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FREDHOLM THEORY OF PIECEWISE CONTINUOUS FOURIER INTEGRAL OPERATORS ON HILBERT SPACE

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Duducava has studied the Fredholm theory of certain convolution integral operators on $L^p(\mathbf{R})$ (and $L^p(\mathbf{R}) \otimes \mathfrak{M}_n$) of a very general kind, and which are determined by piecewise continuous functions [6], [7]. For example if F denotes the Fourier transform then Fredholm criteria and an index theorem are obtained for the operators

(1) $$A = \sum_{i=1}^{n} M_{\varphi_i} D_{\psi_i} M_{\theta_i},$$

where φ_i, ψ_i, θ_i are piecewise continuous, where M_φ denotes multiplication by φ, and where D_ψ denotes the Fourier multiplier $F^{-1}M_\psi F$. It is easily checked that special cases of A include convolution integral operators, singular integral operators, and, when $\theta_i = 1$ for $i = 1, 2, \ldots, n$, the Fourier integral operator

(2) $$\mathrm{Op}(a) = \frac{1}{\sqrt{2\pi}} \int a(x, y) \mathrm{e}^{\mathrm{i}xy} Ff(y)\, \mathrm{d}y,$$

with symbol function $a(x, y) = \sum_{i=1}^{n} \varphi_i(x)\psi_i(y)$.

In the present paper we concentrate on the Hilbert space case and the operators $\mathrm{Op}(a)$. By restricting to certain (locally simple) symbols, whose operators behave locally like Hermitian operators, we see that the essential spectrum of $\mathrm{Op}(a)$ is a line segment augmentation of the asymptotic range of $a(x, y)$, and thus a finite union of closed curves. A natural winding number provides the Fredholm index. This is an explicit generalization of the continuous case [2], and, although implicitly present, it is by no means apparent in [6], [7].

Being on a Hilbert space we can employ C^*-techniques and obtain further information about the generated pseudo-differential C^*-algebra. For example the character space can be determined and the corresponding character spectrum for $\mathrm{Op}(a)$ identified with the asymptotic range of $a(x, y)$. This is also shown to be true when φ_i and ψ_i, $i = 1, 2, \dots, n$, are piecewise slowly oscillating. Our main technique, which replaces the local principle of Gohberg and Krupnik, employed by Duducava, is to use Douglas's localization theorem for a C^*-algebra with centre, and to characterize the local algebras as $\mathcal{M}_2(\mathcal{C}[0, 1])$, the 2×2 matrices over continuous functions on $[0, 1]$. Some further applications of this identification appear in [15]. We expect the method to work with little essential change, for Op(a) with piecewise slowly oscillating symbols. The analysis of the corresponding Toeplitz and Hankel operators can be seen in [5], [12] and [11].

In order to state and describe the result we first introduce some notation and definitions. Let $\mathcal{PC}$ denote the space of functions on the real line which are continuous on the right and posses limits from the left at each point, including infinity. If φ_i, ψ_i for $i = 1, 2, \dots, n$, belong to $\mathcal{PC}$ then the *asymptotic range* of $a(x, y)$, denoted ran $a(x, y)$ or ran a, is the set of limit points of $a(x, y)$ as $|x| + |y|$ converges to $+\infty$. It is convenient to think of this as the closure of the range of $a(x, y)$ regarded as a piecewise continuous function on

$$W = (\mathbf{R} \cup \{-\infty, +\infty\})\times(\mathbf{R} \cup \{-\infty, +\infty\})\setminus(\mathbf{R}\times\mathbf{R}).$$

This set is homeomorphic to, and identified with, the boundary of the unit square of Figure 1. The asymptotic range, therefore, possesses the natural orientation induced from the (indicated) orientation of W.

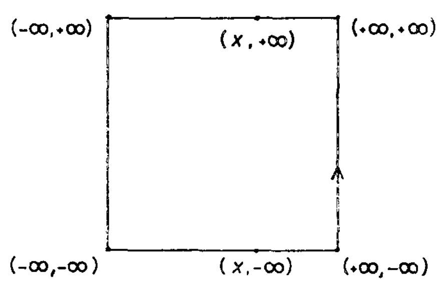

Fig. 1. The space W.

Definition 1. Let φ_i, ψ_i, $1 \leqslant i \leqslant n$, be functions in $\mathcal{PC}$ possessing only a finite number of discontinuities, and for fixed x, y in $\mathbf{R}$ let $\alpha = a(x+, +\infty)$, $\beta = a(x+, -\infty)$, $\gamma = a(x-, +\infty)$, $\delta = a(x-, -\infty)$ and $\alpha' = a(+\infty, y+)$, $\beta' = a(-\infty, y+)$, $\gamma' = a(+\infty, y-)$, $\delta' = a(-\infty, y-)$.

(i) The function $a(x, y)$ is called *locally simple* if for each x the points $\alpha, \beta, \gamma, \delta$ are colinear and $\alpha - \beta = \gamma - \delta$, and if for each y the points $\alpha', \beta', \gamma', \delta'$ are colinear and $\alpha' - \beta' = \gamma' - \delta'$.

(ii) The *local segments at* (x, ∞) of the locally simple function $a(x, y)$ are the two non overlapping finite segments determined by the quadruple $\alpha, \beta, \gamma, \delta$. The *local segments at* (∞, y) for $a(x, y)$ are analogously defined.

(iii) The *curve* $a^{\#}$ associated with the locally simple function $a(x, y)$ is the union of ran a and all the local segments. (Actually we could omit ran a here because it appears as the closure of the union of the degenerate local segments.)

The curve $a^{\#}$ is the union of ran a and the non degenerate local segments, it has a unique orientation compatible with that of ran a, and appears as the union of

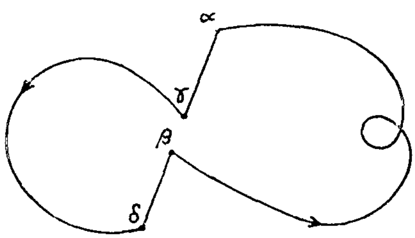

Fig. 2.

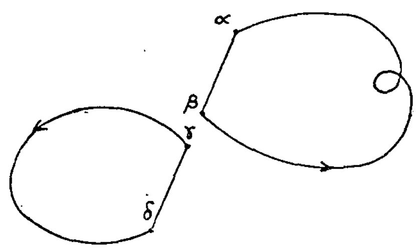

Fig. 3.

a finite number of closed curves. Let us illustrate what is happening in the case where $a(x, y)$ has no discontinuities on W except at $(x, +\infty)$ and (necessarily, by (i) above) at $(x, -\infty)$. If $[\alpha, \gamma]$ and $[\beta, \delta]$ are the local segments here then $a^{\#}$ will be a closed curve as in Figure 2, whereas in the alternative case, Figure 3, $a^{\#}$ is the union of two closed curves.

In general $a^{\#}$ is the union of $|n_1 - n_2|$ closed curves where n_1 (resp. n_2) is the number of discontinuities of $a(x, y)$ at (x, ∞) (resp. (∞, y)) of the second kind (Fig. 3).

THEOREM 1. *If* $a(x, y)$ *is locally simple then* Op(a) *is a Fredholm operator if and only if* $a^{\#}$ *does not pass through the origin. In this case the Fredholm index of* Op(a) *is the winding number of* $a^{\#}$.

In general Op(a) is *not* essentially normal and some specialization of symbol function seems necessary in order to have a natural index theorem. The class we have chosen, the locally simple symbols, correspond to those essentially normal operators Op(a) (with φ_i and ψ_i having a finite number of discontinuities) which behave locally like hermitian operators. This statement should become clear on reading Section 3.

We need some more notation. The space $\mathscr{C}$ (resp. $\mathscr{C}_\infty$) consists of those continuous functions on the real line possessing equal (resp. possibly different) limits at $+\infty$ and $-\infty$. If A is a subalgebra of L^∞ then $M(A)$ denotes the topological space of multiplicative linear functionals on A. This notation is sometimes used when A is a C^*-algebra.

As in [13], if A and B are subalgebras of $L^\infty(\mathbf{R})$ we define $\Psi(A, B)$ to be the C^*-algebra generated by the collection of M_φ and D_ψ for φ in A and ψ in B, respectively.

Throughout we let $\mathcal{K}$ denote the ideal of compact operators acting on $L^2(\mathbf{R})$, and the spectrum and essential spectrum of an operator T are denoted by $\sigma(T)$ and $\sigma_e(T)$ respectively.

1. SPECTRAL INCLUSION OF ran a

The essential spectrum of the operator $\mathrm{Op}(a)$, as given in (2), always contains the asymptotic range of $a(x, y)$. This has been shown to be true in a general setting [13, Theorem 6.5], but let us just consider piecewise slowly oscillating symbols, since it seems likely that the result of this paper generalizes to this case.

A function φ on $\mathbf{R}$ is said to be *piecewise slowly oscillating* if it is continuous except at a finite number of points $x_1, x_2, \ldots, x_n$ where

$$\mathrm{osc}(\varphi, [x_i + \delta/2,\ x_i + \delta]) \to 0 \ \ (\text{as } \delta \to 0)$$

and

$$\mathrm{osc}(\varphi, [x_i - \delta, x_i - \delta/2]) \to 0 \ \ (\text{as } \delta \to 0)$$

for $i = 1, 2, \ldots, n$, and if, moreover,

$$\mathrm{osc}(\varphi, [x, 2x]) \to 0$$

as x tends to $+\infty$ or to $-\infty$. Here we use the notation $\mathrm{osc}(\varphi, I)$ to denote the maximum of $|\varphi(s) - \varphi(t)|$ for s, t in I. Let $\mathcal{PSO}$ denote the C^*-algebra of functions that they generate. In view of the theorems of [13], [14] the space $M(\Psi(\mathcal{PSO}, \mathcal{PSO}))$ is the subspace of points $z = (x, y)$ of $M(\mathcal{PSO}) \times M(\mathcal{PSO})$ such that $\|M_\varphi D_\psi\| = 1$ whenever $0 \leqslant \varphi, \psi \leqslant 1$ with $\varphi(x) = \psi(y) = 1$. This is precisely the subset

$$M(\mathcal{PSO}) \times M_\infty(\mathcal{PSO}) \cup M_\infty(\mathcal{PSO}) \times M(\mathcal{PSO})$$

where $M_\infty(\mathcal{PSO})$ denotes those characters which annihilate functions which vanish at infinity. The identification is implemented by $z(M_\varphi) = \varphi(x)$, $z(D_\psi) = \psi(y)$, and because of this it becomes clear that the character spectrum $\{z(\mathrm{Op}(a)) : z \in M(\Psi(\mathcal{PSO}, \mathcal{PSO}))\}$ is equal to ran a. Since all characters annihilate $\mathcal{K}$ it follows that ran a is contained in $\sigma_e(\mathrm{Op}(a))$.

2. THE LOCALIZATION OF $\Psi/\mathcal{K}$

We now apply the localization method to $\Psi(\mathcal{PC}, \mathcal{PC})/\mathcal{K}$ and identify certain local C^*-algebras. This technique allows us to consider the discontinuities of $a(x, y)$ separately and thus obtain the essential spectrum of $\mathrm{Op}(a)$ as the union of ran a and all the (non degenerate) local spectra. The local spectra, for locally simple functions, turn out to be the local line segments, as in Definition 1, and so we obtain $\sigma_e(\mathrm{Op}(a)) = a^{\#}$.

LEMMA 1. (i) *$\Psi(\mathscr{C}, \mathscr{C})/\mathscr{K}$ is a commutative C^*-algebra whose maximal ideal space is naturally homeomorphic to $M(\mathscr{C}) \times M(\mathscr{C})\setminus \mathbf{R}\times\mathbf{R}$.*

(ii) *$\Psi(\mathscr{C},\mathscr{C})/\mathscr{K}$ is contained in the centre of $\Psi(\mathscr{P}\mathscr{C}, \mathscr{P}\mathscr{C})/\mathscr{K}$.*

Proof. (i) We have noted in the introduction that $M_\varphi D_\psi$ is a compact operator if φ and ψ are continuous functions with compact support. Consequently, by a trivial approximation argument this is also true if φ, ψ belong to $\mathscr{C}$ with $\varphi(\infty) = \psi(\infty) = 0$. Plainly then $M_\varphi D_\psi - D_\psi M_\varphi$ is compact if φ, ψ belong to $\mathscr{C}$ and so the quotient algebra of (i) is commutative. That $M(\Psi(\mathscr{C}, \mathscr{C})/\mathscr{K})$ is homeomorphic to $M(\mathscr{C})\times M(\mathscr{C})\setminus\mathbf{R}\times\mathbf{R}$ is well known and a proof may be found in [14]. The identification of these spaces is exactly analogous to the identification of the previous section.

(ii) Let $H^2(\mathbf{R})$ be the usual Hardy space of $L^2(\mathbf{R})$ associated with analytic functions in the upper half plane, and let Q be the orthogonal projection onto this subspace. The Paley-Wiener theorem asserts that $Q = D_\chi$, where χ is the characteristic function of $(0, \infty)$, and so $D_\chi M_\varphi - M_\varphi D_\chi = QM_\varphi - M_\varphi Q$ is a compact operator when φ belongs to $\mathscr{C}$ (e.g. see [3]). It follows from this that $D_\rho M_\varphi - M_\varphi D_\rho$ is compact if ρ is the characteristic function of (a, ∞) and so it follows that M_φ commutes, modulo the compact operators, with D_ψ, for any ψ in $\mathscr{P}\mathscr{C}$. Now (ii) follows easily.

THEOREM 2 [3, 7.47]. *Let A be a commutative C^*-algebra which is contained in the centre of a C^*-algebra B, and for z in $M(A)$ let $B^z = B/B_z$ where B_z is the closed two sided ideal in B generated by $\{a \text{ in } A \mid z(a) = 0\}$. Then*

(i) *The natural mapping from B to $\bigoplus_{z\in M(A)} B^z$ is a star isometric isomorphism.*

(ii) *The spectrum of an element b of B is the union of the spectra of the elements $b + B_z$ of B^z for z in $M(A)$.*

We shall henceforth identify $M(\Psi(\mathscr{C}, \mathscr{C}))$ with the set $Z = M(\mathscr{C})\times M(\mathscr{C})\setminus\mathbf{R}\times\mathbf{R}$ as in Lemma 1. Let $\Psi = \Psi(\mathscr{P}\mathscr{C}, \mathscr{P}\mathscr{C})$. Theorem 2 can now be used to obtain the following localization of $\Psi/\mathscr{K}$ to the points of Z.

For $z = (x, y)$ in Z let I^z be the two sided closed ideal in Ψ generated by the collection of M_φ, D_ψ such that φ, ψ belong to $\mathscr{C}$ and $x(\varphi) = y(\psi) = 0$. (Each such ideal contains $\mathscr{K}$.) Let $\Psi^z = \Psi/I^z$.

THEOREM 3. (i) *The natural mapping from $\Psi/\mathscr{K}$ to $\bigoplus_{z\in Z} \Psi^z$ is a star isometric isomorphism.*

(ii) *If T belongs to Ψ then the essential spectrum of T is the union of the spectra of $T + I^z$, in Ψ^z, the union being taken over all z in Z.*

We now proceed to identify all the local algebras Ψ^z and represent them as more elementary algebras. A special role is played by $\Psi^{(\infty,\infty)}$ which we now show is simply $\mathbf{C}^4$. (Here, of course, ∞ denotes the functional on $\mathscr{C}$ of evaluation at infinity and (∞, ∞) denotes the corresponding character on $\Psi(\mathscr{C}, \mathscr{C})$.) In fact the following lemma also shows that the local spectrum at infinity of $\mathrm{Op}(a)$ is not particularly interesting since it is included in ran a.

LEMMA 2. *There is a C^*-isomorphism from $\Psi^{(\infty, \infty)}$ to $\mathbf{C}^4$ such that*

$$M_\varphi + I^{(\infty,\infty)} \to \varphi(+\infty) \oplus \varphi(+\infty) \oplus \varphi(-\infty) \oplus \varphi(-\infty)$$

$$D_\psi + I^{(\infty,\infty)} \to \psi(+\infty) \oplus \psi(-\infty) \oplus \psi(+\infty) \oplus \psi(-\infty)$$

for φ, ψ in $\mathscr{PC}$.

Proof. It is a result of Cordes and Herman [2], which has also been proved in [13], that $\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)/\mathscr{K}$ is a commutative C^*-algebra. In fact $M(\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)/\mathscr{K})$ (which also is $M(\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty))$) is naturally homeomorphic to W (Fig. 1), and the correspondences

$$\mathrm{Op}(a) + K \to a(x, y)|_W$$

extends to a C^*-isomorphism. It should be reasonably clear then why the following analogue of Lemma 2 is true. Let $J^{(\infty,\infty)}$ be the closed two-sided ideal in $\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)$ generated by those M_φ and D_ψ with symbols φ, ψ in $\mathscr{C}$ satisfying $\varphi(\infty) = \psi(\infty) = 0$. Then $\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)/J^{(\infty,\infty)}$ is naturally isomorphic to $\mathbf{C}^4$ by an isomorphism analogous (exactly) to that of the lemma. Indeed this quotient algebra is, by what we have already said, isomorphic to the continuous functions on W modulo the ideal of functions vanishing on the corners of W.

It suffices then to show that $\Psi^{(\infty,\infty)}$ is naturally isomorphic to $\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)/J^{(\infty,\infty)}$. (In other, rather vague, words, it makes no difference to the quotient algebra at infinity whether we include functions with discontinuities at real points.)

Note first that $\Psi^{(\infty,\infty)}$ is naturally isomorphic to $\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)/I^{(\infty,\infty)}$. Indeed, if h is a strictly increasing function in $\mathscr{C}_\infty$ then the cosets of M_h and D_h actually generate $\Psi^{(\infty,\infty)}$, and so $\Psi^{(\infty,\infty)} = \Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)/I^{(\infty,\infty)}$. We complete the proof by showing that $J^{(\infty,\infty)} = I^{(\infty,\infty)} \cap \Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)$. The inclusion of $J^{(\infty,\infty)}$ is clear, so let T belong to the latter ideal. We now show that T belongs to $J^{(\infty,\infty)}$.

Let φ_n, belonging to $\mathscr{C}$, be a function which vanishes on $[-n, n]$, satisfies $0 \leqslant \varphi \leqslant 1$, and has limit 1 at infinity. The cosets of M_{φ_n} and of D_{φ_n}, for each n, act as the identity in the quotient algebra $\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)/J^{(\infty,\infty)}$ and so, since T is in $\Psi(\mathscr{C}_\infty, \mathscr{C}_\infty)$, it follows that

$$\mathrm{d}(T, J^{(\infty,\infty)}) = \mathrm{d}(M_{\varphi_n} D_{\varphi_n} T, J^{(\infty,\infty)}) \tag{3}$$

where $\mathrm{d}(\cdot, J^{(\infty,\infty)})$ signifies distance to $J^{(\infty,\infty)}$. However T also belongs to $I^{(\infty,\infty)}$ and we claim that this ensures that

$$\lim_{n\to\infty} \mathrm{d}(M_{\varphi_n} D_{\varphi_n} T, J^{(\infty,\infty)}) = 0. \tag{4}$$

Combining (3) and (4) then gives $T \in J^{(\infty, \infty)}$ as desired. To see (4) first note that as T is in $I^{(\infty, \infty)}$ it may be approximated arbitrarily closely by an operator

$$T' = \sum_{i=1}^{n} \left\{ \prod_{j=1}^{m} M_{\theta_{ij}} D_{\omega_{ij}} \right\} M_{\theta_i} D_{\omega_i} + K_1 ,$$

where θ_i, ω_i belong to $\mathscr{C}$ and vanish at ∞, where θ_{ij}, ω_{ij} are $\mathscr{PC}$ functions with only a finite number of discontinuities, and where K_1 is a compact operator. Now we "absorb" the (real) discontinuities of θ_{ij}, ω_{ij}. First, consider N large enough so that these functions have no discontinuities at real points t with $|t| > N$. Now write $\varphi_N = f^{mn}$ with f, positive and vanishing on $[-N, N]$, so that $\theta'_{ij} = f \cdot \theta_{ij}$ and $\varphi'_{ij} = f \cdot \varphi_{ij}$ are $\mathscr{C}_\infty$ functions. Thus, using Lemma 1 (ii), we obtain

$$M_{\varphi_N} D_{\varphi_N} T' = \sum_{i=1}^{n} \left\{ \prod_{j=1}^{m} M_{\theta'_{ij}} D_{\varphi'_{ij}} \right\} M_{\theta_i} D_{\omega_i} + K_2 ,$$

where K_2 is compact. Thus $M_{\varphi_N} D_{\varphi_N} T'$ belongs to $J^{(\infty, \infty)}$, and so (4) follows.

To describe the other local algebras we must introduce the C^*-algebra which is generated by two projections p, q such that the spectrum of pqp is [0, 1]. It follows from [8], and also from [9, Section 3], that this algebra is independent of the particular generators. We shall set

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad q = \begin{bmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{bmatrix},$$

which belong to the C^*-algebra of 2×2 matrices over $\mathscr{C}[0,1]$, the continuous functions on the unit interval. We let $\mathscr{M}$ denote the unital C^*-algebra generated by p and q. It is the algebra of matrices whose off-diagonal functions vanish at 0 and 1. Let χ denote the characteristic function of $(0, \infty)$.

LEMMA 3. *Let t be a real number.*

(i) *There exists a C^*-isomorphism from $\Psi^{(t, \infty)}$ to $\mathscr{M}$ such that*

$$M_{\chi_{(t, \infty)}} + I^{(t, \infty)} \to p ,$$

$$D_\chi + I^{(t, \infty)} \to q .$$

(ii) *There exists a C^*-isomorphism from $\Psi^{(\infty, t)}$ to M such that*

$$M_\chi + I^{(\infty, t)} \to p ,$$

$$D_{\chi_{(t, \infty)}} + I^{(\infty, t)} \to q .$$

Proof. (i) In view of the already mentioned fact that M does not depend on the specific realization of the generating projections we need only show that the spectrum of $M_{\chi_{(t,\infty)}}D_\chi M_{\chi_{(t,\infty)}} + I^{(t,\infty)}$ is the unit interval. (A little reflection shows that the cosets given in (i) generate $\Psi^{(t,\infty)}$.) It is enough, in view of a trivial translation argument to show that $[0, 1]$ is the spectrum of $M_\chi D_\chi M_\chi + I^{(0,\infty)}$.

Let φ be a function in $\mathscr{C}$ such that $\varphi(0) = 0$, $\varphi(\infty) = 1$, $\varphi(x) = \varphi(-x)$ and $\operatorname{Im}\varphi(y) > 0$ for $0 < y < \infty$. Then $S = M_\chi D_{\varphi\chi} M_\chi$ is equivalent (as an $L^2(0,\infty)$ operator, and under the Fourier-Plancherel transform) to a Toeplitz operator with symbol $\varphi\chi$ and therefore has essential spectrum $[0,1] \cup \{\varphi(x) \ x \text{ in } \mathbf{R}\}$ (see [4] for example). However, by Theorem 3 (iii)

$$\sigma_e(S) = \bigcup_{z \in Z} \sigma(S + I^z).$$

By simple arguments, and also by Lemma 2, one can show that if $z = (x, y)$ belongs to Z but $z \neq (0, \infty)$ then $\sigma(S + I^z)$ is contained in the range of φ. We must therefore have $\sigma(S + I^{(0,\infty)}) = [0, 1]$. Since $S + I^{(0,\infty)} = M_\chi D_\chi M_\chi + I^{(0,\infty)}$ we are done.

(ii) Similar.

3. THE ESSENTIAL SPECTRUM

Let φ_i, ψ_i, $1 \leqslant i \leqslant n$, belong to $\mathscr{PC}$ and determine the operator $\mathrm{Op}(a)$ as in (1). Then it follows from Lemma 3 (i) that, for a fixed x in $\mathbf{R}$, the local operator $\mathrm{Op}(a) + I^{(x,\infty)}$ is algebraically equivalent to

$$S = \sum_{i=1}^{n} (\varphi_i(x+)p + \varphi_i(x-)p^\perp)(\psi_i(+\infty)q + \psi_i(-\infty)q^\perp) =$$
$$= \alpha pq + \beta pq^\perp + \gamma p^\perp q + \delta p^\perp q^\perp$$

where α, β, γ and δ are as given in Definition 1. Thus we have

$$S = \begin{bmatrix} \beta + (\alpha - \beta)x & (\alpha - \beta)\sqrt{x(1-x)} \\ (\gamma - \delta)\sqrt{x(1-x)} & \gamma + (\delta - \gamma)x \end{bmatrix}.$$

and, with a little computation we find

$$\det(\lambda - S) = x(\lambda - \alpha)(\lambda - \delta) + (1 - x)(\lambda - \beta)(\lambda - \gamma).$$

LEMMA 4. *If $\alpha, \beta, \gamma, \delta$ are collinear and $\alpha - \beta = \gamma - \delta$, then the spectrum of S is $[a, b] \cup [c, d]$ where $\{a, b, c, d\}$ is the quadruple $\{\alpha, \beta, \gamma, \delta\}$ written in an order determined by their common line.*

Proof. Suppose first that $\alpha, \beta, \gamma, \delta$ are real, so that S is hermitian. The spectrum of S is then the set of real values λ for which $\det(\lambda - S) = 0$. However, this is precisely the real values of t where the polynomials $(t - \alpha)(t - \beta)$ and $(t - \gamma)\cdot$

$\cdot(t-\delta)$ either both vanish or have differing sign, and this is soon seen to be the set $[a, b] \cup [c, d]$. The general case now follows by a simple translation argument which involves choosing suitable complex numbers μ and η so that the local quadruple for $\mu + \eta a(x, y)$ is real.

LEMMA 5. *If $a(x, y)$ is a locally simple symbol then the essential spectrum of* $\mathrm{Op}(a)$ *is the set* $a^{\#}$.

Proof. By Theorem 3 (ii) the essential spectrum is the union of $\sigma(\mathrm{Op}(a) + I^z)$ for all z in Z. But Lemma 3 and Lemma 4 show that each of these sets is just the local segments of $a(x, y)$ at z, and so the lemma follows.

REMARKS. In general, and even if $\alpha, \beta, \gamma, \delta$ are colinear, the spectrum of S does not consist of line segments. The locally simple symbols correspond precisely to those $\mathrm{Op}(a)$ which are locally translates of multiples of hermitian operators. We leave open here a natural question: What is the analogue of Theorem 1 for the essentially normal $\mathrm{Op}(a)$?

The Lemmas 2,3 actually lead to much more than Theorem 1 because in principal, and in practice, one may analyze any operator which can be shown to belong to $\Psi(\mathscr{PC}, \mathscr{PC})$ by analyzing it locally. This is indicated in a subsequent paper [15].

4. PROOF OF THEOREM 1

The following lemma allows a reduction to the case where $a(x, y)$ is continuous on W and φ_i, ψ_i, $1 \leqslant i \leqslant n$, belong to $\mathscr{C}_\infty$. The index theorem for this case is well known (e.g., [2], [13]). Let L be the collection of locally simple functions.

LEMMA 6. *If a belongs to L and $a^{\#}$ does not pass through the origin then there exists a homotopy a_t, $0 \leqslant t \leqslant 1$, in L such that*

(i) $0 \notin a_t^{\#}$, $0 \leqslant t \leqslant 1$.

(ii) $a_0 = a$.

(iii) a_1 *is continuous on* W.

(iv) $a_1^{\#}$ *and* $a^{\#}$ *have the same winding number.*

Proof. It is clearer here to explain the proof rather than to be totally explicit with the technicalities. The main idea is to perform a homotopy which shrinks the local segments to points, so that the function a_1 has (using obvious notation) $\alpha_1 = \gamma_1 = (\alpha + \gamma)/2$ and $\beta_1 = \delta_1 = (\beta + \delta)/2$ for each quadruple $\alpha_1, \beta_1, \gamma_1, \delta_1$, and similarly for the y discontinuities. These conditions ensure that a_1 is continuous on W. In order to keep a_t in L, $0 \leqslant t \leqslant 1$, the segments $[\alpha, \gamma]$ and $[\beta, \delta]$ are simultaneously shrunk to their midpoints.

A slight difficulty occurs in this procedure if we have a discontinuity, at (x, ∞) say, of the type illustrated in Figure 3 and where 0 belongs to $[\alpha, \gamma]$. In this case we would get $0 \in a_t^\#$ for some t. However, by making a small translation of $a(x, y)$ we may assume that 0 does not lie on the convex hull of any of the local data.

It is simple enough to perform the homotopy in L. Consider first the case of a single pair of discontinuities of $a(x,y)$ at $(x,+\infty)$ and $(x, -\infty)$. The values of a_t need only differ from the values of $a(x,y)$ on the four sets $(x-\eta, x) \times (+\infty)$, $(x, x+\eta) \times (+\infty)$, $(x-\eta, x) \times (-\infty)$, $(x, x+\eta) \times (-\infty)$ with η small and fixed. A little induction with this case will give the lemma.

To complete the proof of Theorem 1 we now let $w(a^\#)$ denote the winding number of $a^\#$. It is known, as we have already noted, that the index of $\mathrm{Op}(a_1) = w(a_1^\#)$. Thus, since $\mathrm{Op}(a_t)$ is a homotopy of Fredholm operators we have

$$\mathrm{Index}\,\mathrm{Op}(a) = \mathrm{Index}\,\mathrm{Op}(a_1) = w(a_1^\#) = w(a^\#).$$

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