0 & n & 0 & 0 \\ 0 & 0 & n^2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Assume that $r \geqslant 1$ and T_n has property $D(r)$ for $n = 1, 2, \ldots$. Consider the linear functional φ_n defined on $\mathscr{A}(T_n)$ by $\varphi_n((a_{ij})) = a_{23} + a_{14}$. Clearly, $\|\varphi_n\| \leqslant 2$ for $n = 1, 2, \ldots$; thus there are vectors f_n, g_n in $\mathbf{C}^{(4)}$ with $\|f_n\| = 1$, $\|g_n\| \leqslant 3r$ such that $\varphi_n(A) = (Af_n, g_n)$ for every A in $\mathscr{A}(T_n)$. Fix n and write $f_n = (s, t, u, v)$ and $g_n = (w, x, y, z)$. By computing $\varphi_n(T_n^k)$ for $k = 1, 2, 3$, we obtain the equations

$$ntw + n^2ux + vy = n^2$$

$$n^3uw + n^2vx = 0$$

$$n^3vw = n^3 .$$

Since the absolute values of t, u and v are no greater than 1 and those of w, x and y are no greater than $3r$, these equations imply (respectively)

$$|u|3r \geqslant |u|\,|x| \geqslant 1 - (3r/n) - (3r/n^2)$$

$$|uw| \leqslant 3r/n$$

$$|w| \geqslant |wv| = 1 .$$

The latter inequalities imply

$$1 - (3r/n) - (3r/n^2) \leqslant |u|\,|w|\,3r \leqslant 9r^2/n\,.$$

It is clear that this last statement cannot hold for all positive integers n. This contradiction shows that there is no $r \geqslant 1$ such that every T_n has property D(r). For each positive integer n let $S_n = (1 + T_n)/n^3$, and let $S = S_1 \oplus S_2 \oplus \ldots$. It is clear that S is a compact 4-normal operator, and it follows from the Riesz functional calculus that

$$\mathscr{A}(S) = \mathscr{A}(S_1) \oplus \mathscr{A}(S_2) \oplus \ldots = \mathscr{A}(T_1) \oplus \mathscr{A}(T_2) \oplus \ldots.$$

It follows from Proposition 4.1(4) that $\mathscr{A}(S)$ does not have property D_σ. However, it follows from Corollary 4.4 that the weak and ultraweak topologies agree on $\mathscr{A}(S)$. Thus $\mathscr{A}(S)$ does not have property D. This completes the proof.

REMARKS 5.10. (a) Applying reasoning similar to that in the proof of the preceding theorem to the matrices

$$\begin{bmatrix} 0 & n^2 & 0 & 0 \\ 0 & 0 & n^3 & 0 \\ 0 & 0 & 0 & n \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad n = 1, 2, \ldots,$$

it is possible to construct a reflexive 4-normal operator that does not have property D.

(b) An example of a commutative reflexive algebra of 12×12 matrices that is not super-reflexive can be constructed as follows. In [2, Remark 4.4] E. Azoff gave an example of a commutative algebra $\mathscr{A}_0$ of 6×6 matrices such that $\mathscr{A}_0^{(2)}$ is not reflexive. Let $\mathscr{A}$ be a maximal commutative algebra containing $\mathscr{A}_0$. Since $\mathrm{Alg}(\mathrm{Lat}\,\mathscr{A}^{(2)}) \subset (\mathscr{A}')^{(2)} = \mathscr{A}^{(2)}$ (e.g., see Lemma 4.3 in [2]), it follows that $\mathscr{A}^{(2)}$ is reflexive but not super-reflexive. In particular, this means that $\mathscr{A}^{(2)}$ does not have property D.

A characterization of reflexivity for operators on a finite-dimensional space was obtained by Deddens and Fillmore [8]. The following theorem shows that an easy argument extends their result to algebraic operators. The obvious analogue of this result holds on an arbitrary Banach space. Suppose $T \in B(H)$ and T is an algebraic operator and $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then H is an algebraic direct sum of closed invariant subspaces $M_1, M_2, \ldots, M_n$ for T such that $\sigma(T|M_k) = \{\lambda_k\}$ for $1 \leqslant k \leqslant n$. It is easily shown [20, Proposition 9.7] that T is reflexive if and only if the operators $T|M_1, T|M_2, \ldots, T|M_n$ are reflexive. Hence it suffices to consider the case when T is nilpotent. Although the characterization of reflexive nilpotent matrices in [8] is stated in terms of the Jordan canonical form, it is clearly equivalent to the following.

THEOREM 5.11. *Suppose T is a nilpotent operator on H of index n ($\geqslant 2$). Then T is reflexive if and only if* $\dim(\ker T^{n-2})^{\perp} > 2$.

Proof. If $\dim(\ker T^{n-2})^{\perp} \leqslant 2$, then the proof in [8] carries over directly to show that T is not reflexive. On the other hand suppose $\dim(\ker T^{n-2})^{\perp} > 2$, and suppose $A \in \mathrm{Alg}(\mathrm{Lat}T)$. Choose linearly independent vectors f_1, f_2, f_3 in $(\ker T^{n-2})^{\perp}$ with $f_1 \in (\ker T^{n-1})^{\perp}$. Suppose $g_1, g_2, \ldots, g_k \in H$. Let M be the smallest invariant subspace of T that contains $f_1, f_2, f_3, g_1, g_2, \ldots, g_k$. Since T is nilpotent, M is finite-dimensional. Also $A|M \in \mathrm{Alg}(\mathrm{Lat}T|M)$. Since $f_1, f_2, f_3 \in M$, we conclude that $T|M$ is nilpotent of index n and $M \ominus \ker(T|M)^{n-2}$ has dimension greater than 2. Thus, by [8], $T|M$ is reflexive. Hence there is a polynomial p such that $p(T|M) = A|M$. In particular, $p(T)g_i = Ag_i$ for $1 \leqslant i \leqslant k$. Thus $\{p(T) : p \text{ a polynomial}\}$ intersects every strong neighborhood of A. Hence $A \in \mathscr{A}(T)$. Therefore, T is reflexive.

Note that it follows from Corollary 5.3 and Proposition 2.5 that every reflexive algebraic operator is super-reflexive.

6. HYPOREFLEXIVITY

A commutative algebra $\mathscr{A}$ of operators is *hyporeflexive* if $\mathscr{A} = \mathscr{A}' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{A})$ and an arbitrary algebra is *dc-reflexive* if $\mathscr{A} = \mathscr{A}'' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{A})$. Note that if $\mathscr{B} \subset \mathscr{A}$, then $\mathscr{B}'' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{B}) \subset \mathscr{A}'' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{A})$ always holds, but $\mathscr{B}' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{B}) \subset \mathscr{A}' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{A})$ does not generally hold (e.g., when $\mathscr{A} = B(H)$). An operator T is *hyporeflexive* (resp. *dc-reflexive*) if $\mathscr{A}(T)$ is. The results on super-reflexivity in the preceding sections carry over directly for dc-reflexivity. We list here some of the important analogues, but we leave the proofs to the interested reader.

THEOREM 6.1. (1) *If $\mathscr{A}$ is a unital weakly closed dc-reflexive subalgebra of $B(H)$ that has property* D, *then every unital weakly closed subalgebra of $\mathscr{A}$ is dc-reflexive.*

(2) *A direct integral of a measurable family of unital weakly closed dc-reflexive subalgebra of $B(H)$ is dc-reflexive.*

Although we cannot prove the analogue of (1) in the preceding theorem for hyporeflexivity, we can still say something about hyporeflexive algebras.

THEOREM 6.2. (1) *A direct integral of a measurable family of weakly closed unital hyporeflexive subalgebras of $B(H)$ is hyporeflexive.*

(2) *A weakly closed unital hyporeflexive subalgebra of a commutative reflexive algebra is reflexive.*

(3) *If $\mathscr{A}$ is a unital weakly closed commutative subalgebra of $B(H)$ such that $\mathscr{A}' \cap \mathrm{Alg}(\mathrm{Lat}\mathscr{A})$ has property* D, *then $\mathscr{A}$ is hyporeflexive.*

(4) *If* $r \geqslant 1$, (X, μ) *is a sigma-finite measure space, if* $\varphi_1, \varphi_2, \ldots \in L^\infty(\mu, B(H))$, *and if* $\mathscr{A}(\varphi_1(x), \varphi_2(x), \ldots)$ *is hyporeflexive and has property* D(r) a.e., *then* $\mathscr{A}(M_{\varphi_1}, M_{\varphi_2}, \ldots)$ *is hyporeflexive.*

Proof. The proofs of (1) and (2) are left to the reader.

(3). This follows from Proposition 2.5 (4).

(4). Let $\mathscr{A} = \mathscr{A}(M_{\varphi_1}, M_{\varphi_2}, \ldots)$ and let $\mathscr{B}$ be the direct integral of the $\mathscr{A}(\varphi_1(x), \varphi_2(x), \ldots)$'s. It follows from [3] that $\mathrm{Alg}(\mathrm{Lat}\mathscr{A}) \subset \mathrm{Alg}(\mathrm{Lat}\,\mathscr{B})$ and $\mathrm{Alg}(\mathrm{Lat}\,\mathscr{A}) \cap \mathscr{A}' \subset \subset \mathrm{Alg}(\mathrm{Lat}\,\mathscr{B}) \cap \mathscr{A}' = \mathrm{Alg}(\mathrm{Lat}\,\mathscr{B}) \cap \mathscr{B}'$. It follows from (1) that the latter algebra is $\mathscr{B}$; and it follows from Theorem 3.6 that $\mathscr{B}$ has property D(r). Thus, by Proposition 2.5(3), $\mathrm{Alg}(\mathrm{Lat}\,\mathscr{A}) \cap \mathscr{A}'$ has property D(r). It follows from (3) that $\mathscr{A}$ is hyporeflexive.

We conclude this section with two examples of hyporeflexive operators. Note that Brickmann and Fillmore [4] proved that every operator in a finite-dimensional space is hyporeflexive.

PROPOSITION 6.3. (1) *Every algebraic operator is hyporeflexive.*

(2) *Every* 2-*normal operator is hyporeflexive.*

Proof. (1). Imitate the proof of Theorem 5.11.

(2). Use Proposition 5.7, Theorem 6.2 (4), and the hyporeflexivity of 2×2 matrices [4].

6. QUESTIONS AND COMMENTS

(1) Properties D, D(r) (resp. D_σ, $D_\sigma(r)$) can be defined for weakly (resp. ultraweakly) closed linear manifolds in $B(H)$ in the obvious way. There is also a notion of reflexivity for submanifolds of $B(H)$. A linear submanifold $\mathscr{S}$ of $B(H)$ is *reflexive* (see [16]) provided $T \in \mathscr{S}$ whenever $T \in B(H)$ and $Tf \in (\mathscr{S}f)^-$ for every f in H, and $\mathscr{S}$ is *hereditarily reflexive* (see [16]) if every weakly closed submanifold of $\mathscr{S}$ is reflexive. Note that for unital algebras of operators the two notions of reflexivity coincide, but it is not clear that every super-reflexive algebra is hereditarily reflexive. Loginov and Sulman [16, Theorem 2.3] proved that a reflexive linear manifold in $B(H)$ is hereditarily reflexive if and only if it has property D. This leads to the question of whether every super-reflexive algebra has property D. It is therefore worth knowing if the operator in Remark 5.10 (a) is super-reflexive.

(2) We noted before that Sarason and Rosenthal have asked whether every operator is hyporeflexive and that Azoff [2] has given an example of a commutative algebra of 6×6 matrices that is not dc-reflexive. We have used property D(r) to show that every 2-normal operator is hyporeflexive (Proposition 6.3(2)), but our proof breaks down for n-normal operators when $n \geqslant 4$. Is the difficulty in our proof or is there really an n-normal operator that is not hyporeflexive? A related question is whether every unital weakly closed subalgebra of a singly generated algebra is

hyporeflexive. A negative answer to this latter question would be implied by the existence of a reflexive operator that is not super-reflexive. Thus, determining whether the operator of Remark 5.10(a) is super-reflexive would either provide an answer to this latter question or to the last question in (1) above.

(3) The von Neumann algebras with property D were characterized in [16]. It was also shown in [16] that an algebra $\mathscr{A}$ has property D if, for each f_1, f_2 in H there is an f in H and operators A_1, A_2 in the commutant of $\mathscr{A}$ such that $A_i f = f_i$ for $i = 1, 2$. In particular if $\mathscr{A}'$ is strictly cyclic, then $\mathscr{A}$ has property D ([16, Theorem 3.7, Corollary 2.4]); this implies our Corollary 5.2.

(4) Characterizing the reflexive algebras of $n \times n$ matrices is a difficult (if not impossible) task. Perhaps it is easier to characterize the super-reflexive algebras of $n \times n$ matrices. Finite-dimensionality reduces the problem to finding the maximal super-reflexive algebras of $n \times n$ matrices. What are the maximal algebras (linear manifolds) with property D? Call two linear submanifolds $\mathscr{M}_1$, $\mathscr{M}_2$ of $B(H)$ *equivalent* if there are invertible operators A, B in $B(H)$ such that $A\mathscr{M}_1 = \mathscr{M}_2 B$. It is clear that equivalence preserves property D. This reduces the problem of finding all manifolds with property D to the problem of finding which equivalence classes have property D.

(5) If $\mathscr{A}$ is a unital weakly (resp. ultraweakly) closed algebra with property D (resp. D_σ), then does there exist an $r \geqslant 1$ such that $\mathscr{A}$ has property $D(r)$ (resp. $D_\sigma(r)$)? At first glance it seems that some application of the open mapping theorem is in order, but the set of weakly continuous linear functionals seem to have no completeness properties. Note that Proposition 4.1(4) implies that an ultraweakly closed unital algebra that has property D_σ and whose center contains infinitely many projections has property $D_\sigma(r)$ for some $r \geqslant 1$.

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