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CONTRACTION SEMIGROUPS AND THE SPECTRUM OF $A_1 \otimes I + I \otimes A_2$

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1. INTRODUCTION

In the following we will be mostly concerned with a strongly continuous semigroup of operators $\{P(t) : t \geqslant 0\}$ on a separable Hilbert space $\mathscr{H}$ satisfying

$$\|P(t)\| \leqslant e^{kt}. \tag{1.1}$$

We introduce the generator A of $P(t)$ by writing

$$P(t) = e^{-tA}.$$

We will analyze the relationship of the spectrum of the operator $P(t)$ to that of A and to the behavior of the resolvent $(z - A)^{-1}$ as a function of z.

The spectral mapping equation

$$\sigma(e^{-tA}) \setminus \{0\} = e^{-t\sigma(A)}; \quad t > 0 \tag{1.2}$$

is known to hold for a large class of A [3]. (Here $\sigma(B)$ is the spectrum of B and $e^{-t\sigma(A)} = \{e^{-tz} : z \in \sigma(A)\}$.) For example, if $P(t)$ is a holomorphic semigroup, (1.2) is known to be valid [3]. While the inclusion

$$\sigma(e^{-tA}) \supseteq e^{-t\sigma(A)} \tag{1.3}$$

is always true [3], (1.2) can fail in a dramatic way. In fact, Hille and Phillips [3] have constructed an operator A with e^{-tA} satisfying (1.1) such that $\sigma(A)$ is empty while e^{-tA} has circles in its spectrum.

In a recent paper [2], Gearhart has related the spectrum of an operator e^{-tA} satisfying our assumptions to the behavior of $(z - A)^{-1}$ when z is near infinity. Explicitly, Gearhart shows among other results [2]:

THEOREM 1.1. *Suppose* $\{e^{-tA} : t \geqslant 0\}$ *is a strongly continuous semigroup of operators on a separable Hilbert space* $\mathscr{H}$ *satisfying* (1.1). *Then* e^{-z_0} *is in the resolvent set of* e^{-A} *if and only if* $z_0 + 2\pi i n$ *is in the resolvent set of* A *for all integers* n *and*

$$\sup_{n \in \mathbf{Z}} \|(z_0 + 2\pi i n - A)^{-1}\| < \infty . \tag{1.4}$$

In the next section we make use of Gearhart's theorem to show that if (1.2) fails, it must fail rather dramatically.

If A_1 and A_2 are generators of holomorphic semigroups on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, $\{e^{-tA_1} \otimes e^{-tA_2} : t \geqslant 0\}$ is also a holomorphic semigroup on $\mathscr{H}_1 \otimes \mathscr{H}_2$. We denote its generator by $A = A_1 \otimes I + I \otimes A_2$. The spectral mapping formula (1.2) is correct [3] for generators of holomorphic semigroups so that using

$$\sigma(C \otimes D) = \sigma(C)\sigma(D) \tag{1.5}$$

which holds for arbitrary bounded operators C and D, we have for $t > 0$

$$e^{-t\sigma(A)} = e^{-t\sigma(A_1)} e^{-t\sigma(A_2)} . \tag{1.4}$$

This leads by a simple argument [6] to a theorem of Ichinose [5]:

THEOREM 1.2. *Suppose* A_1 *and* A_2 *generate holomorphic semigroups on Hilbert spaces* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *respectively. Then*

$$\sigma(A_1 \otimes I + I \otimes A_2) = \sigma(A_1) + \sigma(A_2). \tag{1.7}$$

If A_1 and A_2 generate semigroups satisfying (1.1) which are not holomorphic, then (1.7) can also fail in a dramatic way as we show in Section 3 (see Example 2). We give two extensions of Theorem 1.2 in Section 3, one of which is based on the results of Section 2. An application of one of these results (Theorem 3.1) is given in [4] where it is used to determine the spectra of certain operators which describe resonances of atoms and molecules in an external electric field.

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2. CONSEQUENCES OF THE FAILURE OF THE SPECTRAL MAPPING EQUATION (1.2)

Consider a strongly continuous semigroup of operators $\{e^{-tA} : t \geqslant 0\}$ on a separable Hilbert space satisfying (1.1). The generator A will be said to have property $\mathbb{P}$ if given any $E \in \mathbf{R}$ there is a $y_E > 0$ so that if

$$W_E = \{z : \operatorname{Re} z \leqslant E,\ |\operatorname{Im} z| \geqslant y_E\}$$

then

(i) $W_E \cap \sigma(A) = \emptyset$

and

(ii) $\sup_{z \in W_E} \|(z - A)^{-1}\| < \infty$.

One reason that property P is important is the following corollary of Gearhart's theorem (Theorem 1.1):

LEMMA 2.1. *Suppose* $\{e^{-tA} : t \geqslant 0\}$ *is as in Theorem* 1.1 *and A has property* P. *Then for* $t > 0$

$$\sigma(e^{-tA}) \setminus \{0\} = e^{-t\sigma(A)}.$$

Proof. Since $\sigma(e^{-tA}) \supseteq e^{-t\sigma(A)}$ is always true, we must only show that there is no z_0 with $e^{-tz_0} \in \sigma(e^{-tA}) \setminus e^{-\sigma(tA)}$. But this is a direct consequence of Gearhart's theorem. ▨

The main result of this section is the following theorem:

THEOREM 2.2. *Suppose* $\{e^{-tA} : t \geqslant 0\}$ *is as in Theorem* 1.1. *Suppose there is a* $z_0 \in \mathbf{C}$ *and a* $t_0 > 0$ *so that* $e^{-t_0 z_0} \in \sigma(e^{-t_0 A}) \setminus e^{-t_0 \sigma(A)}$. *Then we have*

$$\sigma(e^{-t(A-z_0)}) \supseteq \{\omega : |\omega| = 1\} \tag{2.1}$$

for almost all $t \in [0, \infty)$ *(with respect to Lebesgue measure).*

The condition that for no $z_0 \in \mathbf{C}$ *can* (2.1) *hold for almost all* $t \in [0, \infty)$ *is equivalent to the condition that A has property* P.

As will be seen from the proof of Theorem 2.2, there is another situation where (2.1) holds for almost all $t > 0$, namely if there is a sequence $x_n + iy_n + z_0 \in \sigma(A)$ with $x_n \to 0$, $|y_n| \to \infty$ and $x_n, y_n \in \mathbf{R}$. However, (2.1) can occur even if $\sigma(A)$ is empty as the example of Hille and Phillips shows [3].

Theorem 2.2 can be useful if one has apriori knowledge about the spectrum of e^{-tA}. It will be crucial in our proof of Theorem 3.1 in Section 3. Our proof of Theorem 2.2 will make use of Gearhart's theorem and the following proposition:

PROPOSITION 2.3. *Suppose* $B \subseteq \mathbf{R}$. *Then* $Q_t = \{e^{ity} : y \in B\}$ *is dense in the unit circle for almost all* $t \in \mathbf{R}$ *if and only if B is unbounded.*

Proposition 2.3 follows from the following ergodic type lemma:

LEMMA 2.4. *Suppose* $\{n_j\}_{j=1}^{\infty}$ *is a sequence of integers with* $n_j \neq n_k$ *if* $j \neq k$. *Then for almost all* $t \in \mathbf{R}$, *the set* $S_t = \{e^{2\pi i n_j t} : j \geqslant 1\}$ *is dense in the unit circle.*

Deduction of Proposition 2.3 *from Lemma* 2.4. If B is unbounded choose an unbounded sequence $\{y_j\}_{j=1}^{\infty}$ from B. By going to a subsequence, we can assume there is a sequence of integers $\{n_j\}_{j=1}^{\infty}$ with $n_j \neq n_k$ if $j \neq k$ such that for some y_0, $y_j - 2\pi n_j \to y_0$. This follows from the fact that a subsequence of $\{e^{iy_j}\}_{j=1}^{\infty}$ converges. Let $F = \{t \in \mathbf{R} : S_t$ is dense in the unit circle$\}$, where S_t is defined in Lemma 2.4. Then by Lemma 2.4, $\mathbf{R} \setminus F$ has measure 0. If $|\omega| = 1$ and $t \in F$ then there is a subsequence $\{\sigma(j)\}_{j=1}^{\infty}$ of the positive integers with $\lim_{j\to\infty} \sigma(j) = \infty$ such that

$$e^{2\pi i n_{\sigma(j)} t} \to \omega e^{-iy_0 t}.$$

Then $e^{ity_{\sigma(j)}} \to \omega$. Thus Q_t is dense in the unit circle for $t \in F$. The converse is trivial. ▨

Proof of Lemma 2.4. Let $C_1 = \{z \in \mathbf{C} : |z| = 1\}$ and $d\mu$ normalized Lebesgue measure on C_1. We consider for each $t \in [0,1]$, the operator

$$T(t)f(z) = f(ze^{2\pi i t})$$

defined on $L^2(C_1, d\mu)$. We define the averaging operators

$$V_N(t) = N^{-1} \sum_{j=1}^{N} (T(t))^{n_j}.$$

Note that $V_N(t)$ has an orthonormal basis of eigenvectors, namely the functions $\varphi_n(z) = z^n$:

$$V_N(t)\varphi_n = \lambda_{N,n}(t)\,\varphi_n \ ; \ n \in \mathbf{Z}$$

$$\lambda_{N,n}(t) = N^{-1} \sum_{j=1}^{N} \exp(2\pi i n_j n t).$$

Consider the function

$$f_N(t) = \frac{1}{2} \sum_{|n| \geqslant 1} |\lambda_{N,n}(t)|\, 2^{-|n|}.$$

We have

$$\|f_N\|_{L^2([0,1], dt)} \leqslant 1/\sqrt{N} \sum_{n=1}^{\infty} 2^{-n} = 1/\sqrt{N}$$

since by orthogonality $\|\lambda_{N,n}\|_{L^2([0,1], dt)} = 1/\sqrt{N}$ if $n \neq 0$. Thus there is a subsequence $\sigma(N)$ and a set $E \subseteq [0, 1]$ with measure 1 so that for all $t \in E$, $\lim_{N\to\infty} \lambda_{\sigma(N),n}(t) = 0$ if $n \neq 0$. Since $\lambda_{\sigma(N),0}(t) = 1$ and $\|V_{\sigma(N)}(t)\| \leqslant 1$, if P is the orthogonal projection onto the constant functions in $L^2(C_1, \mathrm{d}\mu)$ we have

$$V_{\sigma(N)}(t) \xrightarrow[N\to\infty]{s} P, \quad t \in E.$$

Choose a point $t \in E$ and suppose S_t is not dense in C_1. Then there is a non empty open set $O \subseteq C_1$ and a neighborhood $U \subseteq C_1$ of the point 1 so that for all $z \in U$ we have

$$(zS_t) \cap O = \emptyset.$$

Here $zS_t = \{zz' : z' \in S_t\}$. Define $f \in L^2(C_1, \mathrm{d}\mu)$ by

$$f(z) = \begin{cases} 1 & \text{if } z \in O \\ 0 & \text{otherwise}. \end{cases}$$

Then $f(ze^{2\pi i n_j t}) = 0$ if $z \in U$ and thus

$$\text{s-}\lim_{N\to\infty} V_{\sigma(N)}(t)f = g$$

is zero for almost all $z \in U$. However $g = Pf = \int f(w)\mathrm{d}\mu(w) > 0$. This shows that for $t \in E$, S_t is dense in C_1. The result now follows by a simple argument. ▨

Proof of Theorem 2.2. Suppose $\mathrm{e}^{-z_0} \in \sigma(\mathrm{e}^{-A}) \setminus \mathrm{e}^{-\sigma(A)}$. Then by Theorem 1.1 there is a sequence of integers $\{n_j\}_{j=1}^{\infty}$ with $n_j \neq n_k$ if $j \neq k$ such that

$$\lim_{j\to\infty} \|(z_0 + 2\pi i n_j - A)^{-1}\| = \infty . \tag{2.2}$$

Let $F = \{t > 0 : S_t \text{ is dense in the unit circle}\}$. Suppose $t \in F$ and $\omega \in C_1$. We want to show that $\omega^{-1} \in \sigma(\mathrm{e}^{-t(A-z_0)})$. Choose θ so that $\omega = \mathrm{e}^{i\theta}$, and a subsequence $\{m_j\}_{j=1}^{\infty}$ of $\{n_j\}_{j=1}^{\infty}$ so that $|m_j| \to \infty$ and $\mathrm{e}^{2\pi i m_j t} \to \omega$. Then there are integers l_j, $j = 1, 2, \ldots$ so that

$$|2\pi i m_j t - 2\pi i l_j - i\theta| \to 0 .$$

If for some j, $2\pi i l_j + i\theta \in \sigma(t(A - z_0))$, then by (1.3) $\omega^{-1} = \mathrm{e}^{-i\theta} \in \sigma(\mathrm{e}^{-t(A-z_0)})$. If for all j, $2\pi i l_j + i\theta \notin \sigma(t(A - z_0))$ while

$$\sup_j \|(t(A - z_0) - 2\pi i l_j - i\theta)^{-1}\| = M < \infty$$

then

$$t(A - z_0) - t(2\pi i m_j) = t(A - z_0) - 2\pi i l_j - i\theta + \delta_j =$$

$$= [t(A - z_0) - 2\pi i l_j - i\theta](1 + [t(A - z_0) - 2\pi i l_j - i\theta]^{-1}\delta_j)$$

so that if $|\delta_j| = |2\pi i l_j + i\theta - 2\pi i m_j t| < \frac{1}{2}$

$$\|(t(A - z_0) - t2\pi i m_j)^{-1}\| \leqslant 2M$$

and thus for large enough j, $\|(A - z_0 - 2\pi i m_j)^{-1}\| \leqslant 2Mt$. This contradicts (2.2) and thus by Theorem 1.1 $\omega^{-1} \in \sigma(e^{-t(A-z_0)})$. We have thus shown that $e^{-z_0} \in \sigma(e^{-A}) \setminus e^{-\sigma(A)}$ implies that for almost all $t > 0$

$$\sigma(e^{-t(A-z_0)}) \supseteq \{\omega : |\omega| = 1\}.$$

Similarly if $e^{-z_0 t_0} \in \sigma(e^{-t_0 A}) \setminus e^{-t_0 \sigma(A)}$ then again (2.1) holds for almost all $t > 0$.

Now suppose that given any $z_0 \in \mathbf{C}$, (2.1) cannot hold for almost all $t > 0$. Choose $E \in \mathbf{R}$ and suppose $z_n = x_n + iy_n \in \sigma(A)$ with $x_n \leqslant E$ and $|y_n| \to \infty$. By going to a subsequence we can assume $x_n \to x_0$ (since by (1.1), $x_n \geqslant -k$). Let

$$R_t = \{e^{-tx_n} e^{-ity_n} : n \geqslant 1\}$$

$$Q_t = \{e^{-ity_n} : n \geqslant 1\}$$

$$F = \{t > 0 : Q_t \text{ is dense in the unit circle}\}.$$

Then $[0, \infty) \setminus F$ has measure zero and if $t \in F$, then given any ω with $|\omega| = 1$, there is a sequence $\{e^{-ity_{n_j}}\}_{j=1}^{\infty}$ such that $n_j \to \infty$ and $e^{-ity_{n_j}} \to \omega$. Hence the closure of R_t contains $\{e^{-tx_0}\omega : |\omega| = 1\}$. Since $\sigma(e^{-tA})$ is closed it contains the closure of R_t and hence for all $t \in F$

$$\sigma(e^{-t(A-x_0)}) \supseteq \{\omega : |\omega| = 1\}.$$

This contradicts our assumption and shows there is a y_E so that

$$W_E \cap \sigma(A) = \emptyset.$$

If $z_n = x_n + iy_n \in W_E$ and

$$\lim_{n\to\infty} \|(z_n - A)^{-1}\| = \infty \tag{2.3}$$

then by the analyticity of $(z - A)^{-1}$ for $z \notin \sigma(A)$ we must have $|y_n| \to \infty$. Again going to a subsequence we can assume $x_n \to x_0$, and $y_j - 2\pi m_j \to y_0$ with

$|2\pi m_j + y_0| \geqslant y_E$. From (2.3) it is clear that

$$\sup_j \|(x_0 + iy_0 + 2\pi i m_j - A)^{-1}\| = \infty$$

so that there is a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\{m_j\}_{j=1}^{\infty}$ with $n_j \neq n_k$ if $j \neq k$ so that

$$\lim_{j\to\infty} \|(z_0 + 2\pi i n_j - A)^{-1}\| = \infty\,, \quad z_0 = x_0 + iy_0.$$

By the argument given after (2.2), this implies

$$\sigma(e^{-t(A-z_0)}) \supseteq \{\omega : |\omega| = 1\}$$

for almost all $t > 0$. This contradicts our assumption and thus we have shown

$$\sup_{z\in W_E} \|(z - A)^{-1}\| < \infty\,.$$

Conversely if for each E there is a W_E with the given properties then by Lemma 2.1, for each $t > 0$, $\sigma(e^{-tA})\setminus\{0\} = e^{-t\sigma(A)}$. In addition given any $z_0 \in \mathbf{C}$ the set $B = \{y \in \mathbf{R} : iy + z_0 \in \sigma(A)\}$ is bounded. We thus have

$$\sigma(e^{-t(A-z_0)}) \cap \{\omega : |\omega| = 1\} = e^{-t(\sigma(A)-z_0)} \cap \{\omega : |\omega| = 1\} = \{e^{-ity} : y \in B\}$$

which by Proposition 2.3 cannot be all of the unit circle for almost all $t > 0$. ▨

3. THE SPECTRUM OF $A_1 \otimes I + I \otimes A_2$

We give two extensions of Theorem 1.2. The first applies to the generators of semigroups satisfying (1.1), while the second is more abstract.

THEOREM 3.1. *Suppose* $\{e^{-tA_j} : t \geqslant 0\}$, $j = 1,2$ *are strongly continuous semigroups of operators on separable Hilbert spaces* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *respectively. Suppose* $\|e^{-tA_j}\| \leqslant e^{kt}$. *Suppose in addition that both* A_1 *and* A_2 *have property* P.

Let A *be the generator of* $e^{-tA_1} \otimes e^{-tA_2}$, *then* A *has property* P *and*

$$\sigma(A) = \sigma(A_1) + \sigma(A_2). \tag{3.1}$$

In our next theorem we no longer restrict ourselves to generators of semigroups but substitute other conditions.

THEOREM 3.2. *Let* A_1, A_2 *be closed, densely defined operators on Hilbert spaces* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *respectively. Define the operator* C *with domain* $\mathscr{D}(A_1) \otimes \mathscr{D}(A_2) \equiv \equiv \operatorname{span}\{u\otimes v : u \in \mathscr{D}(A_1),\ v \in \mathscr{D}(A_2)\}$ *by* $C(u \otimes v) = (A_1u) \otimes v + u \otimes (A_2v)$ *for* $u \in \mathscr{D}(A_1)$, $v \in \mathscr{D}(A_2)$. *Then* C *is closable. Denote its closure by* $A = A_1 \otimes I + I \otimes A_2$.

Suppose the resolvent sets of A_1, A_2, and A are not empty. Define

$$\mathscr{A}_j = \{(z - A_j)^{-1} : z \notin \sigma(A_j)\}''$$

where $''$ indicates the double commutant and $\mathscr{A}$ = the smallest norm closed algebra on $\mathscr{H}_1 \otimes \mathscr{H}_2$ containing operators of the form $a \otimes I$ and $I \otimes b$ with $a \in \mathscr{A}_1$, $b \in \mathscr{A}_2$.

Then if $(z_0 - A)^{-1} \in \bar{\mathscr{A}}$ for some $z_0 \notin \sigma(A)$

$$\sigma(A) = \sigma(A_1) + \sigma(A_2).$$

A similar theorem is proved in [8] with $\mathscr{A}_j$ replaced by the Banach algebra generated by $\{(z - A_j)^{-1} : z \notin \sigma(A_j)\}$. Our use of double-commutant algebras gives stronger results.

COROLLARY 3.3. *Suppose $\{e^{-tA_j} : t \geqslant 0\}$, $j = 1,2$ are strongly continuous semigroups of operators on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Let $e^{-tA} = e^{-tA_1} \otimes e^{-tA_2}$. Define $\mathscr{A}_j$ and $\bar{\mathscr{A}}$ as in Theorem 3.2.*

If there is a $z_0 \notin \sigma(A)$ such that $(z_0 - A)^{-1} \in \bar{\mathscr{A}}$ then

$$\sigma(A) = \sigma(A_1) + \sigma(A_2).$$

It would be interesting to elucidate the relationship of Corollary 3.3 to Theorem 3.1. In this connection see Proposition 3.7. It is not obvious for example that Theorem 3.1 does not follow from Corollary 3.3.

Proof of Corollary 3.3. We show that the hypotheses of Theorem 3.2 are satisfied. It is easy to see by differentiating $e^{-tA}(u \otimes v)$ that if $u \in \mathscr{D}(A_1)$ and $v \in \mathscr{D}(A_2)$ then $u \otimes v \in \mathscr{D}(A)$ and $A(u \otimes v) = (A_1 u) \otimes v + u \otimes (A_2 v)$. To see that $\mathscr{D} \equiv \mathscr{D}(A_1) \otimes \otimes \mathscr{D}(A_2)$ is a core for A first note that $e^{-tA_j} : \mathscr{D}(A_j) \to \mathscr{D}(A_j)$ so that $e^{-tA} : \mathscr{D} \to \mathscr{D}$. The latter condition is shown in [7], p. 241, to imply that $\mathscr{D}$ is a core for A. (Although the theorem in [7] is stated only for contraction semigroups, the proof is valid for arbitrary strongly continuous semigroups.)

Since the resolvent sets of A, A_1, and A_2 contain a half-plane, the proof is complete. ▨

Proof of Theorem 3.1. As noted earlier we have

$$\sigma(e^{-tA}) = \sigma(e^{-tA_1})\,\sigma(e^{-tA_2})$$

and thus by Lemma 2.1, if $t > 0$

$$\sigma(e^{-tA}) \setminus \{0\} = e^{-t\sigma(A_1)}\, e^{-t\sigma(A_2)}. \tag{3.2}$$

By assumption, A_1 and A_2 have property P. Suppose $e^{-t_0 z_0} \in \sigma(e^{-t_0 A}) \setminus e^{-t_0 \sigma(A)}$. Then choose $E = \mathrm{Re} z_0 + k$ and $y_E > 0$ so that with

$$W_E = \{z : \mathrm{Re} z \leqslant E,\ |\mathrm{Im} z| \geqslant y_E\}$$

we have

$$W_E \cap \sigma(A_j) = \emptyset$$

and

$$\sup_{z \in W_E} \|(z - A_j)^{-1}\| < \infty .$$

Because of (2.1) and (3.2), for almost every $t \in (0, \infty)$ and all $\theta \in \mathbf{R}$ there exist $z_j \in \sigma(A_j)$ with

$$e^{i\theta} e^{-tz_0} = e^{-t(z_1 + z_2)}. \tag{3.3}$$

Since $\mathrm{Re} z_0 = \mathrm{Re} z_1 + \mathrm{Re} z_2$ and $\mathrm{Re} z_j \geqslant -k$ we have $\mathrm{Re} z_i \leqslant E$ and thus since $z_j \in \sigma(A_j)$, $|\mathrm{Im} z_j| \leqslant y_E$. For suitable θ this contradicts (3.3) for all small $t > 0$. Thus for all $t > 0$

$$\sigma(e^{-tA}) \setminus \{0\} = e^{-t\sigma(A)} . \tag{3.4}$$

The formula (3.2) and the fact that A_1 and A_2 have property P imply that for no $z_0 \in \mathbf{C}$ can (2.1) hold for almost all $t > 0$. For if $e^{i\pi/2} \in \sigma(e^{-t_n(A - z_0)})$ for a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n > 0$, $\lim_{n\to\infty} t_n = 0$ then by (3.2) there exist $z_n' \in \sigma(A_1)$, $z_n'' \in \sigma(A_2)$ with $i = e^{-t_n(z_n' + z_n'' - z_0)}$. This means that $\mathrm{Re} z_n' + \mathrm{Re} z_n'' = \mathrm{Re} z_0$ so that $\mathrm{Re} z_n' \leqslant \text{const}$, $\mathrm{Re} z_n'' \leqslant \text{const}$ and hence because of property P, $|\mathrm{Im} z_n'| \leqslant \text{const}$, $|\mathrm{Im} z_n''| \leqslant \text{const}$. Thus $\lim_{n\to\infty} e^{-t_n(z_n' + z_n'' - z_0)} = 1$ which contradicts $i = e^{-t_n(z_n' + z_n'' - z_0)}$. Thus by Theorem 2.2, A has property P.

Combining (3.2) and (3.4) we have for all $t > 0$

$$e^{-t\sigma(A)} = e^{-t(\sigma(A_1) + \sigma(A_2))} . \tag{3.5}$$

We now use an argument of [6] to show that (3.5) implies (3.1). Suppose $\lambda \in \sigma(A)$. Then for each $t > 0$ there is a $\lambda_j(t) \in \sigma(A_j)$ and an integer $n(t)$ so that

$$\lambda = \lambda_1(t) + \lambda_2(t) + 2\pi i n(t) t^{-1} . \tag{3.6}$$

Since (3.6) implies $\mathrm{Re}\lambda_j(t) \leqslant \mathrm{Re}\lambda + k$, the fact that A_1 and A_2 have property P implies $|\mathrm{Im}\lambda_j(t)| \leqslant \text{const}$ and thus for all $t > 0$

$$|\mathrm{Im}\lambda| \geqslant |2\pi n(t) t^{-1}| - \text{const}$$

which implies $n(t) = 0$ for sufficiently small $t > 0$. Thus $\sigma(A) \subseteq \sigma(A_1) + \sigma(A_2)$.

A similar argument using the fact that A has property P gives the reverse inclusion. ▨

Before proving Theorem 3.2 we will need to say a few words about commutative Banach algebras $\mathscr{A}$ of operators on a Hilbert space $\mathscr{H}$. In what follows all Banach algebras will contain the identity. The set of all multiplicative linear functionals on $\mathscr{A}$, i.e., the Gel'fand spectrum of $\mathscr{A}$, will be denoted $\Delta_{\mathscr{A}}$.

Suppose $\mathscr{A}_j$, $j = 1,2$ are commutative Banach algebras of operators on the Hilbert spaces $\mathscr{H}_j$, $j = 1,2$. We define an algebra of operators on $\mathscr{H}_1 \otimes \mathscr{H}_2$ by

$$\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2 = \left\{ \sum_{i=1}^{N} a_i \otimes b_i \ : \ a_i \in \mathscr{A}_1, b_i \in \mathscr{A}_2, N < \infty \right\} \tag{3.7}$$

and denote its *norm* closure by $\bar{\mathscr{A}}$.

The following result will be crucial in our proof of Theorem 3.2. Results similar to it are proved in [10].

LEMMA 3.5. *Given $\ell_1 \in \Delta_{\mathscr{A}_1}$, there is a unique continuous map $\hat{\ell}_1 : \bar{\mathscr{A}} \to \mathscr{A}_2$ such that for $a_i \in \mathscr{A}_1$, $b_i \in \mathscr{A}_2$*

$$\hat{\ell}_1\left(\sum_{i=1}^{N} a_i \otimes b_i \right) = \sum_{i=1}^{N} \ell_1(a_i) b_i . \tag{3.8}$$

The map $\hat{\ell}_1$ is linear, multiplicative (i.e., *for each $a, b \in \bar{\mathscr{A}}$, $\hat{\ell}_1(ab) = \hat{\ell}_1(a)\, \hat{\ell}_1(b)$*) *and a contraction* ($\| \hat{\ell}_1(a) \| \leqslant \| a \|$).

Proof. We define $\hat{\ell}_1$ on $\mathscr{A}$ by (3.8). To show that $\hat{\ell}_1$ is well-defined on $\mathscr{A}$ and a contraction, it is enough to show

$$\left\| \sum_{i=1}^{N} \ell_1(a_i) b_i \right\| \leqslant \left\| \sum_{i=1}^{N} a_i \otimes b_i \right\| . \tag{3.9}$$

First note that for any $\varepsilon > 0$ we can find vectors u and v in $\mathscr{H}_2$ of norm 1 so that

$$\left\| \sum_{i=1}^{N} \ell_1(a_i) b_i \right\| \leqslant \left| \left(u, \sum_{i=1}^{N} \ell_1(a_i) b_i v \right) \right| + \varepsilon/2 . \tag{3.10}$$

But

$$\left(u, \sum_{i=1}^{N} \ell_1(a_i) b_i v \right) = \ell_1 \left(\sum_{i=1}^{N} (u, b_i v) a_i \right) \leqslant \left\| \sum_{i=1}^{N} (u, b_i v) a_i \right\|$$

since $|\ell_1(a)| \leqslant \|a\|$ for any $a \in \mathscr{A}_1$. Now there are vectors u' and v' in $\mathscr{H}_1$ of norm

1 so that

$$\left\| \sum_{i=1}^{N} (u, b_i v) a_i \right\| \leqslant \left(u', \sum_{i=1}^{N} (u, b_i v) a_i v' \right) + \varepsilon/2 =$$

(3.11)

$$= \left(u' \otimes u, \left(\sum_{i=1}^{N} a_i \otimes b_i \right) v' \otimes v \right) + \varepsilon/2 \leqslant \left\| \sum_{i=1}^{N} a_i \otimes b_i \right\| + \varepsilon/2 .$$

Thus,

$$\left\| \sum_{i=1}^{N} \ell_1(a_i) \otimes b_i \right\| \leqslant \left\| \sum_{i=1}^{N} a_i \otimes b_i \right\| + \varepsilon . \tag{3.12}$$

This proves (3.9). We extend $\hat{\ell}_1$ to $\bar{\mathscr{A}}$ by continuity. A short computation shows that $\hat{\ell}_1$ is multiplicative on $\mathscr{A}$ and thus by continuity on $\bar{\mathscr{A}}$. ▨

The following corollary of Lemma 3.5 is known. See [8, 10].

COROLLARY 3.6. *$\Delta_{\bar{\mathscr{A}}} = \Delta_{\mathscr{A}_1} \otimes \Delta_{\mathscr{A}_2}$ in the sense that given $\ell_j \in \Delta_{\mathscr{A}_j}$, $j = 1,2$ there is an $\ell \in \Delta_{\bar{\mathscr{A}}}$ such that for $a_i \in \mathscr{A}_1, b_i \in \mathscr{A}_2$*

$$\ell\left(\sum_{i=1}^{N} a_i \otimes b_i \right) = \sum_{i=1}^{N} \ell_1(a_i) \ell_2(b_i) \tag{3.13}$$

and conversely for any $\ell \in \Delta_{\bar{\mathscr{A}}}$ there are $\ell_j \in \Delta_{\mathscr{A}_j}$, $j = 1,2$ satisfying (3.13).

When the relation (3.13) is satisfied we write $\ell = \ell_1 \otimes \ell_2$.

Proof. Given $\ell_j \in \Delta_{\mathscr{A}_j}$, define $\ell = \ell_2 \circ \hat{\ell}_1$. By Lemma 3.5, $\ell \in \Delta_{\bar{\mathscr{A}}}$ and satisfies (3.13). Conversely given $\ell \in \Delta_{\bar{\mathscr{A}}}$ define $\ell_1(a) = \ell(a \otimes I)$ and $\ell_2(b) = \ell(I \otimes b)$ for $a \in \mathscr{A}_1$, $b \in \mathscr{A}_2$. ▨

Proof of Theorem 3.2. The closability of C and the inclusion $\sigma(A) \supseteq \sigma(A_1) + \sigma(A_2)$ are proved by Ichinose in [5]. For the sake of completeness we give a proof here. Some of the ideas go back to Brown and Pearcy [1].

The closability of C follows from the fact that C has a densely defined adjoint. In fact, as is easily seen, $\mathscr{D}(C^*) \supseteq \mathscr{D}(A_1^*) \otimes \mathscr{D}(A_2^*)$.

To prove the inclusion

$$\sigma(A) \supseteq \sigma(A_1) + \sigma(A_2) \tag{3.14}$$

we first introduce some notation. We denote the approximate point spectrum of a closed operator C by $\sigma_{ap}(C)$ and define $\sigma_r(C) = \sigma(C) \setminus \sigma_{ap}(C)$. We remind the reader that $\lambda \in \sigma_{ap}(C)$ if and only if there is a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of unit vectors in

$\mathscr{D}(C)$ with $\lim_{n\to\infty} \|(C-\lambda)\varphi_n\| = 0$. In addition we will use the fact that $\sigma_r(C)$ is an open subset of $\mathbf{C}$ and that if $\lambda \in \sigma_r(C)$ then $\bar{\lambda}$ is an eigenvalue of C^*.

Suppose $\mu_i \in \sigma(A_i)$ and $z_1 \notin \sigma(A_1)$. Define the set

$$K = \{t \geqslant 0 : \mu_1 + s(z_1 - \mu_1) \in \sigma(A_1) \text{ and } \mu_2 - s(z_1 - \mu_1) \in \sigma(A_2) \text{ for all } s \in [0, t]\}.$$

Note that K contains 0 and that since $z_1 \notin \sigma(A_1)$, $K \subseteq [0,1]$. Let $t_0 = \sup K$. Then because $\sigma(A_i)$ is closed, $\lambda_1 = \mu_1 + t_0(z_1 - \mu_1) \in \sigma(A_1)$, $\lambda_2 = \mu_2 - t_0(z_1 - \mu_1) \in \sigma(A_2)$. Note that $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$. It is clear from the construction that either $\lambda_1 \in$ boundary $(\sigma(A_1))$ or $\lambda_2 \in$ boundary $(\sigma(A_2))$. Without loss of generality we assume $\lambda_1 \in$ boundary $(\sigma(A_1))$. Since $\sigma(A_1^*) = \{\bar{z} : z \in \sigma(A_1)\}$, we also have $\bar{\lambda}_1 \in$ boundary $(\sigma(A_1^*))$. Since $\sigma_r(A_1)$ and $\sigma_r(A_1^*)$ are open sets we have $\lambda_1 \in \sigma_{ap}(A_1)$, $\bar{\lambda}_1 \in \sigma_{ap}(A_1^*)$. Thus there are sequences $\{u_n\}_{n=1}^\infty$, $\{u_n^*\}_{n=1}^\infty$ of unit vectors in $\mathscr{D}(A_1)$ and $\mathscr{D}(A_1^*)$ respectively with $\|(A_1 - \lambda_1)u_n\| \to 0$ and $\|(A_1^* - \bar{\lambda}_1)u_n^*\| \to 0$. There are two cases to consider. If $\lambda_2 \in \sigma_{ap}(A_2)$ then there is a sequence $\{v_n\}_{n=1}^\infty$ of unit vectors in $\mathscr{D}(A_2)$ with $\|(A_2 - \lambda_2)v_n\| \to 0$. Thus in this case,

$$\lim_{n\to\infty} \|(A - \lambda_1 - \lambda_2)u_n \otimes v_n\| \leqslant \lim_{n\to\infty} \|(A_1 - \lambda_1)u_n\| + \lim_{n\to\infty} \|(A_2 - \lambda_2)v_n\| = 0$$

and $\mu_1 + \mu_2 = \lambda_1 + \lambda_2 \in \sigma(A)$. If $\lambda_2 \in \sigma_r(A_2)$ then there is a unit vector $v \in \mathscr{D}(A_2^*)$ with $(A_2^* - \bar{\lambda}_2)v = 0$. Let $\varphi_n = u_n^* \otimes v$. Then for all $\psi \in \mathscr{D}(A_1) \otimes \mathscr{D}(A_2)$,

$$(\varphi_n, (A - \lambda_1 - \lambda_2)\psi) = ([(A_1^* - \bar{\lambda}_1)u_n^*] \otimes v, \psi).$$

Since $\mathscr{D}(A_1) \otimes \mathscr{D}(A_2)$ is a core for A this extends to all $\psi \in \mathscr{D}(A)$ and thus $\varphi_n \in \mathscr{D}(A^*)$ with

$$(A^* - \bar{\lambda}_1 - \bar{\lambda}_2)\varphi_n = [(A_1^* - \bar{\lambda}_1)u_n^*] \otimes v.$$

Thus

$$\lim_{n\to\infty} \|(A^* - \bar{\lambda}_1 - \bar{\lambda}_2)\varphi_n\| = \lim_{n\to\infty} \|(A_1^* - \bar{\lambda}_1)u_n^*\| = 0$$

so that $\bar{\lambda}_1 + \bar{\lambda}_2 \in \sigma(A^*)$. This again gives $\mu_1 + \mu_2 \in \sigma(A)$ and thus (3.14) follows.

To prove the reverse inclusion suppose z_0, z_1, and z_2 are in the resolvent sets of A, A_1, and A_2 respectively. Our analysis will be based on the following identity:

$$\begin{aligned}(z_1 - A_1)^{-1} &\otimes (z_2 - A_2)^{-1} = \\ &= (z_0 - z_1 - z_2)(z_0 - A)^{-1}\{(z_1 - A_1)^{-1} \otimes (z_2 - A_2)^{-1}\} + \\ &+ (z_0 - A)^{-1}\{(z_1 - A_1)^{-1} \otimes I + I \otimes (z_2 - A_2)^{-1}\}\end{aligned} \tag{3.15}$$

which can be seen by applying $z_0 - A$ to the vector $(z_1 - A_1)^{-1}u \otimes (z_2 - A_2)^{-1}v$, using the formula

$$A(u' \otimes v') = (A_1 u') \otimes v' + u' \otimes (A_2 v') \qquad \text{for } u' \in \mathscr{D}(A_1),\ v' \in \mathscr{D}(A_2),$$

and then multiplying by $(z_0 - A)^{-1}$. This gives (3.15) on vectors of the form $u \otimes v$. Since linear combinations of such vectors are dense in $\mathscr{H}_1 \otimes \mathscr{H}_2$ and both sides of (3.15) are bounded operators, (3.15) follows. Suppose $\lambda \in \sigma(A)$. Then by the spectral mapping theorem for resolvents $(z_0 - \lambda)^{-1} \in \sigma((z_0 - A)^{-1})$. Since $(A - z_0)^{-1} \in \bar{\mathscr{A}}$ there exists an $\ell \in \Delta_{\bar{\mathscr{A}}}$ with $\ell((z_0 - A)^{-1}) = (z_0 - \lambda)^{-1}$ for if $a \in \mathscr{A}$, $\{\ell(a) : \ell \in \Delta_{\bar{\mathscr{A}}}\} \supseteq \sigma(a)$. By Corollary 3.6, $\ell = \ell_1 \otimes \ell_2$ with $\ell_j \in \Delta_{\mathscr{A}_j}$. We claim that $\ell_1((z_1 - A_1)^{-1})\, \ell_2((z_2 - A_2)^{-1}) \neq 0$. If for example $\ell_1((z_1 - A_1)^{-1}) = 0$, applying $\hat{\ell}_1$ to (3.15) gives

$$\hat{\ell}_1((z_0 - A)^{-1})(z_2 - A_2)^{-1} = 0$$

which implies $\hat{\ell}_1((z_0 - A)^{-1}) = 0$. This implies $\ell_2 \circ \hat{\ell}_1((z_0 - A)^{-1}) = \ell((z_0 - A)^{-1}) = 0$ which contradicts $\ell((z_0 - A)^{-1}) = (z_0 - \lambda)^{-1}$. Similarly $\ell_2((z_2 - A_2)^{-1}) \neq 0$. Thus there are $\lambda_j \in \mathbf{C}$ with $\ell_j((z_j - A_j)^{-1}) = (z_j - \lambda_j)^{-1}$. Because of the structure of the double commutant algebras $\mathscr{A}_j$, if $a_j \in \mathscr{A}_j$ and $w_j \notin \sigma(a_j)$ then $(w_j - a_j)^{-1} \in \mathscr{A}_j$. Hence $\sigma(a_j) = \{\mu(a_j) : \mu \in \Delta_{\mathscr{A}_j}\}$. This implies $(z_j - \lambda_j)^{-1} \in \sigma((z_j - A_j)^{-1})$ and hence by the spectral mapping theorem for resolvents, $\lambda_j \in \sigma(A_j)$.

Applying $\ell = \ell_1 \otimes \ell_2$ to both sides of (3.15) results in

$$(z_1 - \lambda_1)^{-1}(z_2 - \lambda_2)^{-1} = (z_0 - z_1 - z_2)(z_0 - \lambda)^{-1}(z_1 - \lambda_1)^{-1}(z_2 - \lambda_2)^{-1} + (z_0 - \lambda)^{-1}\{(z_1 - \lambda_1)^{-1} + (z_2 - \lambda_2)^{-1}\}$$

and solving for λ we find $\lambda = \lambda_1 + \lambda_2$. Thus

$$\sigma(A) \subseteq \sigma(A_1) + \sigma(A_2). \tag{3.16}$$

The combination of (3.14) and (3.16) completes the proof. ▨

Theorem 3.2 extends Theorem 1.2 because if the strongly continuous semigroups $\{e^{-tA_j} : t \geqslant 0\}, j = 1,2$ are both norm continuous for $t > 0$ (as is the case for holomorphic semigroups) then for large enough $\lambda > 0$,

$$\int_0^\infty e^{-tA_1} \otimes e^{-tA_2}\, e^{-\lambda t} dt = (\lambda + A)^{-1}$$

is a norm convergent limit of sums of the form

$$\sum_{j=1}^{N} \alpha_j\, e^{-t_j A_1} \otimes e^{-t_j A_2}$$

and thus in $\mathscr{A}$. We give a further example to show how Theorem 3.2 may be applied:

PROPOSITION 3.7. *Suppose* $\{e^{-tA_j} : t \geqslant 0\}$ *is a strongly continuous semigroup on* $\mathscr{H}_j$, $j = 1,2$. *Suppose that* $\|e^{-tA_j}\| \leqslant c_j e^{kt}$ *with* $c_2 = 1$ *and*

$$\lim_{y\to\pm\infty} \|(A_1 + k + 1 + iy)^{-1}\| = 0. \tag{3.17}$$

Then defining A by $e^{-tA} = e^{-tA_1} \otimes e^{-tA_2}$ *we have*

$$\sigma(A) = \sigma(A_1) + \sigma(A_2).$$

Note that if e^{-tA_1} is norm continuous for $t > 0$ then (3.17) follows from the Riemann-Lebesgue lemma if we write $(A_1 + k + 1 + iy)^{-1}$ as the Fourier transform of $e^{-tA_1} e^{-t(k+1)}$.

Proof. We assume without loss that $\|e^{-tA_j}\| \leqslant c_j$ with $c_2 = 1$ and verify the hypotheses of Corollary 3.3. Note first that given strongly continuous semigroups $\{e^{-tA_j} : t \geqslant 0\}$ with $\|e^{-tA_j}\| \leqslant$ const we have (with no further assumptions)

$$(1 + A)^{-1}(I \otimes (\lambda A_2 + 1)^{-1}) \in \mathscr{A} \tag{3.18}$$

for all $\lambda > 0$. This can be seen as follows. Since if $\mu_j > 0$, $e^{-tA_j}(\mu_j A_j + 1)^{-1} \in \mathscr{A}_j$ and norm continuous in t for $t \geqslant 0$, we have

$$F(t) = e^{-tA}((\mu A_1 + 1)^{-1} \otimes (\lambda A_2 + 1)^{-1})e^{-t} \in \mathscr{A}$$

and norm continuous in t. Thus for $\mu, \lambda > 0$

$$\int_0^\infty F(t)\mathrm{d}t = (1 + A)^{-1}((\mu A_1 + 1)^{-1} \otimes (\lambda A_2 + 1)^{-1}) \in \mathscr{A}. \tag{3.19}$$

From (3.15) we see that for $\lambda > 0$

$$\tilde{C}_\lambda = \|(A_1 \otimes I)(1 + A)^{-1}(I \otimes (\lambda A_2 + 1)^{-1})\| < \infty$$

so that

$$\|(1 + A)^{-1}\{[(\mu A_1 + 1)^{-1} - 1] \otimes (\lambda A_2 + 1)^{-1}\}\| \leqslant \mu\,(\text{const})\,\tilde{C}_\lambda.$$

This gives (3.18). We now want to take λ to zero in (3.18). We will make use of

$$\lim_{y\to\pm\infty} \|(A_1 + 1 + iy)^{-1}\| = 0 \tag{3.20}$$

and the fact that $\{e^{-tA_2} : t \geqslant 0\}$ is a *contraction* semigroup to estimate

(3.21) $$\|(1 + A)^{-1}\{I \otimes [(\lambda A_2 + 1)^{-1} - 1]\}\| .$$

Let e^{-itH} be a unitary dilation [9] of e^{-tA_2} so that e^{-itH} is a unitary operator on the Hilbert space $\mathscr{H}_3 \supseteq \mathscr{H}_2$ and the orthogonal projection P of $\mathscr{H}_3$ onto $\mathscr{H}_2$ satisfies

(3.22) $$Pe^{-itH} P = e^{-tA_2} P ; \quad t \geqslant 0.$$

(3.21) can be written as

$$\left\| \int_0^\infty dt \int_0^\infty ds\, e^{-tA_1} \otimes \{e^{-tA_2}[e^{-s\lambda A_2} - 1]\} e^{-(t+s)} (I \otimes P) \right\| =$$

(3.23) $$= \left\| \int_0^\infty dt \int_0^\infty ds\, e^{-tA_1} \otimes \{Pe^{-itH}[e^{-s\lambda iH} - 1]P\} e^{-(t+s)} \right\| =$$

$$= \|(I \otimes P)(1 + \tilde{A})^{-1}(I \otimes [(i\lambda H + 1)^{-1} - 1])(I \otimes P)\| \leqslant$$

$$\leqslant \|((1 + \tilde{A})^{-1})I \otimes [(i\lambda H + 1)^{-1} - 1] \|$$

where $\tilde{A}$ is defined by $e^{-t\tilde{A}} = e^{-tA_1} \otimes e^{-itH}$. We claim that

(3.24) $$\|(1 + \tilde{A})^{-1} I \otimes [(i\lambda H + 1)^{-1} - 1]\| \leqslant \sup_{y \in \mathbf{R}} \{\|(1+A_1+iy)^{-1}\| \, |(i\lambda y+1)^{-1} - 1|\}.$$

By the spectral theorem for the self-adjoint operator H we can assume $\mathscr{H}_3 = L^2(M, d\mu)$ and that for $f \in \mathscr{H}_3$, $(e^{-itH}f)(x) = e^{-it\alpha(x)}f(x)$ for some real valued function α. Denote the right side of (3.24) by γ and let

$$B = (1 + \tilde{A})^{-1}\{I \otimes [(i\lambda H + 1)^{-1} - 1]\}.$$

To prove (3.24) it is enough to show that

(3.25) $$|(\psi, B\varphi)| \leqslant \gamma\|\varphi\| \, \|\psi\|$$

for φ and ψ in a dense set. Thus let $\varphi = \sum_{i=1}^N \varphi_i \otimes f_i$, $\psi = \sum_{i=1}^N \psi_i \otimes g_i$ and define $\varphi(x) = \sum_{i=1}^N f_i(x)\varphi_i$, $\psi(x) = \sum_{i=1}^N g_i(x)\psi_i$. Note that $\varphi(x)$ and $\psi(x)$ are in $\mathscr{H}_1$ for

μ-almost every x, and that

$$\int \|\varphi(x)\|^2 d\mu(x) = \|\varphi\|^2, \quad \int \|\psi(x)\|^2 d\mu(x) = \|\psi\|^2.$$

Let $\tilde{\varphi} = (I \otimes [(i\lambda H + 1)^{-1} - 1])\varphi$. We have $\tilde{\varphi} = \sum_{i=1}^{N} \varphi_i \otimes \tilde{f}_i$ with $\tilde{f}_i(x) = [(i\lambda\alpha(x) + 1)^{-1} - 1] f_i(x)$. We compute

$$(\psi, B\varphi) = (\psi, (1 + \tilde{A})^{-1}\tilde{\varphi}) = \int_0^\infty dt\, e^{-t} (\psi, e^{-tA_1} \otimes e^{-itH}\tilde{\varphi}) =$$

$$= \int_0^\infty dt \sum_{i,j} \int (\psi_i, e^{-t(A_1 + i\alpha(x)+1)}\varphi_j)\, \bar{g}_i(x)\tilde{f}_j(x) d\mu(x).$$

Since $\bar{g}_i f_j d\mu$ is a σ-finite measure we can use Fubini's theorem to write

$$(\psi, B\varphi) = \sum_{i,j} \int (\psi_i, (A_1 + i\alpha(x) + 1)^{-1}\varphi_j)\bar{g}_i(x)\tilde{f}_j(x) d\mu(x) =$$

(3.26)

$$= \int (\psi(x), (A_1 + i\alpha(x) + 1)^{-1} \varphi(x))[(i\lambda\alpha(x) + 1)^{-1} - 1] d\mu(x).$$

Thus using the Schwarz inequality and $\|(A_1 + i\alpha(x) + 1)^{-1}[(i\lambda\alpha(x) + 1)^{-1} - 1]\| \leq \gamma$, we have

$$|(\psi, B\varphi)| \leq \gamma \int \|\psi(x)\| \|\varphi(x)\| d\mu(x) \leq$$

$$\leq \gamma \left(\int \|\psi(x)\|^2 d\mu(x) \right)^{\frac{1}{2}} \left(\int \|\varphi(x)\|^2 d\mu(x) \right)^{\frac{1}{2}} = \gamma \|\psi\| \|\varphi\|$$

which is (3.25).

By (3.20) the right hand side of (3.24) has limit zero as $\lambda \to 0$. This shows $(1 + A)^{-1} \in \bar{\mathscr{A}}$ and the proof is complete. ▨

We now show by two examples how $\sigma(A) = \sigma(A_1) + \sigma(A_2)$ can fail.

EXAMPLE 1. Suppose H_1 and H_2 are self-adjoint operators on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Then $e^{-itH} = e^{-itH_1} \otimes e^{-itH_2}$ defines a self-adjoint operator H which has spectrum

$$\sigma(H) = \overline{\sigma(H_1) + \sigma(H_2)} \tag{3.27}$$

where here the bar denotes closure. (3.27) is an easy consequence of the spectral theorem. It is easy to manufacture H_1 and H_2 so that $\sigma(H_1) + \sigma(H_2)$ is not closed.

This is a natural way for $\sigma(A) = \sigma(A_1) + \sigma(A_2)$ to fail and it would thus be interesting to find general conditions which insure the validity of (3.27).

A more drastic failure is shown in the next example:

EXAMPLE 2. Suppose $\{e^{-tA_1} : t \geqslant 0\}$ is a strongly continuous contraction semigroup on a separable Hilbert space $\mathscr{H}_1$ such that $\sigma(A_1)$ is empty but there is a $z_0 \in \mathbf{C}$ such that $e^{-z_0} \in \sigma(e^{-A_1})$. See [3, p. 663] for the existence of such an operator. Let A_2 be multiplication by ix on $\mathscr{H}_2 = L^2(\mathbf{R}, dx)$. Then

$$\sigma(A_1) + \sigma(A_2) = \emptyset$$

but

$$\sigma(A_1 \otimes I + I \otimes A_2) \supseteq \{z_0 + ix : x \in \mathbf{R}\}\,. \tag{3.28}$$

To see (3.28) note that by Gearhart's theorem, there is a sequence of integers $\{n_j\}_{j=1}^{\infty}$ so that $\|(z_0 + 2\pi i n_j - A_1)^{-1}\| \to \infty$ and thus a sequence of unit vectors $\{\varphi_j\}_{j=1}^{\infty}$ in the domain of A_1 with $\|(z_0 + 2\pi i n_j - A_1)\varphi_j\| \to 0$. Choose $x_0 \in \mathbf{R}$ and a sequence $\{\psi_j\}_{j=1}^{\infty}$ of unit vectors in $\mathscr{H}_2$ so that $\|(A_2 - ix_0 + 2\pi i n_j)\psi_j\| \to 0$. If $f_j = \varphi_j \otimes \psi_j$ then it is easy to see that with $A = A_1 \otimes I + I \otimes A_2$,

$$\|(A - z_0 - ix_0)f_j\| \to 0$$

so that $z_0 + ix_0 \in \sigma(A)$.

In closing we want to remark on an extension of Theorem 1.2 proved by Ichinose [5]. Suppose A_j, $j = 1, 2$, is a closed operator in a Hilbert space $\mathscr{H}_j$ with $\sigma(A_j) \subseteq S_j \equiv \{z : |\arg z| \leqslant \theta_j\}$, $\theta_1 + \theta_2 < \pi$, and $\theta_j > 0$. Suppose in addition that $\sup\{\|z(z - A_j)^{-1}\| : z \notin S_j\} < \infty$. Define the operator $A = A_1 \otimes I + I \otimes A_2$ as in Theorem 3.2. Then Ichinose shows [5] that $\sigma(A) = \sigma(A_1) + \sigma(A_2)$.

For the reader's convenience we sketch a proof of this fact based on Theorem 3.2. We will show that A satisfies the assumptions of Theorem 3.2 by verifying that $-2 \notin \sigma(A)$ and that $(2 + A)^{-1} \in \mathscr{A}$. Let Γ be the contour $\{te^{-i\theta_1} : t \leqslant 0\} \cup \cup \{te^{i\theta_1} : t \geqslant 0\}$ traversed from $e^{-i\theta_1}(-\infty)$ to $e^{i\theta_1}(+\infty)$. It is easily seen that the operator

$$B = (2\pi i)^{-1} \int_{\Gamma} (-1 + z - A_1)^{-1} \otimes (-1 - z - A_2)^{-1} dz$$

is a norm convergent limit of sums of the form

$$\sum_{n=1}^{N} a_n(-1 + z_n - A_1)^{-1} \otimes (-1 - z_n - A_2)^{-1}$$

and thus in $\mathscr{A}$. In addition it is easy to verify that if $u_j \in \mathscr{D}(A_j)$ then $B(u_1 \otimes u_2) \in \mathscr{D}(A)$ and that

$$-B(A + 2)(u_1 \otimes u_2) = -(A + 2)\, B(u_1 \otimes u_2) = u_1 \otimes u_2 .$$

It follows from these relations that $-2 \notin \sigma(A)$ and $-(2 + A)^{-1} = B$. We leave the details of these arguments to the reader. Thus Theorem 3.2 implies $\sigma(A) = \sigma(A_1) + \sigma(A_2)$.

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