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SUPPORT FUNCTIONS FOR MATRIX RANGES: ANALOGUES OF LUMER'S FORMULA

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1. INTRODUCTION

If $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras and $\mathfrak{M}_m$ is the set of complex $m \times m$ matrices with identity matrix I_m and identity map $\tilde{I}_m : \mathfrak{M}_m \to \mathfrak{M}_m$, a linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is said to be *completely positive* if the associated maps,

$$\varphi \otimes \tilde{I}_m : \mathscr{A} \otimes \mathfrak{M}_m \to \mathscr{B} \otimes \mathfrak{M}_m, \quad m \geqslant 1,$$

are all positive. Stinespring introduced completely positive maps and proved an elegant, useful representation theorem for them in [17]. Such maps have recently played a role in classifying C^*-algebras [8, 9].

Arveson [2, p. 301] used completely positive maps from a C^*-algebra $\mathscr{A}$ (with identity I) into $\mathfrak{M}_n$ to define generalized state spaces,

$$(1.1) \qquad \mathcal{S}_n = \{\varphi : \mathscr{A} \to \mathfrak{M}_n \mid \varphi \text{ is completely positive, } \varphi(I) = I_n\}$$

and, for $T \in \mathscr{A}$, the matrix ranges,

$$(1.2) \qquad W_n(T) = \{p \in \mathfrak{M}_n : p = \varphi(T),\ \varphi \in \mathcal{S}_n\}\,.$$

He then showed the importance of matrix ranges by proving that if T is a compact, irreducible, linear operator on a separable Hilbert space, then the set $\{W_n(T),\ n = 1, 2, \ldots\}$ constitutes a complete set of unitary invariants for T [2, Section 2.5].

Matrix ranges are generalizations of the numerical range; indeed, $W_1(T)$ is the numerical range of T (i.e. in the Banach algebra sense; it is the closure of the usual Hilbert space numerical range). Each matrix range $W_n(T)$ shares with $W_1(T)$ the property of being a compact, convex subset of a finite dimensional vector space [2, p. 301] which may be viewed as real; $\mathbf{R}^2 \approx \mathbf{C}$ for $W_1(T)$ and $\mathbf{R}^{2n^2} \approx \mathfrak{M}_n$ for $W_n(T)$. Such subsets are characterized by their support functions (cf. [13], Chapter 13; Section 2 of this paper).

The support function for $W_1(T)$ is defined by

$$H_1(z) \equiv \sup\{\mathrm{Re}(\bar{z}\omega) : \omega \in W_1(T)\}, \quad z \in \mathbf{C}. \tag{1.3}$$

Because $\bar{z}W_1(T) = W_1(\bar{z}T)$, (1.3) can be computed via Lumer's formula [12]:

$$H_1(z) = \lim_{\alpha \to 0^+} \left\{ \frac{\|I + \alpha \bar{z} T\| - 1}{\alpha} \right\}. \tag{1.4}$$

The geometric interpretation of (1.4) is simply that $H(e^{i\theta})$ is the directed distance from the origin to the line tangent to $W_1(T)$ with outward normal $e^{i\theta}$.

The purpose of this paper is to find formulae for the support function of $W_n(T)$, including the natural generalization of Lumer's derivative formula. In carrying this out, natural correspondences between all completely positive maps from $\mathscr{A}$ to $\mathfrak{M}_n$ and all positive linear functionals on $\mathscr{A} \otimes \mathfrak{M}_n$ and between all unital completely positive maps and certain positive linear functionals on $\mathscr{A} \otimes \mathfrak{M}_n$ are given. The former correspondence is also found in a work of Lance [11, Lemma 3.1]. In addition, a metric characterization of $W_n(T)$ is found. Finally, Choi's structural theorems [7] are generalized.

OUTLINE AND SUMMARY. In Section 2, the support function for $W_n(T)$ is defined and certain preliminary results are given. Section 3 begins by giving the correspondences mentioned above. These are then used to obtain formulae for the support function of $W_n(T)$, a metric characterization of $W_n(T)$, and bounds on the support function. The support functions for normal elements of $\mathscr{A}$ are discussed in Section 4. Finally, Section 5 starts with a theorem similar to Choi's structural theorems. This result is then used to concretely realize certain of the formulae for the support function obtained in Section 2 in terms of the inner product on a certain tensor product space.

NOTATION. $\mathscr{A}$, $\mathfrak{S}_n$, T, $W_n(T)$, and $H_n(\cdot)$ will denote, respectively: a fixed unital C^*-algebra with identity I; the set of all unital completely positive maps from $\mathscr{A}$ to $\mathfrak{M}_n$ given in (1.1); a fixed element of $\mathscr{A}$; the matrix range of T given in (1.2); and, the support function for $W_n(T)$. $\mathscr{B}^*$ is the dual of a space $\mathscr{B}$; $\mathscr{B}^+$ is the set of positive elements in $\mathscr{B}$. The notation $\|\cdot\|$ will be used for norms on all C^*-algebras and for vectors in a complex Hilbert space. The inner product on any complex Hilbert space will be $\langle\ ,\ \rangle$. Other notation will be introduced as needed.

2. PRELIMINARIES

If K is a convex subset of a finite dimensional real vector space V, which has V^* as its dual, the *support function* [13, p. 112) for K is a mapping $H : V^* \to \{R, +\infty\}$ defined by

$$H(\lambda) \equiv \sup\{\lambda \cdot p \mid p \in K\}, \ \lambda \in V^*. \tag{2.1}$$

To discuss the support function for $W_n(T)$, it is necessary to discuss linear functionals on $\mathcal{M}_n$.

Since $\mathcal{M}_n$ is finite dimensional, its real dual is isomorphic to $\mathcal{M}_n$. The linear functionals on $\mathcal{M}_n$ are well known to have the form

$$\lambda \cdot p = \mathrm{Re}(\mathrm{tr}(\lambda^* p)), \tag{2.2}$$

where "tr" is the trace and λ^* is the adjoint of λ. These functionals may also be viewed in terms of a certain positive linear functional on $\mathcal{M}_n \otimes \mathcal{M}_n$.

Let $\xi_1, \ldots, \xi_n$ be the canonical basis for $\mathbf{C}^n$; that is, ξ_j is a column vector with one in the j^{th} entry and zeros elsewhere. If $\langle \cdot, \cdot \rangle$ denotes the usual complex inner product on a space — in this case $\mathbf{C}^n \otimes \mathbf{C}^n$ — define

$$\rho(\alpha) \equiv \langle \alpha\xi, \xi \rangle, \quad \xi = \sum_{j=1}^{n} \xi_j \otimes \xi_j, \quad \alpha \in \mathcal{M}_n \otimes \mathcal{M}_n. \tag{2.3}$$

It is clear that $\rho(\cdot)$ is a positive linear functional on $\mathcal{M}_n \otimes \mathcal{M}_n$. The relationship between $\rho(\cdot)$ and the functionals defined by (2.2) is given in the following proposition.

Proposition 2.1. *Let $\rho(\cdot)$ be the positive linear functional defined by* (2.3). *If $p, q \in \mathcal{M}_n$, and q^t is the transpose of q, then*

$$\mathrm{tr}(q^t p) = \rho(p \otimes q). \tag{2.4}$$

In particular, the linear functional $\lambda \cdot p$ is

$$\lambda \cdot p = \mathrm{Re}(\rho(p \otimes \bar{\lambda})) = \rho(\mathrm{Re}(p \otimes \bar{\lambda})). \tag{2.5}$$

Finally, if $E_{jk} = \xi_j \xi_k^t$ is the elementary $n \times n$ matrix with one in the (j, k) position and zeros elsewhere,

$$\rho(p \otimes E_{jk}) = p_{jk}. \tag{2.6}$$

Proof. Apply (2.3) and use the elementary properties of the tensor product and inner product.

Remark 2.1. The dot product defined by (2.2) and the norm defined by

$$|p|_2 \equiv (p \cdot p)^{1/2} = (\mathrm{tr} p^* p)^{1/2} \tag{2.7}$$

make $\mathcal{M}_n$ into a real $2n^2$-dimensional Hilbert space. The norm in (2.7) coincides with the usual Hilbert-Schmidt norm on $\mathcal{M}_n$.

In the special case of $W_n(T)$, the *support function*, $H_n(\lambda)$, is given by

$$H_n(\lambda) \equiv \sup\{\lambda \cdot p \mid p \in W_n(T)\}, \ \lambda \in \mathcal{M}_n, \tag{2.8}$$

or, in terms of the state space $\mathcal{S}_n$,

$$H_n(\lambda) = \sup\{\lambda \cdot \varphi(T) \mid \varphi \in \mathcal{S}_n\}, \quad \lambda \in \mathcal{M}_n . \tag{2.9}$$

Elementary properties of the support function $H_n(\lambda)$ are given in the following propositions:

PROPOSITION 2.2. *$H_n(\lambda)$ is a convex function defined on $\mathcal{M}_n$. If $c \in \mathbf{R}$, $c > 0$, then*

$$H_n(c\lambda) = cH_n(\lambda) . \tag{2.10}$$

PROPOSITION 2.3. *$H_n(\lambda) = \sup\{\lambda \cdot \varphi(T) \mid \varphi$ is an extreme point of $\mathcal{S}_n\}$.*

PROPOSITION 2.4. *$p \in W_n(T)$ if and only if $\lambda \cdot p \leqslant H_n(\lambda)$ for all $\lambda \in \mathcal{M}_n$.*

Propositions 2.2 and 2.4 are proved in [13, p. 112—114] for general support functions. Proposition 2.3 is a direct consequence of the definition of $H_n(\lambda)$ and the convexity of $\mathcal{S}_n$. Note that Proposition 2.4 implies that $H_n(\lambda)$ completely determines $W_n(T)$.

There is an important connection between the support function for $W_n(T)$ and certain positive linear functionals related to those defined by (2.3). First of all, observe that

$$\lambda \cdot \varphi(T) = \mathrm{Re}(\rho(\varphi(T) \otimes \bar{\lambda})) = \mathrm{Re}\rho((\varphi \otimes \tilde{I}_n)(T \otimes \bar{\lambda})).$$

Because φ is a completely positive map,

$$\varphi \otimes \tilde{I}_n : \mathscr{A} \otimes \mathcal{M}_n \to \mathcal{M}_n \otimes \mathcal{M}_n$$

is a positive map. The composition of ρ with $\varphi \otimes \tilde{I}_n$ thus results in a positive linear functional on $\mathscr{A} \otimes \mathcal{M}_n$. These remarks prove:

THEOREM 2.1. *Let $\varphi : \mathscr{A} \to \mathcal{M}_n$ be a completely positive map; define*

$$\rho_\varphi(S) \equiv \rho((\varphi \otimes \tilde{I}_n)(S)), \quad S \in \mathscr{A} \otimes \mathcal{M}_n . \tag{2.11}$$

The map $\rho_\varphi(\cdot)$ is a positive linear functional on $\mathscr{A} \otimes \mathcal{M}_n$. In addition,

$$H_n(\lambda) = \sup\{\rho_\varphi(\mathrm{Re}(T \otimes \bar{\lambda})) \mid \varphi \in \mathcal{S}_n\} . \tag{2.12}$$

In the next section, the linear functionals defined by (2.11) will be completely characterized; (2.12) and an argument using the Hahn-Banach theorem will then yield formulae for $H_n(\lambda)$.

3. THE MAIN RESULTS

Essential to obtaining formulae for the support function $H_n(\lambda)$ and a metric characterization for $W_n(T)$ are the relationships among positive linear functionals on $\mathscr{A} \otimes \mathfrak{M}_n$, completely positive maps from $\mathscr{A}$ to $\mathfrak{M}_n$, and states in $\mathcal{S}_n$. These relationships are given in the next theorem and its corollary.

THEOREM 3.1. *Let $\varphi : \mathscr{A} \to \mathfrak{M}_n$ be a completely positive map and let $\rho_\varphi(\cdot)$ be the positive linear functional defined by (2.11). φ may be recovered from ρ_φ via*

$$(3.1) \qquad [\varphi(T)]_{jk} = \rho_\varphi(T \otimes E_{jk}), \ T \in \mathscr{A},$$

where E_{jk} is the matrix defined in Proposition 2.1. Conversely, if $\theta : \mathscr{A} \otimes \mathfrak{M}_n \to \mathbf{C}$ is a positive linear functional on $\mathscr{A} \otimes \mathfrak{M}_n$, then the map

$$(3.2) \qquad [\varphi(T)]_{jk} \equiv \theta(T \otimes E_{jk}), \ T \in \mathscr{A},$$

defines a completely positive map from $\mathscr{A}$ into M_n. θ may be recovered from φ via

$$(3.3) \qquad \theta(S) = \rho_\varphi(S), \ S \in \mathscr{A} \otimes \mathfrak{M}_n .$$

Proof. Formula (3.1) is a direct consequence of (2.6) and (2.11); similarly, assuming (3.2) defines a completely positive map, (3.3) is a restatement of (2.11). All that need be shown is that (3.2) defines a completely positive map; that is, for all $r \geqslant 1$, $\varphi \otimes \tilde{I}_r : \mathscr{A} \otimes \mathfrak{M}_r \to \mathfrak{M}_n \otimes \mathfrak{M}_r$ is positive.

Let $G \in (\mathscr{A} \otimes \mathfrak{M}_r)^+$. The map $\varphi \otimes \tilde{I}_r$ will be positive if $(\varphi \otimes \tilde{I}_r)(G)$ is a positive matrix in $\mathfrak{M}_n \otimes \mathfrak{M}_r$. To see that this is so, let $v \in \mathbf{C}^n \otimes \mathbf{C}^r$ and let $\{\xi_j\}$, $\{\eta_\nu\}$ be canonical bases for $\mathbf{C}^n$ and $\mathbf{C}^r$, respectively. (For the remainder of the proof, Latin subscripts run from 1 to n, Greek from 1 to r. Repeated indices are summed.) Expand v in the basis $\{\xi_j \otimes \xi_\nu\}$:

$$(3.4) \qquad v = \sum v_{j,\nu} \xi_j \otimes \eta_\nu .$$

A straightforward matrix computation then gives

$$(3.5) \qquad \langle \varphi \otimes \tilde{I}_r(G)v, v\rangle = \sum v_{j,\nu} \bar{v}_{k,\mu} \langle \varphi(G_{\mu\nu})\xi_j, \xi_k\rangle .$$

The inner product in the sum on the right above is just $[\varphi(\cdot)]_{kj}$. Use this and (3.2) to put (3.5) in the form

$$(3.6) \qquad \langle \varphi \otimes \tilde{I}_r(G)v, v\rangle = \theta(\sum v_{j,\nu} \bar{v}_{k,\mu} G_{\mu\nu} \otimes E_{kj}) .$$

By [11, Proposition 2.1], $G \in (\mathscr{A} \otimes \mathfrak{M}_r)^+$ if and only if it is a finite sum of terms having the form $\sum X_\mu^* X_\nu \otimes E_{\mu\nu}$, $X_\mu \in \mathscr{A}$; because of this and the linearity of the maps involved, it is sufficient to check the case $G_{\mu\nu} = X_\mu^* X_\nu$. In this case, (3.6) becomes

$$\langle \varphi \otimes \tilde{I}_r(G)v, v\rangle = \theta(\sum V_k^* V_j \otimes E_{kj}), \tag{3.7}$$

where,

$$V_j = \sum v_{j,\nu} X_\nu . \tag{3.8}$$

Again, by [11, Proposition 2.1], the argument of θ in (3.7) belongs to $(\mathscr{A} \otimes \mathfrak{M}_n)^+$. Since θ is assumed to be a positive linear functional on $\mathscr{A} \otimes \mathfrak{M}_n$, the left side of (3.5) is positive: $\varphi \otimes \tilde{I}_r(G)$ is a positive matrix; φ is a completely positive map.

REMARK. This correspondence between positive linear functionals and completely positive maps was found earlier by Lance [11, Lemma 3.1].

The association $\varphi \leftrightarrow \theta = \rho_\varphi$ gives a one-to-one correspondence between completely positive maps and positive linear functionals. The corollary which follows explicitly characterizes those positive linear functionals on $\mathscr{A} \otimes \mathfrak{M}_n$ which correspond to states (i.e. unital completely positive maps) in $\mathfrak{S}_n$.

COROLLARY 3.1. *Let $\theta = \rho_\varphi$ where $\varphi : \mathscr{A} \to \mathfrak{M}_n$ is a completely positive map. φ is a state in $\mathfrak{S}_n$ if and only if θ is a linear functional which satisfies the following properties:*

(i) $\theta(I \otimes I_n) = \|\theta\| = n$.

(ii) $\theta(I \otimes q) \doteq \operatorname{tr} q$ *for all* $q \in \mathfrak{M}_n$.

Proof. If $\varphi \in \mathfrak{S}_n$, then $\varphi(I) = I_n$. From (2.11) and (2.4),

$$\theta(I \otimes q) = \rho(\varphi(I) \otimes q) = \rho(I_n \otimes q) = \operatorname{tr} q . \tag{3.9}$$

Hence (ii) holds. In particular

$$\theta(I \otimes I_n) = n. \tag{3.10}$$

By Theorem 2.1, $\theta = \rho_\varphi$ is a positive linear functional on $\mathscr{A} \otimes \mathfrak{M}_n$. Such functionals attain their norms on the identity [10, p. 31]:

$$\theta(I \otimes I_n) = \|\theta\|. \tag{3.11}$$

Combining (3.10) and (3.11) gives (i).

Conversely, suppose $\theta : \mathscr{A} \otimes \mathfrak{M}_n \to \mathbf{C}$ is a linear functional on $\mathscr{A} \otimes \mathfrak{M}_n$ and satisfies (i) and (ii). Since θ satisfies (i), it is a positive linear functional [10, p. 31]; hence φ defined by (3.2) is a completely positive map. In addition, (3.2) and (ii) imply that

$$[\varphi(I)]_{jk} = \theta(I \otimes E_{jk}) = \operatorname{tr} E_{jk} = \delta_{jk} , \tag{3.12}$$

where δ_{jk} is the Kronecker delta function. Thus $\varphi(I) = I_n$ and $\varphi \in \mathcal{S}_n$. This completes the proof.

DEFINITION 3.1. A functional on $\mathscr{A} \otimes \mathfrak{M}_n$ which satisfies conditions (i) and (ii) of Corollary 3.1 will be termed an *associated state functional;* the set of all such functionals will be denoted by Σ_n.

Corollary 3.1 allows replacement of the state space by the space of associated state functionals, Σ_n:

COROLLARY 3.2. *If Σ_n is the space of associated state functionals and $H_n(\lambda)$ is the support function for $W_n(T)$, then*

$$(3.13)\qquad H_n(\lambda) = \sup\{\theta(\mathrm{Re}(T \otimes \bar{\lambda})) \mid \theta \in \Sigma_n\}.$$

Proof. Apply Theorem 2.1 and Corollary 3.1.

It is now possible to use the Hahn-Banach theorem to obtain a formula for the support function:

THEOREM 3.2. *The support function for $W_n(T)$ is given by the formula,*

$$(3.14)\qquad H_n(\lambda) = \inf\{n\|\mathrm{Re}(T \otimes \bar{\lambda}) + I \otimes m\| - \mathrm{tr}\,m \mid m \in \mathfrak{M}_n,\ m = m^*\}.$$

Proof. Let $\mathrm{Re}(\mathscr{A} \otimes \mathfrak{M}_n)$ denote the set of all self-adjoint elements in $\mathscr{A} \otimes \mathfrak{M}_n$. Viewed as a real vector space, $\mathrm{Re}(\mathscr{A} \otimes \mathfrak{M}_n)$ has the set of all self-adjoint linear functionals in $(\mathscr{A} \otimes \mathfrak{M}_n)^*$ for its dual [14, p. 9]. Since the associated state functionals are all self-adjoint, $\Sigma_n \subseteq [\mathrm{Re}(\mathscr{A} \otimes \mathfrak{M}_n)]^*$; in addition, on the subspace $I \otimes \mathrm{Re}(\mathfrak{M}_n)$, each $\theta \in \Sigma_n$ coincides with the functional which assigns the trace to each $m \in \mathrm{Re}(\mathfrak{M}_n)$. Finally, note that $\theta \in \Sigma_n$ has $\|\theta\| = n$; thus, $|\theta(X)| \leqslant n\|X\|$ for all X. Define the real sublinear functional $p(X) \equiv n\|X\|$ and mimic the proof of the Hahn-Banach theorem. The result is that for each $X \in \mathrm{Re}(\mathscr{A} \otimes \mathfrak{M}_n)$

$$\sup\{\theta(X) \mid \theta \in \Sigma_n\} = \inf\{p(X + I \otimes m) - \theta(I \otimes m) \mid m \in \mathfrak{M}_n,\ m = m^*\}.$$

Using the definition of $p(\cdot)$ and noting that $\theta(I \otimes m) = \mathrm{tr}\,m$ then gives, for $X \in \mathrm{Re}(\mathscr{A} \otimes \mathfrak{M}_n)$,

$$(3.15)\qquad \sup\{\theta(X) \mid \theta \in \Sigma_n\} = \inf\{n\|X + I \otimes m\| - \mathrm{tr}\,m \mid m \in \mathfrak{M}_n,\ m = m^*\}.$$

The formula (3.14) follows from (3.13) and (3.14), with $X = \mathrm{Re}(T \otimes \bar{\lambda})$.

To obtain other formulae for $H_n(\lambda)$ and to get the metric characterization of $W_n(T)$, it is necessary to discuss certain semi-norms and norms on C^*-algebras and to prove a technical lemma.

Let $\mathscr{B}$ be a C^*-algebra with unit I and let F be a class of norm-one positive linear functionals on $\mathscr{B}$. Given $Y \in \mathscr{B}$, define

$$\|Y\|_F \equiv \sup\{\psi(Y^*Y)^{1/2} \mid \psi \in F\}. \tag{3.16}$$

REMARK 3.1. A routine calculation using the Cauchy-Schwarz inequality shows that $\|\cdot\|_F$ is a semi-norm on $\mathscr{B}$. In addition, if F consists of all norm 1, positive linear functionals on $\mathscr{B}$, then $\|\cdot\|_F = \|\cdot\|$ [10, p. 48].

The following is a useful technical result:

LEMMA 3.1. *Let $\mathscr{B}$ be a unital C^*-algebra, F a class of norm-one positive linear functionals on $\mathscr{B}$, and let $\|\cdot\|_F$ be as in* (3.16). *Fix $Y \in \mathscr{B}$. For every $c \in \mathbf{R}$,*

$$g(c) \equiv \|Y + cI\|_F - c \tag{3.17}$$

is a decreasing function of c. In addition,

$$\lim_{c\to+\infty} g(c) = \sup\{\psi(\mathrm{Re}(Y)) \mid \psi \in F\}. \tag{3.18}$$

Proof. Since every $\psi \in F$ is positive and norm-one,

$$\psi(I) = \|\psi\| = 1. \tag{3.19}$$

From (3.16), it follows that

$$\|I\|_F = 1. \tag{3.20}$$

That $g(c)$ is decreasing is a consequence of the triangle inequality for $\|\cdot\|_F$ and (3.20): Let $\delta > 0$, then

$$\|Y + (c+\delta)I\|_F - (c+\delta) \leqslant \|Y + cI\|_F - c + \delta\|I\|_F - \delta = g(c).$$

The last term cancels by virtue of (3.20); the result is that $g(c+\delta) \leqslant g(c)$.

To obtain (3.18), first note that

$$\frac{\psi((Y+cI)^*(Y+cI)) - c^2}{2c} = \frac{1}{2c}\psi(Y^*Y) + \psi(\mathrm{Re}\,Y). \tag{3.21}$$

Employ (3.16) and the positivity of ψ to get:

$$\sup_{\psi\in F}\{\psi(\mathrm{Re}\,Y)\} \leqslant \frac{\|Y+cI\|_F^2 - c^2}{2c} \leqslant \frac{1}{2c}\|Y\|_F^2 + \sup_{\psi\in F}\{\psi(\mathrm{Re}\,Y)\}. \tag{3.22}$$

Letting $c \to +\infty$ results in

$$\lim_{c\to\infty}\left(\frac{\|Y+cI\|_F^2 - c^2}{2c}\right) = \sup_{\psi\in F}\{\psi(\mathrm{Re}\,Y)\}. \tag{3.23}$$

Finally, observe that the term in parentheses on the left in (3.23) has the form

$$(3.24)\qquad (\cdot) = g(c)\left(\frac{\|Y + cI\|_F + c}{2c}\right).$$

A simple argument using the triangle inequality gives that

$$(\|Y + cI\|_F + c)(2c)^{-1} \to 1 \quad \text{as } c \to +\infty.$$

This fact, (3.23) and (3.24) establish (3.18).

The lemma just proved leads to a second formula for the support function.

COROLLARY 3.3. *If $H_n(\cdot)$ is the support function for $W_n(T)$, then*

$$(3.25)\qquad H_n(\lambda) = \inf\{n\|T \otimes \bar{\lambda} + I \otimes m\| - \mathrm{tr}m \mid m \in \mathfrak{M}_n \text{ and } m = m^*\}.$$

Proof. In Lemma 3.1, take $\mathscr{B} = \mathscr{A} \otimes \mathfrak{M}_n$, and set $F = P$, the set of all positive norm-one functionals on $\mathscr{A} \otimes \mathfrak{M}_n$. By Remark 3.1, $\|\cdot\|_F = \|\cdot\|$. Define

$$(3.26)\qquad D(Y, m) \equiv n\|Y + I \otimes m\| - \mathrm{tr}m,$$

where $Y \in \mathscr{A} \otimes \mathfrak{M}_n$, $m \in \mathfrak{M}_n$, $m = m^*$. A short computation gives that for every $c \in \mathbf{R}$,

$$(3.27)\qquad D(Y, m + c) = n[\|Y + I \otimes m + cI \otimes I_n\| - c] - \mathrm{tr}m\,.$$

By Lemma 3.1, $D(Y,\ m + c)$ is a decreasing function of c and satisfies

$$(3.28)\qquad \lim_{c\to\infty} D(Y, m + c) = \sup_{\psi\in P}[n\psi(\mathrm{Re}(Y) + I \otimes m)] - \mathrm{tr}m\,.$$

Given the decreasing nature of $D(Y, m + c)$ as a function of c and (3.28), it is clear that

$$(3.29)\qquad \begin{aligned} &\inf\{D(Y, m) \mid m \in \mathfrak{M}_n\,,\ m = m^*\} = \\ &= \inf_m\{n\sup_{\psi\in P}[\psi(\mathrm{Re}(Y) + I \otimes m)] - \mathrm{tr}m\}\,. \end{aligned}$$

Formula 3.25 follows on taking $Y = T \otimes \bar{\lambda}$ in (3.29), then choosing $Y = \mathrm{Re}(T \otimes \bar{\lambda})$ in (3.29), comparing the results and applying (3.14).

REMARK 3.2. The proof of the corollary yields the formula,

$$(3.30)\qquad H_n(\lambda) = \inf_m\{\sup_\psi\{n\psi(\mathrm{Re}(T \otimes \bar{\lambda}) + I \otimes m) - \mathrm{tr}m\}\}\,,$$

where ψ runs over all norm-one positive linear functionals and m over all self-adjoint matrices in $\mathfrak{M}_n$.

REMARK 3.3. A further examination of the proof shows that the infima in (3.14), (3.25) and (3.30) need only be taken over positive matrices in $\mathfrak{M}_n$. Indeed, these may be required to have arbitrarily large lowest eigenvalue.

As a second application of Lemma 3.1, the natural analogue of Lumer's "derivative" formula will now be obtained. Again take $\mathscr{B} = \mathscr{A} \otimes \mathfrak{M}_n$ and put $F = \{\psi \mid n\psi \in \Sigma_n\}$. The semi-norm defined by (3.16), $\|\cdot\|_F$, will be denoted by $\|\cdot\|_\sigma$; that is, if $Y \in \mathscr{A} \otimes \mathfrak{M}_n$,

$$\|Y\|_\sigma \equiv \sup\{\sqrt{\psi(Y^*Y)} \mid n\psi \in \Sigma_n\}. \tag{3.31}$$

(The subscript "σ" is used to bring out the connection with associated state functionals.)

PROPOSITION 3.1. *$\|\cdot\|_\sigma$ is a norm on $\mathscr{A} \otimes \mathfrak{M}_n$.*

Proof. Let f be a norm-one positive linear functional on $\mathscr{A}$ and let $\tau(\cdot) = \frac{1}{n}\operatorname{tr}(\cdot)$ be the normalized trace on $\mathfrak{M}_n$. Define $\psi_f \in (\mathscr{A} \otimes \mathfrak{M}_n)^*$ by requiring that $\psi_f(a \otimes m) = f(a)\tau(m)$ for all $a \in \mathscr{A}$, $m \in \mathfrak{M}_n$. Note that $n\psi_f \in \Sigma_n$, so if $\|Z\|_\sigma = 0$ for some $Z = \sum_{j,k} z_{jk} \otimes E_{jk} \in \mathscr{A} \otimes \mathfrak{M}_n$, then (3.31) implies that $\psi_f(Z^*Z) = 0$. In this equation, express Z and Z^* in component-form, use the definition of ψ_f, and make a computation to get: $\frac{1}{n}\,[\sum_{j,k} f(z_{jk}^* z_{jk})] = 0$. Since f and its arguments are positive, each term in the equation vanishes; since f is arbitrary, each of the arguments must also vanish, so $z_{jk}^* z_{jk} = 0$. It follows that $Z = 0$; hence, the semi-norm $\|\cdot\|_\sigma$ is a norm.

COROLLARY 3.4. *If $\|\cdot\|_\sigma$ is the norm on $\mathscr{A} \otimes \mathfrak{M}_n$ defined by* (3.31) *and $H_n(\lambda)$ is the support function for $W_n(T)$, then*

$$H_n(\lambda) = n \cdot \lim_{\alpha \to 0^+} \left\{ \frac{\|I \otimes I_n + \alpha T \otimes \bar{\lambda}\|_\sigma - 1}{\alpha} \right\}. \tag{3.32}$$

Proof. In Lemma 3.1, take $\mathscr{B} = \mathscr{A} \otimes \mathfrak{M}_n$, $F = \{\psi \mid n\psi \in \Sigma_n\}$, $\|\cdot\|_F = \|\cdot\|_\sigma$. Formula (3.18) then gives

$$\sup\{\psi(\operatorname{Re}(T \otimes \bar{\lambda})) \mid n\psi \in \Sigma_n\} = \lim_{c \to \infty}\{\|cI \otimes I_n + T \otimes \bar{\lambda}\|_\sigma - c\}. \tag{3.33}$$

On the other hand, formula (2.12) implies that

$$H_n(\lambda) = n\sup\{\psi(\operatorname{Re}(T \otimes \bar{\lambda})) \mid n\psi \in \Sigma_n\}. \tag{3.34}$$

The formulae in (3.33) and (3.34) give,

$$H_n(\lambda) = n\lim_{c \to \infty}\{\|cI \otimes I_n + T \otimes \bar{\lambda}\|_\sigma - c\}. \tag{3.35}$$

Adjusting the limit by scaling then yields (3.32).

REMARKS. (a). In case $n = 1$, $\|\cdot\| = \|\cdot\|_\sigma$ on $\mathscr{A}$. Formula (3.32) then becomes

$$H_1(e^{i\theta}) = \lim_{\alpha\to 0+} \left\{ \frac{\|I + \alpha e^{-i\theta} T\| - 1}{\alpha} \right\},$$

which is Lumer's derivative formula. In the present case, this was obtained by an application of the Hahn-Banach theorem; that such an application was possible was pointed out to the authors by Professor Effros.

(b). The limit in (3.32) is simply the directional derivative of $\|\cdot\|_\sigma$ evaluated at the identity and taken in the direction of $T \otimes \bar{\lambda}$.

The theorems and corollaries proved earlier can now be applied to obtain a metric characterization of the matrix range $W_n(T)$.

THEOREM 3.3. *Let* $|\cdot|_2$ *be the Hilbert-Schmidt norm on* $\mathfrak{M}_n$ *given in* (2.7) *and let* $\|\cdot\|_\sigma$ *be the norm on* $\mathscr{A} \otimes \mathfrak{M}_n$ *given in* (3.31). *A necessary and sufficient condition for* $p \in \mathfrak{M}_n$ *to belong to* $W_n(T)$ *is that* p *satisfy*

(3.36)
$$|I_n + pq|_2 \leqslant \sqrt{n}\,\|I \otimes I_n + T \otimes q^t\|_\sigma$$

for every $q \in \mathfrak{M}_n$.

Proof. Proposition 2.4 implies that $p \in W_n(T)$ if and only if

(3.37)
$$\lambda \cdot p = \mathrm{Re}(\mathrm{tr}(\lambda^* p)) \leqslant H_n(\lambda),$$

for every $\lambda \in \mathfrak{M}_n$. Consider formula (3.35). By Lemma 3.1, the term in braces on the right is a decreasing function of c. This formula can be rewritten as

(3.38)
$$H_n(\lambda) = n \cdot \inf_{c \in \mathbf{R}^+} \{\|cI \otimes I_n + T \otimes \bar{\lambda}\|_\sigma - c\}.$$

The combination of (3.37) and (3.38) then gives that $p \in W_n(T)$ if and only if for every $c > 0$ and $\lambda \in \mathfrak{M}_n$,

(3.39)
$$\mathrm{Re}(\mathrm{tr}(\lambda^* p)) \leqslant n\|cI \otimes I_n + T \otimes \bar{\lambda}\|_\sigma - nc\,.$$

After interchange of matrices within the trace, this inequality can be put in the form

(3.40)
$$\mathrm{Re}(\mathrm{tr}(p\lambda^* + cI_n) \leqslant n\|cI \otimes I_n + T \otimes \bar{\lambda}\|_\sigma\,.$$

Finally, divide both sides by c and put $q = c^{-1}\lambda^*$ to obtain:

(3.41)
$$\mathrm{Re}(\mathrm{tr}(pq + I_n)) \leqslant n\|I \otimes I_n + T \otimes q^t\|_\sigma\,.$$

Again, $p \in W_n(T)$ if and only if (3.41) holds for all $q \in \mathfrak{M}_n$.

Suppose that p satisfies (3.36) for all $q \in \mathfrak{M}_n$. Apply Schwarz inequality to $\mathrm{Re}(\mathrm{tr}(pq + I_n))$ and use (3.36):

$$\mathrm{Re}(\mathrm{tr}(pq + I_n)) \leqslant \sqrt{n}\,|qp + I_n|_2 \leqslant n\|I \otimes I_n + T \otimes q^t\|_\sigma .$$

Thus, $p \in W_n(T)$.

On the other hand, if $p \in W_n(T)$, there exists $\varphi \in \mathfrak{S}_n$ such that $p = \varphi(T)$. Let $q \in \mathfrak{M}_n$, set $R = I \otimes I_n + T \otimes q^t$, and consider the associated state functional corresponding to φ, $\rho_\varphi(\cdot) = \rho(\varphi \otimes \tilde{I}_n(\cdot))$. (See Theorem 3.1.) Since $\rho_\varphi \in \Sigma_n$, (3.31) implies that

$$\|R\|_\sigma^2 \geqslant \frac{1}{n}\rho_\varphi(R^*R) = \frac{1}{n}\rho(\varphi \otimes \tilde{I}_n(R^*R)). \tag{3.42}$$

It is easy to see that $\varphi_n \equiv \varphi \otimes \tilde{I}_n$ is a unital $[\varphi_n(I \otimes I_n) = I_n \otimes I_n]$, completely positive map from $\mathscr{A} \otimes \mathfrak{M}_n$ to $\mathfrak{M}_n \otimes \mathfrak{M}_n$; hence, the generalized Schwarz inequality

$$\varphi_n(R^*R) \geqslant \varphi_n(R)^*\varphi_n(R), \tag{3.43}$$

holds. Note that ρ, which is defined by (2.11), is positive so (3.42) and (3.43) yield

$$\|R\|_\sigma^2 \geqslant \frac{1}{n}\rho(\varphi_n(R)^*\varphi_n(R)). \tag{3.44}$$

To compute the right side of (3.44), do the following: Substitute $\varphi_n(R) = I_n \otimes I_n + p \otimes q^t$ into (3.44); perform the multiplication; apply (2.4); manipulate the result using properties of the trace; and, finally, use the definition of the Hilbert-Schmidt norm on $\mathfrak{M}_n$. The result is,

$$\rho(\varphi_n(R)^*\varphi_n(R)) = |I_n + pq|_2^2 . \tag{3.45}$$

This and (3.44) imply (3.36).

REMARK. The metric characterization for $W_n(T)$ given in Theorem 3.3 is the natural generalization of the corresponding result for the numerical range obtained by Stampfli and Williams [14, Theorem 4].

Given the number of formulae for the support function, it seems wise to list them in one place:

THEOREM 3.4. *Let $T \in \mathscr{A}$, $\lambda \in \mathfrak{M}_n$ and let $H_n(\lambda)$ be the support function for $W_n(T)$. All of the following expressions are equal to $H_n(\lambda)$:*

(i) $\sup\{\theta(\mathrm{Re}(T \otimes \bar{\lambda})) \mid \theta \in \Sigma_n\}$

(ii) $\inf_m\{n\|\mathrm{Re}(T \otimes \bar{\lambda}) + I \otimes m\| - \mathrm{tr}\,m\}$

(iii) $\inf_m\{n\|T \otimes \bar{\lambda} + I \otimes m\| - \mathrm{tr}\,m\}$

(iv) $\inf_m\{\sup_{\psi \in P}\{n\psi(\mathrm{Re}(T \otimes \bar{\lambda}) + I \otimes m) - \mathrm{tr}\,m\}\}$

(v) $\lim_{\alpha \to 0^+}\left\{\dfrac{\|I \otimes I_n + \alpha T \otimes \bar{\lambda}\|_\sigma - 1}{\alpha}\right\} \cdot n$.

The infima in (ii), (iii), *and* (iv) *may be taken over all self-adjoint matrices in* $\mathfrak{M}_n$ *or over all positive self-adjoint matrices in* $\mathfrak{M}_n$. *The set* P *in* (v) *is that of all norm-one positive linear functionals on* $\mathscr{A} \otimes \mathfrak{M}_n$.

Proof. In the order given, these formulae (3.13), (3.14), (3.30) and (3.32). The comment concerning infima is a reiteration of Remark 3.3.

To conclude the discussion, upper and lower bounds on $H_n(\lambda)$ will be given; these will prove useful in the next section.

PROPOSITION 3.2. *Let* $T \in \mathscr{A}$, $\lambda \in \mathfrak{M}_n$. *If* $\gamma = \gamma^* \in \mathfrak{M}_n$ *satisfies*

$$\operatorname{Re}(T \otimes \bar{\lambda}) \leqslant I \otimes \gamma, \tag{3.46}$$

then

$$H_n(\lambda) \leqslant \operatorname{tr}\gamma. \tag{3.47}$$

Proof. The difference $I \otimes \gamma - \operatorname{Re}(T \otimes \bar{\lambda})$ is positive. Since $\theta \in \Sigma_n$ is a positive linear functional on $\mathscr{A} \otimes \mathfrak{M}_n$ which reduces to the trace on $I \otimes \mathfrak{M}_n$, it follows that

$$0 \leqslant \theta(I \otimes \gamma - \operatorname{Re}(T \otimes \bar{\lambda})) = \operatorname{tr}\gamma - \theta(\operatorname{Re}(T \otimes \bar{\lambda}))\,.$$

The result follows on taking the supremum and using (3.13).

PROPOSITION 3.3. *Let* $T \in \mathscr{A}$, $\lambda \in \mathfrak{M}_n$. *If* $\{v_j\}_{j=1}^n$ *is an orthonormal set of vectors in* $\mathbf{C}^n$ *and* $w_1, \ldots, w_n$ *are arbitrary points in the numerical range of* T, *then*

$$H_n(\lambda) \geqslant \sum_{j=1}^{n} \langle \operatorname{Re}(w_j\bar{\lambda})v_j\,,\ v_j\rangle\,. \tag{3.48}$$

Proof. Let $\psi_1, \ldots, \psi_n$ be norm-one positive linear functionals on $\mathscr{A}$ such that $\psi_j(T) = w_j$. In addition, define $\chi_j(p) = \langle pv_j, v_j\rangle$, $j = 1, \ldots, n$, $p \in \mathfrak{M}_n$. These are clearly norm-one positive linear functionals on $\mathfrak{M}_n$. It is a simple matter to check that $\theta \equiv \sum_{j=1}^{n} \psi_j \otimes \chi_j$ is in Σ_n. A computation then shows that

$$\theta(\operatorname{Re}(T \otimes \bar{\lambda})) = \sum_{j=1}^{n} \langle \operatorname{Re}(w_j\bar{\lambda})v_j, v_j\rangle.$$

Apply (3.13) to get (3.48).

4. SUPPORT FUNCTIONS FOR NORMAL ELEMENTS

The formulae derived in Section 3 will now be used to discuss the support function for $W_n(T)$ when T is normal. In particular, for self-adjoint T and for those normal elements having disks for their numerical ranges, explicit formulae for their suport functions will be given. These formulae yield characterizations of the corresponding matrix ranges as by-products. The characterizations given were also obtained by Arveson [2, p. 302] as by-products of his work — his methods differ from those used here.

To use the formulae derived in Section 3, it is necessary to compute norms of certain elements in $\mathscr{A} \otimes \mathfrak{M}_n$. The following lemma enables reducing this computation to one of finding norms in $\mathfrak{M}_n$.

LEMMA 4.1. *Let $N_1, \ldots, N_r$ be mutually commuting normal elements of $\mathscr{A}$ and let $\Omega \subseteq \mathbf{C}^r$ be the joint spectrum of $(N_1, N_2, \ldots, N_r)$.* If $m_1, m_2, \ldots, m_r$ *are matrices in $\mathfrak{M}_n$, then*

$$(4.1)\qquad \left\| \sum_{j=1}^{r} N_j \otimes m_j \right\| = \sup\left\{ \left\| \sum_{j=1}^{r} z_j m_j \right\| \,(z_1, \ldots, z_r) \in \Omega\right\}.$$

Proof. Form the commutative C^*-subalgebra of $\mathscr{A}$ generated by $I, N_1, \ldots, N_r$; denote this subalgebra by $\tilde{\mathscr{A}}$. According to the Gel'fand-Naimark theorem, $\tilde{\mathscr{A}}$ is isometrically isomorphic to the space of continuous complex-valued functions defined on a compact Hausdorff topological space X. Making the identification $\tilde{\mathscr{A}} \approx C(X)$, it is clear that $\tilde{\mathscr{A}} \otimes \mathfrak{M}_n$ consists of all continuous matrix-valued functions on X: If $f_1(\cdot), \ldots, f_r(\cdot)$ are the functions corresponding to $N_1, \ldots, N_r$

$$(4.2)\qquad \left\| \sum N_j \otimes m_j \right\| = \sup_{x \in X} \left\| \sum f_j(x) m_j \right\|.$$

On the other hand, from the construction given in the Gel'fand-Naimark theorem, each x corresponds uniquely to a complex $*$-homomorphism of $\tilde{\mathscr{A}}$. In particular, $x(N_j) = f_j(x)$. Thus the set of r-tuples $(f_1(x), \ldots, f_r(x))$, $x \in X$, consists of all points in $\mathbf{C}^r$ of the form $(x(N_1), \ldots, x(N_r))$ where x is a complex $*$-homomorphism. This set is, however, the joint spectrum Ω [5, p. 64]. The proof is complete.

Assuming that T is normal, the joint spectrum of (T, T^*, I) consists of all triples of the form $(z, \bar{z}, 1)$, $z \in \mathrm{sp}(T)$. Lemma 4.1 then implies that for $\lambda, m \in \mathfrak{M}_n$,

$$(4.3)\qquad \|\mathrm{Re}(T \otimes \bar{\lambda}) + I \otimes m\| = \sup\{\|\mathrm{Re}(z\bar{\lambda}) + m\| \mid z \in \mathrm{sp}(T)\},$$

which yields the following result:

PROPOSITION 4.1. *Let $T \in \mathscr{A}$ be normal and let $\lambda \in \mathfrak{M}_n$. The support function for $W_n(T)$ is given by*

$$(4.4)\qquad H_n(\lambda) = \inf_m\{\sup_z\{n\|\mathrm{Re}(z\bar{\lambda}) + m\| - \mathrm{tr}\,m\}\},$$

where z runs over $\mathrm{sp}(T)$ *and m runs over all self-adjoint matrices in $\mathfrak{M}_n$.*

Proof. Apply (4.3) and Theorem 3.2.

This proposition has an interesting consequence:

COROLLARY 4.1. *Let T and T' be normal elements of* $\mathscr{A}$. *The closed convex hulls of the spectra of T and T' are the same if and only if* $W_n(T) = W_n(T')$ *for all n.*

Proof. Define $g(z) \equiv \|\operatorname{Re}(z\bar{\lambda}) + m\|$. It is easy to check that $g(z)$ is a convex function of z. If $K \subseteq \mathbf{C}$ is a compact subset, the convexity of g implies that

$$\sup\{g(z) \mid z \in K\} = \sup\{g(z) \mid z \in \overline{\mathrm{coh}}K\}, \tag{4.5}$$

where $\overline{\mathrm{coh}}K$ is the closed convex hull of K. If $\overline{\mathrm{coh}}(\mathrm{sp}(T)) = \overline{\mathrm{coh}}(\mathrm{sp}(T'))$, then (4.5) yields

$$\sup\{\|\operatorname{Re}(z\bar{\lambda}) + m\| \mid z \in \mathrm{sp}(T)\} = \sup\{\|\operatorname{Re}(z\bar{\lambda}) + m\| \mid z \in \mathrm{sp}(T')\}.$$

The combination of this and (4.4) implies that the support functions for $W_n(T)$ and $W_n(T')$ are identical; Proposition 2.4 then yields $W_n(T) = W_n(T')$.

The converse follows trivially from the well-known fact that the numerical range of T, $W_1(T)$, satisfies: $W_1(T) = \overline{\mathrm{coh}}(\mathrm{sp}(T))$, if T is normal.

REMARKS. (a) The result is certainly false if the assumption of normality is dropped.

(b) The result is also a consequence of Arveson's characterization of the matricial range for normal operators [2, Prop. 2.4.1].

REMARK 4.1. In view of the proof of Corollary 4.1, the supremum in (4.4) may be taken over

$$\overline{\mathrm{coh}}(\mathrm{sp}(T)) = W_1(T), \tag{4.6}$$

instead of $\mathrm{sp}(T)$.

At this point, it is possible to obtain explicit formulae for the support functions of all self-adjoint elements in $\mathscr{A}$ and every normal element of $\mathscr{A}$ which has a disk for its numerical range. Two lemmas will be needed:

LEMMA 4.2. *Let* $T, T' \in \mathscr{A}$ *and suppose* $T' = \alpha T + \beta I$, *where* $\alpha, \beta \in \mathbf{C}$ *and* $\alpha = |\alpha| e^{i\theta}$. *If* $\lambda \in \mathfrak{M}_n$, *then*

$$H_n(\lambda, T') = |\alpha| H_n(e^{-i\theta}\lambda, T) + \operatorname{Re}(\beta \operatorname{tr}\lambda^*). \tag{4.7}$$

Here, $H_n(\lambda, S)$ *is the support function for* $W_n(S)$, $S \in \mathscr{A}$.

Proof. If $\varphi \in S_n$, then $\varphi(T') = \alpha\varphi(T) + \beta I_n$. Hence

$$\lambda \cdot \varphi(T') = |\alpha|(e^{-i\theta}\lambda) \cdot \varphi(T) + \operatorname{Re}(\beta \operatorname{tr} \lambda^*).$$

Take the supremum over all φ to get (4.7).

LEMMA 4.3. *Let $\tau \in \mathfrak{M}_n$ have a decomposition $\tau = \rho u$, $\rho \geqslant 0$, u unitary. If $\theta \in \mathbf{R}$, then*

$$\frac{1}{2}(u^*\rho u + \rho) - \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\tau) = \frac{1}{2}(u^* - \mathrm{e}^{\mathrm{i}\theta}I_n)\rho(u - \mathrm{e}^{-\mathrm{i}\theta}I_n). \tag{4.8}$$

In addition, for all $z \in \mathbf{C}$ such that $|z| \leqslant 1$,

$$\mathrm{Re}(z\tau) \leqslant \frac{1}{2}(u^*\rho u + \rho). \tag{4.9}$$

Proof. Compute.

The following proposition gives the support function for a normal element having a disk for its numerical range.

PROPOSITION 4.2. *Let $T \in \mathscr{A}$ be normal. If the numerical range of T is the disk $|z - \beta| \leqslant c$, then the support function is*

$$H_n(\lambda) = c|\lambda|_1 + \mathrm{Re}(\beta\, \mathrm{tr}\lambda^*), \quad \lambda \in \mathfrak{M}_n, \tag{4.10}$$

where $|\lambda|_1 = \mathrm{tr}\sqrt{\lambda^*\lambda}$ is the trace norm.

Proof. Assume that T is unitary and that $\mathrm{sp}(T)$ is the unit circle. Put $\tau = \bar{\lambda} = \rho u$ and $\gamma = \frac{1}{2}(u^*\rho u + \rho)$ in Lemma 4.3. Formula (4.9) then implies that $\mathrm{Re}(z\bar{\lambda}) \leqslant \gamma$ for all $z \in \mathrm{sp}(T)$. An application of the commutative Gel'fand-Naimark theorem then implies that $\mathrm{Re}(T \otimes \bar{\lambda}) \leqslant I \otimes \gamma$. By Proposition 3.2,

$$H_n(\lambda) \leqslant \mathrm{tr}\gamma. \tag{4.11}$$

Let $\{\mathrm{e}^{-\mathrm{i}\theta_j}\}_{j=1}^n$ and $\{v_j\}_{j=1}^n$ be the eigenvalues and corresponding orthonormal set of eigenvectors for the unitary matrix u. From (4.8), the choice of γ, τ, and the fact that $(u - \mathrm{e}^{-\mathrm{i}\theta_j}I_n)v_j = 0$, $j = 1, \ldots, n$,

$$\gamma v_j = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta_j}\bar{\lambda})v_j, \quad j = 1, \ldots, n. \tag{4.12}$$

Since $\mathrm{sp}(T)$ is the unit circle, $\mathrm{e}^{\mathrm{i}\theta_j} \in \mathrm{sp}(T)$. Proposition 3.3 thus implies that

$$H_n(\lambda) \geqslant \sum_{j=1}^{n} \langle \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta_j}\bar{\lambda})v_j, v_j \rangle. \tag{4.13}$$

Combine this with 4.12 to get

$$H_n(\lambda) \geqslant \sum_{j=1}^{n} \langle \gamma v_j, v_j \rangle = \mathrm{tr}\gamma. \tag{4.14}$$

From (4.11) and (4.14), it follows that $H_n(\lambda) = \operatorname{tr} \gamma$. Standard matrix manipulation then gives $H_n(\lambda) = |\lambda|_1$. Apply Lemma 4.2 and Corollary 4.1 to get the general case.

The result for the self-adjoint case is:

PROPOSITION 4.3. *Let $T \in \mathscr{A}$ be self-adjoint. If the numerical range of T is the interval $[a, b]$, then the support function is*

$$H_n(\lambda) = \frac{(b-a)}{2} |\mathrm{Re}\lambda|_1 + \frac{a+b}{2} \operatorname{tr}(\mathrm{Re}(\lambda^*)). \tag{4.15}$$

Again, $|\cdot|_1$ is the trace norm.

Proof. Assume $\operatorname{sp}(T) = \{1, -1\}$. Take $\tau = \mathrm{Re}(\lambda)$ and let $\rho = \gamma$ be the positive square root of $(\mathrm{Re}\,\bar{\lambda})^2$. The remainder of the proof is roughly the same as that for Proposition 4.2.

The following corollary is an immediate consequence of the two preceding propositions and the well-known formula

$$\sup\{|\operatorname{tr}(\lambda^* p)| : |\lambda|_1 = 1\} = \|p\|, \quad p \in \mathfrak{M}_n.$$

COROLLARY 4.2. *Let $T \in \mathscr{A}$ be normal. If the numerical range of T is the disk $|z - \beta| \leqslant c$, then*

$$W_n(T) = \{p \in \mathfrak{M}_n : \|p - \beta I_n\| \leqslant c\}. \tag{4.16}$$

If T is self-adjoint and has $[a, b]$ for its numerical range, then

$$W_n(T) = \left\{p \in \mathfrak{M}_n : p = p^*, \left\|p - \frac{a+b}{2} I_n\right\| \leqslant \frac{b-a}{2}\right\}. \tag{4.17}$$

REMARK. The characterizations given above are either explicitly given or implicitly contained in the work of Arveson [2, p. 302]. His methods are different from those used here.

5. CONCRETE CHARACTERIZATIONS OF THE SUPPORT FUNCTION

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ be the set of bounded linear operators on $\mathscr{H}$. A result similar to the elegant structural theorems obtained by Choi [7] for completely positive maps from $\mathfrak{M}_m$ to $\mathfrak{M}_n$ holds when $\mathfrak{M}_m$ is replaced by $\mathscr{B}(\mathscr{H})$:

THEOREM 5.1. *If $\varphi : \mathscr{B}(\mathscr{H}) \to \mathfrak{M}_n$ is a unital completely positive map which is an extreme point in the convex set of such maps, then there exists a positive integer $r \leqslant n$ and a net of unital completely positive maps $\varphi_\nu : \mathscr{B}(\mathscr{H}) \to \mathfrak{M}_n$,*

$$\varphi_\nu(T) = \sum_{j=1}^{r} V_{j,\nu}^* T V_{j,\nu}, \quad T \in \mathscr{B}(\mathscr{H}), \tag{5.1}$$

where $V_{j,v}: \mathbf{C}^n \to \mathcal{H}$, *such that*

$$\lim_{v} \varphi_v(T) = \varphi(T). \tag{5.2}$$

The proof will be given later.

Assuming the C^*-algebra $\mathcal{A}$ used earlier is $\mathcal{B}(\mathcal{H})$, this theorem enables both the support function $H_n(\lambda)$ and the norm $\|\cdot\|_\sigma$ defined by (3.31) to be viewed in terms of the inner product on $\mathcal{H} \otimes \mathbf{C}^n \otimes \mathbf{C}^n$. This is also true for the state functional ρ_φ associated with a unital completely positive map $\varphi: \mathcal{B}(\mathcal{H}) \to \mathfrak{M}_n$ which has the form

$$\varphi(T) = \sum_{j=1}^{r} V_j^* T V_j, \quad V_j: \mathbf{C}^n \to \mathcal{H}, \ r \leqslant n, \ T \in \mathcal{B}(\mathcal{H}). \tag{5.3}$$

To be precise, let $S \in \mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_n$. If the components of S are $S_{jk} \in \mathcal{B}(\mathcal{H})$ and E_{jk} are the elementary matrices given in Proposition 2.1,

$$S = \sum_{j,k=1}^{n} S_{jk} \otimes E_{jk}. \tag{5.4}$$

In addition, define $\hat{S} \in \mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_n \otimes \mathfrak{M}_n$ by

$$\hat{S} = \sum_{j,k=1}^{n} S_{jk} \otimes I_n \otimes E_{jk}. \tag{5.5}$$

Finally, if $\xi_1, \ldots, \xi_n$ is the canonical basis for $\mathbf{C}^n$ defined in Section 2, and $\eta_{jk} \in \mathcal{H}$ where $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant n$, then let

$$\eta_k = \sum_{j=1}^{r} \eta_{jk} \otimes \xi_j, \tag{5.6}$$

$$\eta = \sum_{k=1}^{n} \eta_k \otimes \xi_k = \sum_{k=1}^{n} \sum_{j=1}^{n} \eta_{jk} \otimes \xi_j \otimes \xi_k. \tag{5.7}$$

The following proposition exhibits the connection between ρ_φ and the inner product on $\mathcal{H} \otimes \mathbf{C}^n \otimes \mathbf{C}^n$.

PROPOSITION 5.1. *If* $\varphi \in \mathcal{S}_n$ *has the form* (5.3), *if* ρ_φ *is defined by* (2.11), *and if* $\eta_{jk} = V_j \xi_k$, *then*

$$\rho_\varphi(S) = \langle \hat{S}\eta, \eta \rangle \tag{5.8}$$

for all $S \in \mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_n$. *In addition, the set* $\{\eta_k\}$ *is orthonormal in* $\mathcal{H} \otimes \mathbf{C}_n$.

Conversely, if $\{\eta_{nk}\} \subseteq \mathcal{H}$ *is such that the* η_k*'s defined by* (5.6) *form an orthonormal set, then the maps* $V_j : \mathbf{C}^n \to \mathcal{H}$ *given by*

$$V_j \xi_k \equiv \eta_{jk}, \tag{5.9}$$

define a unique $\varphi \in \mathbb{S}_n$ *given by* (5.3).

Proof. (5.8) requires a tedious computation which amounts to showing that both sides reduce to $\sum \langle \varphi(S_{jk})\xi_k, \xi_j\rangle$. The details will be omitted. The connection between the orthonormality of $\{\eta_k\}$ and the unital nature of φ can be seen as follows.

Form the inner product of η_k and η_j; use the orthonormality of $\xi_1, \ldots, \xi_n$ and make a simple manipulation to put this in the form

$$\langle \eta_k, \eta_l\rangle = \left\langle \left(\sum_{j=1}^{r} V_j^* V_j\right) \xi_k, \xi_l \right\rangle. \tag{5.10}$$

Whether (5.3) is a given unital completely positive map or it is a completely positive map defined with V_j as in (5.9) equation (5.10) takes the form

$$\langle \eta_k, \eta_l\rangle = \langle \varphi(I)\xi_k, \xi_l\rangle = [\varphi(I)]_{lk}. \tag{5.11}$$

Since φ is unital if and only if $[\varphi(I)]_{lk} = \delta_{lk}$, (5.11) implies that φ is unital if and only if the η_k's form an orthonormal set. The proof is complete.

The proposition displays the correspondence between unital completely positive maps of the form (5.3) and all positive linear functionals of the form $\langle(\cdot)\eta, \eta\rangle$ on $\mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_n \otimes \mathfrak{M}_n$ which satisfy the condition that the η_k's are orthonormal. Combining this with Theorem 5.1 gives a characterization of the extreme points of Σ_n, the set of all associated state functionals (cf. Definition 3.1).

COROLLARY 5.1. *If* θ *is an extreme point in the set of associated state functionals* Σ_n, *then there is an integer* r, $1 \leqslant r \leqslant n$ *and a net* $\eta^\nu \equiv \sum_{k=1}^{n}\sum_{j=1}^{r} \eta_{jk}^\nu \otimes \xi_j \otimes \xi_k$ *such that* $\eta_k^\nu \equiv \sum_{j=1}^{r} \eta_{jk}^\nu \otimes \xi_j$ *forms an orthonormal set and such that*

$$\lim_\nu \langle \hat{S}\eta^\nu, \eta^\nu\rangle = \theta(S) \tag{5.12}$$

for every $S \in \mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_n$.

Proof. Because of the correspondence between Σ_n and $\mathbb{S}_n$ given by $\theta = \rho_\varphi \leftrightarrow \varphi$, extreme points of $\mathbb{S}_n$ correspond one-to-one with the extreme points of Σ_n. In particular let φ be the extreme point of $\mathbb{S}_n$ which corresponds to $\theta = \rho_\varphi$. If φ_ν is the net given by (5.1), and if $\theta_\nu \equiv \rho_{\varphi_\nu}$, (5.8) implies that for every $S \in \mathcal{B}(\mathcal{H}) \otimes \mathfrak{M}_n$,

$$\theta_\nu(S) = \langle \hat{S}\eta^\nu, \eta^\nu\rangle, \tag{5.13}$$

where $\eta^{\nu}_{jk} \equiv V_{j,\nu}\xi_k$. From (2.11)

$$\theta_\nu(S) = \rho((\varphi_\nu \otimes \tilde{I}_n)(S)) = \sum_{l,m} \rho(\varphi_\nu(S_{l,m}) \otimes E_{l,m}) . \tag{5.14}$$

By Theorem 5.1, $\lim_\nu \varphi_\nu(S_{l,m}) = \varphi(S_{l,m})$. Taking limits in (5.14) then gives

$$\lim \varphi_\nu(S) = \rho(\varphi \otimes \tilde{I}_n(S)) = \theta(S) . \tag{5.15}$$

The combination of (5.13) and (5.15) then yields (5.12); the proof is done.

This characterization of the extreme points of Σ_n allows computation of the suprema given in (3.13) and (3.31) by means of linear functionals of the form $\langle(\cdot)\eta, \eta\rangle$ on $\mathscr{B}(\mathscr{H}) \otimes \mathfrak{M}_n \otimes \mathfrak{M}_n$. To be exact, let η_k and η by given by (5.6) and (5.7). Define the set

$$\Gamma = \{\eta \in \mathscr{H} \otimes \mathbf{C}^n \otimes \mathbf{C}^n \mid \text{the } \eta_k \text{ are orthonormal}\}. \tag{5.16}$$

The following theorem holds:

THEOREM 5.2. *If $S = S^* \in \mathscr{B}(\mathscr{H}) \otimes \mathfrak{M}_n$, and if $\hat{S}$ is given by* (5.5), *then*

$$\sup\{\theta(S) \mid \theta \in \Sigma_n\} = \sup\{\langle \hat{S}\eta, \eta\rangle \mid \eta \in \Gamma\}. \tag{5.17}$$

Proof. The set Σ_n of associated state functionals is clearly a weak *-closed convex subset of the norm-n positive linear functionals on $\mathscr{B}(\mathscr{H}) \otimes \mathfrak{M}_n$. Since the latter set is weak*-compact, Σ_n is as well. That being the case, the supremum in (5.17) is attained by one or more associated state functionals whose totality, denoted by Σ_n', is again weak*-compact and convex. If θ_0 is an extreme point of Σ_n', then it is also an extreme point of Σ_n: Suppose $\theta_0 = \alpha\theta_0' + \beta\theta_0''$, $\theta_0', \theta_0'' \in \Sigma_n$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$. By the usual argument, $\theta_0'(S) = \theta_0''(S) = \theta_0(S)$. Thus θ_0', θ_0'' are both in Σ_n'. Since θ_0 is an extreme point of Σ_n', $\theta_0' \equiv \theta_0'' \equiv \theta_0$; thus, θ_0 is an extreme point of Σ_n. By Corollary 5.2,

$$\theta_0(S) = \lim_\nu \langle \hat{S}\eta^\nu, \eta^\nu\rangle, \tag{5.18}$$

where $\eta^\nu \in \Gamma$. On the other hand, given $\eta \in \Gamma$, Proposition 5.1 implies that there exists $\varphi \in \mathfrak{S}_n$ such that $\rho_\varphi(S) = \langle \hat{S}\eta, \eta\rangle$, and $\rho_\varphi \in \Sigma_n$ by Corollary 3.1. Hence,

$$\langle \hat{S}\eta, \eta\rangle \leqslant \sup\{\theta(S) \mid \theta \in \Sigma_n\} = \theta_0(S). \tag{5.19}$$

Inspection of (5.18) and (5.19) then gives (5.17).

Concrete formulae for $H_n(\lambda)$ and $\|\cdot\|_\sigma$ are immediate consequences of Theorem 5.2:

COROLLARY 5.2. *If $T \in \mathscr{B}(\mathscr{H})$ and if $H_n(\lambda)$ is the support function for the matrix range $W_n(T)$ then for every $\lambda \in \mathfrak{M}_n$,*

(5.20) $$H_n(\lambda) = \sup\{\langle \mathrm{Re}(T \otimes I_n \otimes \bar{\lambda})\eta, \eta\rangle \mid \eta \in \Gamma\},$$

where Γ is defined by (5.16).

Proof. Apply Theorem 5.2 and Corollary 3.2.

COROLLARY 5.3. *If $\mathscr{A} = \mathscr{B}(\mathscr{H})$ and $\|\cdot\|_\sigma$ is defined by* (3.31), *then for every $S \in \mathscr{B}(\mathscr{H}) \otimes \mathfrak{M}_n$,*

(5.21) $$\|S\|_\sigma = \frac{1}{\sqrt{n}} \sup\{\|\hat{S}\eta\| \mid \eta \in \Gamma\}$$

where Γ is as in (5.16).

Proof. Straightforward computation gives that $\widehat{S^*S} = \hat{S}^*\hat{S}$. Apply Theorem 5.2 and formula (3.31) to obtain the result.

REMARKS. (a). The formula for $\|\cdot\|_\sigma$ given in Corollary 5.3 may be used in conjunction with (3.32) and Theorem 3.3 to give concrete versions of the analogue to Lumer's formula and the metric characterization of $W_n(T)$.

(b). The formulae given in Corollaries 5.2 and 5.3 are concrete and depend heavily on the underlying Hilbert space $\mathscr{H}$. Those given in Corollary 3.2 and formula (3.31) are intrinsic; they do not depend on any particular representation of the C^*-algebra $\mathscr{A}$, but they are more abstract.

At this point, only the proof of Theorem 5.1 is lacking. To prove this theorem, it is necessary to prove a technical lemma. The lemma is important in its own right in that it provides a very natural generalization of Choi's results to certain unital completely positive maps.

In what follows, $\mathscr{C}(\mathscr{H})$ denotes the set of all compact linear operators on $\mathscr{H}$.

LEMMA 5.1. *Let $\varphi : \mathscr{B}(\mathscr{H}) \to \mathfrak{M}_n$ be completely positive, $\varphi(I) = K \geqslant 0$, and let φ be an extreme point in the convex set of all such completely positive maps. If F_μ is an approximate identity for $\mathscr{C}(\mathscr{H})$, $0 \leqslant F_\mu \leqslant I$, and if*

(5.22) $$\lim_\mu \|\varphi(F_\mu) - \varphi(I)\| = 0,$$

then there exist bounded linear maps $V_j : \mathbf{C}^n \to \mathscr{H}$, $j = 1, \ldots, r$, $r \leqslant n$, such that for every $T \in \mathscr{B}(\mathscr{H})$,

(5.23) $$\varphi(T) = \sum_{j=1}^{r} V_j^* T V_j .$$

Proof. The extremality of φ is at the heart of the matter. Stinespring's theorem [17] implies that $\varphi(\cdot) = W^*\pi(\cdot)W$, where $\pi(\cdot) = \bigoplus_{j=1}^{r} \pi_j(\cdot)$ is a representation of $\mathscr{B}(\mathscr{H})$ and each π_j is an irreducible representation of $\mathscr{B}(\mathscr{H})$ with underlying Hilbert space $\mathscr{K}_j$. Without the condition of extremality, it is possible for the number of representations to be infinite; with it, r must be finite. (This is a direct consequence of Arveson's characterization of extremality [1, Theorem 1.4.6]. See [14, Lemma 6.4].) Thus, the canonical representation takes the form,

$$\varphi(T) = \sum_{j=1}^{r} W_j^*\pi_j(T)W_j\,, \quad r < \infty, \tag{5.24}$$

where the pairs (π_j, W_j) are assumed *minimal* (Arveson [1], p. 145); that is,

$$[\pi_j(\mathscr{B}(\mathscr{H}))W_j\mathbf{C}^n] = \mathscr{K}_j\,, \quad j = 1, \ldots, r. \tag{5.25}$$

At this point, it is necessary to examine each irreducible representation π_j. If π_j does not annihilate $\mathscr{C}(\mathscr{H})$, then π_j is an irreducible representation of $\mathscr{C}(\mathscr{H})$ [3, Theorem 1.3.4, p. 16] and is therefore equivalent to the identity representation [3, Corollary 2, p. 20]; hence, there exists a unitary operator U_j such that for every $T \in \mathscr{B}(\mathscr{H})$,

$$\pi_j(T) = U_j^* T U_j\,. \tag{5.26}$$

Next, suppose that $\pi_1, \ldots, \pi_l$ annihilate $\mathscr{C}(\mathscr{H})$ and that $\pi_{l+1}, \ldots, \pi_r$ do not. Because $F_\mu \in \mathscr{C}(\mathscr{H})$, $\pi_1(F_\mu) = \ldots = \pi_l(F_\mu) = 0$; also the remaining representations satisfy (5.26), so $\varphi(F_\mu) = \sum_{j=l+1}^{r} W_j^* U_j^* F_\mu U_j W_j$. As $F_\mu \to I$, this formula implies that $\varphi(F_\mu) \to \sum_{j=l+1}^{r} W_j^* W_j$. By (5.22), however, $\varphi(F_\mu) \to \varphi(I) = \sum_{j=1}^{r} W_j^* W_j$. Comparing the two expressions reveals that $W_1^* W_1 = \ldots = W_l^* W_l = 0$; hence, $W_1 = \ldots = W_l = 0$. This contradicts the minimality of (π_k, W_k), $k = 1, \ldots, l$. As a consequence, none of the representations can annihilate $\mathscr{C}(\mathscr{H})$; each must have the form (5.26).

How large can r be? Since all of the subrepresentations in π are equivalent, a short computation shows that the dimension of the commutant of π is at least r^2. On the other hand, from [14, proof of Lemma 6.4], the dimension of the commutant of π is at most n^2. Hence $r \leqslant n$.

To conclude the proof, set $V_j = U_j W_j$, then use (5.24) and (5.26).

Proof of Theorem 5.1. Recall that $\mathscr{C}(\mathscr{H})$ forms a two-sided ideal in $\mathscr{B}(\mathscr{H})$. A result of Bunce and Salinas [6, Lemma 2.6] then implies that the completely positive map φ may be written uniquely as $\varphi = \varphi_1 + \varphi_2$, where both φ_1

and φ_2 are completely positive maps from $\mathscr{B}(\mathscr{H})$ to $\mathfrak{M}_n$ and have the following properties:

(a) $\varphi_j(I) = K_j \geqslant 0, \quad j = 1, 2, \quad K_1 + K_2 = \varphi(I) = I_n$;

(b) $\varphi_2(\mathscr{C}(\mathscr{H})) = 0$;

and,

(c) for an approximate identity F_μ in $\mathscr{C}(\mathscr{H})$, $0 \leqslant F_\mu \leqslant I$,

$$\lim_{\mu} \| \varphi_1(F_\mu) - \varphi_1(I) \| = 0. \tag{5.27}$$

In addition to the properties listed above, it is easy to see that φ being an extreme unital map implies that φ_1 is an extreme point among all completely positive maps from $\mathscr{B}(\mathscr{H})$ to $\mathfrak{M}_n$ which take I to K_1. (The ordinary proof applies, although at first glance the uniqueness of the decomposition given by Bunce and Salinas seems contradicted; it's not.)

Inspection of the properties above reveals that φ_1 satisfies the conditions of Lemma 5.1. Thus there exist bounded linear maps $V_j : \mathbf{C}^n \to \mathscr{H}$, $j = 1, \ldots, r$, such that for each $T \in \mathscr{B}(\mathscr{H})$,

$$\varphi_1(T) = \sum_{j=1}^{r} V_j^* T V_j, \quad r \leqslant n. \tag{5.28}$$

The effect of adding φ_2 to φ_1 must now be taken account of. From property (b), $\varphi_2(\mathscr{C}(\mathscr{H})) = 0$; a lemma of Arveson [4, p. 335] then implies the existence of a norm-bounded net of operators $R_\nu : C^n \to \mathscr{H}$ which satisfy

$$\lim_{\nu} \| \varphi_2(T) - R_\nu^* T R_\nu \| = 0, \quad T \in \mathscr{B}(\mathscr{H}). \tag{5.29}$$

Define the net of completely positive maps $\chi_\nu : \mathscr{B}(\mathscr{H}) \to \mathfrak{M}_n$,

$$\chi_\nu(T) \equiv (V_1 + R_\nu)^* T (V_1 + R_\nu) + \sum_{j=2}^{r} V_j^* T V_j . \tag{5.30}$$

This net will always converge to $\varphi(T)$ provided the cross-terms $R^* T V_1$ and $V_1^* T R$ vanish in the limit.

A technique of Bunce and Salinas [6, p. 750] will be used to show the vanishing of the cross-terms. Let $x \in \mathbf{C}^n$ and note that $\| V_1^* T R_\nu x \|^2 = \langle R_\nu^* T^* V_1 V_1^* T R_\nu x, x \rangle$ converges to $\langle \varphi_2(T^* V_1 V_1^* T) x, x \rangle$. Since $V_1 : \mathbf{C}^n \to \mathscr{H}$, $V_1 V_1^*$ is a finite rank operator; it is compact and so is $T^* V_1 V_1^* T$. Because $\varphi_2(\mathscr{C}(\mathscr{H})) = 0$, $\varphi_2(T^* V_1 V_1^* T) = 0$; hence, $\| V_1^* T R_\nu x \| \to 0$. Since the space $\mathbf{C}^n$ is finite dimensional, this implies

$$\lim_{\nu} \| V_1^* T R_\nu \| = 0. \tag{5.31}$$

The other cross term can be viewed as $(V_1^* T^* R_\nu)^*$. Since $\| V_1^* T^* R_\nu \| = \| (V_1^* T^* R_\nu)^* \|$, equation (5.31) with T replaced by T^* implies

(5.32) $$\lim_\nu \| R_\nu^* T V_1 \| = 0.$$

Thus $\chi_\nu(T) \to \varphi(T)$.

The maps $\chi_\nu(\cdot)$ are not necessarily unital, however $\chi_\nu(I) \to \varphi(I) = I_n$. Thus the $\chi_\nu(I)$ eventually become positive and invertible. Restricting the net to such ν, define

$$\varphi_\nu(T) = \chi_\nu(I)^{-1/2} \chi_\nu(T) \chi_\nu(I)^{-1/2}.$$

These completely positive maps are unital, have the form $\sum_{j=1}^{r} V_{j,\nu}^* T V_{j,\nu}$ and converge to $\varphi(T)$. The proof is complete.

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REFERENCES

1. Arveson, W. B., Subalgebras of C^*-algebras, *Acta Math.*, **123**(1969), 141–224.
2. Arveson, W. B., Subalgebras of C^*-algebras. II, *Acta Math.*, **128**(1972), 271–308.
3. Arveson, W. B., *An invitation to C^*-algebras*, Springer-Verlag, Berlin, 1976.
4. Arveson, W. B., Notes on extensions of C^*-algebras, *Duke Math. J.*, **44**(1977), 329–355.
5. Brown, L. G.; Douglas, R. G.; Fillmore, P., *Unitary equivalence modulo compact operators and extensions of C^*-algebras*, Springer-Verlag, Lecture Note **345**, 1973, Heidelberg.
6. Bunce, J.; Salinas, N., Completely positive maps on C^*-algebras and the left matricial spectra of an operator, *Duke Math. J.*, **43**(1976), 747–774.
7. Choi, M.-D., Completely positive linear maps on complex matrices, *Linear Algebra and Appl.*, **10**(1975), 285–290.
8. Choi, M.-D.; Effros, E. G., Injectivity and operator spaces, *J. Functional Analysis*, **24**(1977), 156–209.
9. Choi, M. -D.; Effros, E. G., The completely positive lifting problem for C^*-algebras, *Ann. of Math.*, **104**(1976), 585–609.
10. Dixmier, J., *C^*-algebras*, North-Holland Publishing Co., Amsterdam, 1977.
11. Lance, C., On nuclear C^*-algebras, *J. Functional Analysis*, **12**(1973), 157–176.
12. Lumer, G., Semi-inner-product spaces, *Trans. Amer. Math. Soc.*, **100**(1961), 29–43.

13. ROCKAFELLAR, R. T., *Convex analysis*, Princeton University Press, Princeton, N. J., 1970.
14. SAKAI, S., *C*-algebras and W*-algebras*, Springer-Verlag, Berlin, 1971.
15. SMITH, R. R.; WARD, J. D., Matrix ranges for Hilbert space operators, *Amer. J. Math.*, to appear.
16. STAMPFLI, J. G.; WILLIAMS, J. P., Growth conditions and the numerical range in Banach algebra, *Tôhoku Math. J.*, **20**(1968), 417–424.
17. STINESPRING, W. F., Positive functions on C^*-algebras, *Proc. Amer. Math. Soc.*, **6**(1955), 211–216.

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