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ON TOEPLITZ OPERATORS WITH LOOPS. II

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INTRODUCTION

Let T_F denote the Toeplitz operator associated with a rational function $F(e^{it})$ of e^{it} with the poles of $F(z)$ lying off $\mathbf{T} = \{z \in \mathbf{C};\, |z| = 1\}$. Suppose that the bounded components of $\mathbf{C} \setminus F(\mathbf{T})$ are denoted $\mathscr{L}_i$, if the index of $T_F - \lambda I$, for $\lambda \in \mathscr{L}_i$, is negative; and ℓ_i if that index, for $\lambda \in \ell_i$ is positive. Label the index of $T_F - \lambda I$ as N_i for $\lambda \in \mathscr{L}_i$ and ν_i for $\lambda \in \ell_i$. The $\mathscr{L}_i$ and ℓ_i are called the *loops* of F.

In [3], the following similarity theorem for T_F is proved.

THEOREM 1. *Suppose F has the further properties:*

(I) *The intersection of the closures of any two loops consists of a finite number of points (called the multiple points of F).*

(II) *The boundary of each loop is an analytic curve except at the multiple points, where it is piecewise smooth, with inner angle $\theta \neq 0, \pi, 2\pi$. No distinct arcs of $\partial F(\mathbf{T})$ meet at angle $\theta = 0$.*

(III) *No multiple point of F is the image $F(z_0)$ of a point $z_0 \in \mathbf{T}$ where $F'(z_0) = 0$.*

(IV) *F never backs up. That is, if τ_j and T_j are the Riemann mapping functions from $|z| < 1$ to ℓ_j and $\overline{\mathscr{L}}_j$, respectively (a bar over a set denotes conjugate) then the arguments*

$$\arg \tau_j^{-1} F(e^{it}) \quad \text{and} \quad \arg T_j^{-1} \bar{F}(e^{it})$$

are monotone decreasing.

Then T_F is similar to

$$\sum{}^{\oplus} T_{\tau_j} \oplus \sum{}^{\oplus} T_{T_j}^{*} \quad \text{on} \quad \sum{}^{\oplus} H^2_{-\nu_j} \oplus \sum{}^{\oplus} H^2_{N_j} \tag{1}$$

where H^2_ν is the vector H^2 space, based on a Hilbert space of dimension ν.

The purpose of this paper is to improve upon Theorem 1 in such a way as to show that similarity properties of T_F depend upon more than the geometry of $F(\mathbf{T})$, the index of T_F and the "backing up" of F.

The improved version of Theorem 1 involves first the removal of the condition $\theta \neq \pi$, 2π, and the last sentence of (II). Specifically, (II) will be replaced by

(II′) *The boundary of each loop is an analytic curve except at the multiple points, where it is piecewise smooth, with inner angle* $\theta \neq 0$.

The generalization is accomplished by overcoming the need for "nontangential approach" of the $d_i(\tau(e^{it}))$, as proved for the case of Theorem 1 in [3, Lemma 1.2]. This same tangential approach was the main obstacle to generalization of (III), a modification of which we now give.

Let $\lambda_0 \in \partial F(\mathbf{T})$ be a multiple point of F, and let $z_0 \in \mathbf{T}$ be an inverse image of λ_0 under F, such that $F(z) - \lambda_0$ vanishes to order $\beta_0 \geqslant 1$ at z_0. There are β_0 solutions $z = d(\lambda)$ of $F(d(\lambda)) = \lambda$, with $d(\lambda) \to z_0$ as $\lambda \to \lambda_0$. Suppose p_0 of them approach z_0 from inside $|z| < 1$ as $\lambda \to \lambda_0$ along $\gamma \subset \partial F(\mathbf{T})$. The numbers β_0 and p_0 will be called, respectively, the β-order and p-order of the triple (λ_0, z_0, γ).

(III′) *For all multiple points* λ_0, *we require*

(a) *For any arc* γ *on* $\partial \mathscr{L}_i$, *terminating at* λ_0, *let* z_0 *be a preimage of* λ_0 *under* F *such that*

(2) *some solution* $d(\lambda)$ *of* $F(d(\lambda)) = \lambda$ *satisfies* $d(\lambda) \to z_0$ *and* $d(\lambda) \in \mathbf{T}$ *as* $\lambda \to \lambda_0$ *along* γ.

Let z_1 *be another preimage of* λ_0, *and let* (β_0, p_0), (β_1, p_1) *be the* β- *and p-orders of* (λ_0, z_0, γ) *and* (λ_0, z_1, γ), *respectively. Then one has the inequalities*

$$\frac{2p_1 - 1}{\beta_1} \leqslant \frac{2p_0 + 1}{\beta_0} \leqslant \frac{2p_1 + 1}{\beta_1} \tag{3}$$

with equality holding in the right inequality if z_1 *as well as* z_0 *satisfies* (2).

(b) *For an arc* γ *lying on* $\partial \overline{\mathscr{L}}_i$ *and terminating at* $\bar{z}_0$, *we require the condition of part* (a) *to hold with* $F(z)$ *replaced by* $f(z) = \bar{F}(\bar{z}^{-1})$.

Our improved version of Theorem 1 is

THEOREM 2. *Under conditions* (I), (II′), (III′) *and* (IV), T_F *is similar to* (1).

It will be shown that (III′) holds whenever all the β-orders of all triples (λ, z, γ) are odd, and therefore, (III′) is indeed a generalization of (III) (in which all β-orders are 1).

Because of the strengthening (II′) of (II), Theorem 2 includes the example of the $2n$-leaved rose ($F(z) = a(z^{n+1} - z^{-(n-1)})$, n even). The similarity result for T_F, stated in [3, Example 1] is therefore correct as stated (but is a consequence of Theorem 2 and not of Theorem 1).

After developing the machinery necessary to prove Theorem 2 (in Section 1 below), we also explore (in Section 2) converse techniques, which we use to construct an example of a rational function F and a rational, orientation preserving homeomorphism φ of $\mathbf{T}$ such that T_F and $T_{F\circ\varphi}$ are not similar. This example shows that similarity properties of T_F may depend upon more than $F(\mathbf{T})$, N_i, ν_i and the backing up of F.

I am grateful to Kevin Clancey, who showed me how to eliminate the "non-tangential approach" requirement of Theorem 1 and to Frank Lether, who, using high speed computing machinery, supplied me with the graph of the counterexample (figure 1).

1. L_Γ OPERATORS

In this section, we consider L_Γ operators in a setting somewhat more abstract than that of [2] or [3], and we obtain the additional estimates necessary for the proofs of Theorems 1 and 2.

Let $d_1(\lambda), \ldots, d_n(\lambda)$ be n algebraic functions of modulus $\leqslant 1$, for $\lambda \in \sigma$, some subset of the complex plane, and assume $|d_i(\lambda)| < 1$ except at finitely many points $\lambda_1, \ldots, \lambda_k$ of σ. Assume that any singularities of the d_i on σ lie among the λ_i. Let $\Gamma = \{g_i(\lambda)\}$ be a set of algebraic functions of modulus $= 1$ on σ, also having all their singularities on σ included among the λ_i.

Fix a point λ_0 in σ and an arc γ of σ, terminating at λ_0. By a *grouping* (around z_0) we will mean a set of d_i's and g_i's satisfying

$$d_i(\lambda_0) = g_j(\lambda_0) = z_0 \in \mathbf{T}.$$

In a grouping $\mathfrak{G}$, the g_i are called *principal terms* of $\mathfrak{G}$. If $\mathfrak{G}$ contains no g_i, but some d_i that tend to z_0 tangentially as $\lambda \to \lambda_0$ on γ, then those d_i are called *principal terms*. We assume that

1. each grouping contains at most one principal term;
2. for each grouping $\mathfrak{G}$, there is a positive integer β such that for all $d_i, g_i \in \mathfrak{G}$ and for $\lambda \in \gamma$, we have

$$|d_i(\lambda) - d_i(\lambda_0)| \sim c|\lambda - \lambda_0|^{1/\beta}, \qquad |d_i'(\lambda)| \leqslant c|\lambda - \lambda_0|^{1/\beta - 1}$$

$$|g_i(\lambda) - g_i(\lambda_0)| \sim c|\lambda - \lambda_0|^{1/\beta}, \qquad |g_i'(\lambda)| \leqslant c|\lambda - \lambda_0|^{1/\beta - 1}$$

where the symbol $\sim$ has its usual meaning:

$$h(\lambda) \sim k(\lambda) \quad \text{if} \quad \lim_{\lambda \to \lambda_0} h(\lambda)/k(\lambda) = 1.$$

For an example of groupings, in the context of this paper, the reader may turn to the examples in §4.

Let $\delta_1, \ldots, \delta_n$ be constants of modulus >1, let $\rho \in H^2$ and let $\tau(z)$ be a function analytic in $|z| < 1$ which extends to a 1-to-1 map of $\mathbf{T}$ into σ. Suppose τ' exists on $\mathbf{T}$, is continuous and nonzero, except possibly at $w_i = \tau^{-1}(\lambda_i)$ and

$$|\tau(z) - \lambda_i| \sim c|z - w_i|^{\alpha_i}, \quad |\tau'(z)| \leqslant |z - w_i|^{\alpha_i - 1}$$

in a neighborhood of the w_i. Define functions $c_1, \ldots, c_n, \xi_1, \ldots, \xi_m$ by

$$\frac{\prod(1-\delta_i^{-1}z)}{\prod(1-d_i(\lambda)z)\prod_{\Gamma}(1-g_i(\lambda)z)} = \sum c_i(\lambda)(1-d_i(\lambda)\,z)^{-1} + \sum_{\Gamma} \xi_i(\lambda)(1-g_i(\lambda)\,z)^{-1}.$$

The operator $L_\Gamma : H^2 \to L^2$ is defined for $|z| = 1$ by

$$(L_\Gamma x)(z) = \rho[\sum \bar{c}_i(\tau(z))\,x(\bar{d}_i(\tau(z))) + \sum_{\Gamma} \bar{\xi}_i(\tau(z))x(\bar{g}_i(\tau(z)))].$$

For a grouping $\mathfrak{G}$ containing, say, g_1 and $d_2, \ldots, d_n$, let

$$L_1 x = \rho\bar{\xi}_1(\tau(z))x(\bar{g}_1(\tau(z))) = \rho_1(z)x(\bar{g}_1(\tau(z)))$$

$$L_i x = \rho\bar{c}_i(\tau(z))x(\bar{d}_i(\tau(z))) = \rho_i(z)x(\bar{d}_i(\tau(z))).$$

LEMMA 1.1. (a). *For L_1 to be bounded in L^2 norm, it is necessary and sufficient that, for $w_j = \tau^{-1}(\lambda_j)$,*

(4) $$|\rho_i(e^{it})| = O((t - w_j)^{\frac{1}{2}(\alpha_j/\beta - 1)}) \text{ near } w_j, \; |\rho_i(e^{it})| = O(1) \text{ away from } w_j$$

hold with $i = 1$.

(b). *For L_i, $i > 1$, to be bounded in L^2 norm, it is sufficient that, for every $a \in \mathbf{T}$,*

(5) $$\int_{a+\frac{h}{2} \geqslant \arg \bar{d}_i(\tau(e^{it})) \geqslant a - \frac{h}{2}} |\rho_i(e^{it})|^2 \, dt = O(h)$$

with $O(h)$ independent of $0 \leqslant a \leqslant 2\pi$.

(c). *For L_i to be bounded $i \geqslant 1$, it is sufficient that* (4) *hold.*

(d). *For L_i, $i > 1$, to be compact, it is sufficient that*

$$\int_{a+\frac{h}{2} \geqslant \arg \bar{d}_i(\tau(e^{it})) \geqslant a - \frac{h}{2}} |\rho_i(e^{it})|^2 \, dt = o(h).$$

(e). *For L_Γ to be bounded, it is sufficient that for each grouping $\mathfrak{G} = \{d_2, \ldots, d_{p+1}, \Gamma \cap \mathfrak{G}\}$, where $\lambda_0 = \tau(e^{iw_0})$ and β are related to $\mathfrak{G}$ through property 2. of groupings, ρ satisfies*

(6) $$|\rho(e^{it})| = O((t - w_0)^{-\frac{1}{2}+[2(p+q)-1]\alpha_0/(2\beta)})$$

(where $q = |\Gamma \cap \mathfrak{G}|$, the number of elements in $\Gamma \cap \mathfrak{G}$).

Proof. (a). Define a variable θ by $e^{i\theta} = \bar{g}_1(\tau(e^{it}))$, and let $e^{it} = \varphi(e^{i\theta})$. We have

$$\|\rho_1 x(\bar{g}_1(\tau(e^{it})))\|^2 = \int |\rho_1|^2 |x(\bar{g}_1(\tau(e^{it})))|^2 \, dt = \int |\rho_1(\varphi(e^{i\theta}))|^2 \, |x(e^{i\theta})|^2 \, |\varphi'(e^{i\theta})| \, d\theta.$$

Since $|\varphi'(e^{i\theta})| \sim c|\theta - \theta_0|^{\beta/\alpha_0 - 1}$ in a neighborhood of $e^{i\theta_0} = \bar{g}_1(\lambda_0)$, part (a) follows.

(b). We apply Carleson's Lemma [1, Theorem 1] with the measure μ given by

$$\int x(z) \, d\mu(z) = \int |\rho_i|^2 x(\bar{d}_i(\tau(e^{it}))) \, dt.$$

Carleson's Lemma states that the (pointwise) identity operator from H^2 to $L^2(d\mu)$ is bounded if and only if

(7) $$\mu(S) \leqslant Ch$$

for all "Carleson Rectangles" $S = \left\{z : |z| > 1 - h, \; a - \frac{h}{2} \leqslant \arg z \leqslant a + \frac{h}{2}\right\}$.

Thus if (7) holds, we can conclude that

$$C\|x\|^2 \geqslant \int |x(z)|^2 \, d\mu(z) = \int |\rho_i(e^{it})|^2 |x(\bar{d}_i(\tau(e^{it})))|^2 \, dt = \|L_i x\|^2,$$

which will prove (b). But the measure of a given S is

(8) $$\mu(S) = \int |\rho_i|^2 \chi_S(\bar{d}_i(\tau(e^{it}))) \, dt \leqslant \int\limits_{a - \frac{h}{2} \leqslant \arg \bar{d}_i(\tau(e^{it})) \leqslant a + \frac{h}{2}} |\rho_i|^2 \, dt$$

proving (b).

(c). We can assume the $d_i(\tau(e^{it}))$ do not converge radially to $d_i(\lambda_0)$ (in the radial case, the results of [2] apply). Since $\arg \bar{d}_i(\tau(\varphi(z)))$ has a nonzero derivative near $z_0 = \varphi^{-1}(w_0)$, and since

$$|\varphi(z) - \varphi(z_0)| \sim c|z - z_0|^{\beta/\alpha_0}, \qquad |\varphi'(z)| \sim c|z - z_0|^{\beta/\alpha_0 - 1},$$

$\mu(S)$ in (b) becomes

$$\mu(S) \leqslant \int\limits_{a-\frac{h}{2} \leqslant \arg \bar{d}_i(\tau(\varphi)) \leqslant a+\frac{h}{2}} |\rho_i(\varphi(e^{it}))|^2 \, d\varphi \leqslant$$

$$\leqslant \int\limits_{a-ch/2}^{a+ch/2} |\rho_i(\varphi(e^{it}))|^2 \, d\varphi(e^{it}) = \int\limits_{a-ch/2}^{a+ch/2} |\rho_i(\varphi(e^{it}))|^2 \varphi'(e^{it}) \, dt = O(h)$$

proving (c) for $i > 1$. Of course for $i = 1$, (c) follows from (a).

(d). Carleson's Lemma, as used in (b), also states that if $\mu(S) \leqslant Ah$, then $\|L_i\| \leqslant CA$, for a universal constant C. Thus if we write $L_i = L_i' + L_i''$, where

$$L_i' x = \rho_i x(\bar{d}_i(\tau))\chi_{(1-\varepsilon,1)}$$

$$L_i'' = L_i - L_i'$$

we have, in case (d), $\|L_i'\| \to 0$ as $\varepsilon \to 0$ and L_i'' is compact, as can easily be seen by computing the Hilbert-Schmidt norm.

(e). By [2, (3.2)—(3.4)], L_Γ is bounded whenever each L_i, $i \geqslant 1$ is bounded, hence by (c), whenever (4) holds for $i \geqslant 1$. Since $\rho_1 = \rho\bar{\xi}_1$ and $\rho_i = \rho c_i$, $i > 1$, and since, by [2, Lemma 2.1],

$$|c_i| \leqslant c|t - w_0|^{-(p+q-1)\alpha_0/\beta}$$

$$|\xi_i| \leqslant c|t - w_0|^{-(p+q-1)\alpha_0/\beta}$$

we have

$$|\rho_i(e^{it})| \leqslant c|\rho||t - w_0|^{-(p+q-1)\alpha_0/\beta} = O(|t - w_0|^{\frac{1}{2}\left(\frac{\alpha_0}{\beta}-1\right)})$$

if the hypotheses if (e) hold. This proves Lemma 1.1.

2. CONVERSE TECHNIQUES

The next two lemmas are proved in preparation for Lemma 2.3, our second main lemma.

LEMMA 2.1. *Let $\gamma_\pm$ denote the curves*

$$\gamma_+(\eta) = \{1 - \eta^{-1}(\log\eta)^{-\alpha}\} e^{i\{\theta + (\log\eta)^{-\beta}\}}$$

$$\gamma_-(\eta) = \{1 - \eta^{-1}(\log\eta)^{-\alpha}\} e^{i\{\theta - (\log\eta)^{-\beta}\}}$$

parametrized by $1 \leqslant \eta < \infty$, *where* $\beta + 1 > \alpha > 1$. *Let* $B(z)$ *denote the infinite Blaschke product with zeros* $\{\gamma_+(1), \gamma_+(2), \ldots; \gamma_-(1), \gamma_-(2), \ldots\}$. *Then for z near* $e^{i\theta}$, *z between* γ_+ *and* γ_-, $B(z)$ *satisfies*

$$|B(z)| \leqslant c|e^{i\theta} - z|^{\varepsilon} \tag{9}$$

for every $\varepsilon < 1$.

Proof. The prescribed sequence satisfies the Blaschke condition, as proved by Somadasa [4]. Further, if $k \leqslant \eta \leqslant k + 1$, we can verify that

$$|[\gamma_+(\eta) - \gamma_+(k)]/[1 - \bar{\gamma}_+(k)\gamma_+(\eta)]| \leqslant c|e^{i\theta} - \gamma_+(\eta)|^{\varepsilon}.$$

Therefore (9) holds for $z \in \gamma_+ \cup \gamma_-$.

Now connect γ_+ and γ_- at some points interior to $|z| < 1$ and let Ω denote the interior of the closed curve so formed. We have $B_1(z) = B(z)(e^{i\theta} - z)^{-\varepsilon} \in H^1(\Omega)$ and, by (9), $B_1 \in L^{\infty}(\gamma_+ \cup \gamma_-)$. Therefore $B_1(z) \in H^{\infty}(\Omega)$ and the lemma follows.

LEMMA 2.2. *If* $\mathfrak{G}$ *is a grouping with principal term* $D(=g_1$ *or* $d_i)$ *at* λ_0 *and if* $N > 0$, *there is an inner function* $B(z)$ *such that*

$$|B(D)| \geqslant c > 0$$

for the principal term D,

$$|B(d_i(\tau(e^{it})))| \leqslant c|t - w_0|^N \tag{10}$$

for the non principal terms d_i, *in a neighborhood of* $z = e^{iw_0}$.

Proof. Assume $z_0 = d(\tau(e^{iw_0})) = 1$. In case $\mathfrak{G} = \{d_2, \ldots, d_{p+1}\}$ (with d_2 approaching $\mathbf{T}$ tangentially at λ_0) note that (since λ_0 is an *algebraic* singularity of the d_i)

$$|d_i(\lambda) - 1| \geqslant c|\lambda - \lambda_0|^{1/\beta}, \quad i = 2, \ldots, p + 1$$

and, since $d_2 \to \mathbf{T}$ tangentially,

$$1 - |d_2(\lambda)| \leqslant c|\lambda - \lambda_0|^{2/\beta}.$$

Therefore,

$$\frac{1 - |d_2(\tau(e^{it}))|}{|d_2(\tau(e^{it})) - 1|} \leqslant c$$

and so if $B(z) = \exp\left[-\dfrac{1+z}{1-z}\right]$, then

$$|B(d_2(\tau(e^{it})))| = \exp -\left[\frac{1 - |d_2(\tau(e^{it}))|}{|d_2(\tau(e^{it})) - 1|}\right] \geqslant c > 0$$

and $d_3, \ldots, d_p$ approach 1 nontangentially, so (10) holds for any N.

In case $\mathfrak{G} = \{g_1, d_2, \ldots, d_{p+1}\}$, if all the $d_i(\tau(e^{it})) \to 1$ nontangentially as $t \to w_0$, then the same $B(z)$ as above works.

If some d_i (say d_2) tends tangentially to $\mathbf{T}$, we observe that since the curve $z = d_2(\tau(e^{it}))$ lies inside some curve of the form

$$\gamma = \{z : 1 - |z| \leqslant |1 - z|^{\alpha}\}$$

it is sufficient to take Somadasa's Example 2 [4, p. 299], as described in Lemma 2.1.

The next lemma gives our "converse techniques" to Lemma 1.1.

LEMMA 2.3. *Assume Γ contains exactly one g_i, which maps σ one-to-one onto* $\mathbf{T}$, *and suppose each grouping contains a principal term.*

(a). *If the principal term of a grouping with $\lambda_0 = \tau(e^{iw_j})$ is d_2, then L_2 bounded implies* (5) *holds.*

(b). *For L_Γ to be bounded (resp. bounded below), it is necessary that* (6) *hold at each w_j (resp. necessary that*

$$|\rho| \geqslant c|t - w_i|^{-\frac{1}{2}+[2(p+q)-1]\alpha_i/(2\beta_i)}).$$

(c). *For L_Γ to be bounded, it is necessary that L_i, corresponding to the principal term in each grouping, be bounded.*

To prove (a), take $a = \arg z_0$ in (5). Since $d_2 \to d_2(\lambda_i)$ tangentially,

$$1 - |d_2(\lambda)| \leqslant |z_0 - d_2(\lambda)| \leqslant c|\arg d_2(\lambda) - \arg z_0|$$

if $\lambda = \tau(e^{it})$ is close to λ_0. Therefore in (5), we can conclude

$$\arg z_0 - h/2 \leqslant \arg d_2(\tau(e^{it})) \leqslant \arg z_0 + h/2 \Rightarrow |d_2(\tau(e^{it}))| \geqslant 1 - h,$$

so that the inequality in (8) may be replaced by equality.

To prove part (b), pick an inner function B_{w_0} for each point e^{iw_0} where some $d_i(\tau(e^{it})) \to 1$, so that

$$|B_{w_0}(d_i(\tau(e^{it})))| \leqslant c|t - w_0|^{[\frac{1}{2}+(p+q-1)]\alpha_0/\beta_i}$$

near $t = w_0$. Let $B = \prod B_{w_0}$. Applying L_Γ to an H^2 function of the form Bx, we get for L_i,

$$L_i Bx = c_i \rho B(\bar{d}_i(\tau(e^{it})))x(d_i(\tau(e^{it}))).$$

Regarding this operator as acting on x, with $\rho_i = c_i \rho B(\bar{d}_i)$, and examining the integral in (5), we see that

$$\int\limits_{a+\frac{h}{2} \geq \arg \bar{d}_i(\tau(e^{it})) \geq a - \frac{h}{2}} |\rho_i|^2 dt = \int |c_i|^2 |B(\bar{d}_i(\tau(e^{it})))\rho|^2 \, dt \leqslant$$

$$\leqslant c \int |t - w_0|^{\alpha_0/\beta} |\rho|^2 \, dt \leqslant ch \int\limits_{a+\frac{h}{2} > \arg \bar{d}_i(\tau(e^{it})) > a - \frac{h}{2}} |\rho|^2 \, dt,$$

since $a - h/2 < \arg \bar{d}_i(\tau(e^{it})) < a + h/2$ implies $|t - w_0|^{\alpha_0/\beta} < h$. The last integral tends to 0 since $\rho = L_\Gamma 1 \in L^2$. Therefore, each L_i, restricted to BH^2, is compact (since the operator $Bx \to x$ from BH^2 to H^2 is an isometry) and $L_1|_{BH^2}$ is bounded (resp. essentially bounded below). But

$$\|L_1 Bx\|^2 = \|\rho_1 B(g_1) x(g_1)\|^2 = \|\rho_1 x(g_1)\|^2 =$$

$$= \int |\rho_1(\varphi(e^{it}))|^2 |x|^2 \varphi'(e^{it}) dt$$

(by the proof of Lemma 1.1 (a)). Therefore, $L_1|_{BH^2}$ is bounded if and only if $|\rho_1(e^{it})| \leqslant c|t - w_0|^{\frac{1}{2}(\alpha_0/\beta - 1)}$ (and is essentially bounded below if and only if it is bounded below, in which case $|\rho_1(e^{it})| \geqslant c|t - w_0|^{\frac{1}{2}(\alpha_0/\beta - 1)}$). Since $\rho_1 = \xi_1 \rho$, part (b) follows.

To prove part (c), we proceed as in part (b). By the result of part (b), L_Γ bounded implies L_1 bounded and, by Lemma 1.1(b), each L_i, for the grouping containing g_1, is bounded. Now pick a different grouping $\mathfrak{G}$ and proceed to pick $B(z)$ as in Lemma 2.2 so that

$$|B(\bar{d}_i(\tau(e^{it})))| \leqslant |t - w_0|^{\left[\frac{1}{2} + (p+q-1)\right] \alpha_0/\beta}$$

for every d_i (in every grouping) except the principal term d_2 of $\mathfrak{G}$. By choice of B, $L_i|_{BH^2}$ is bounded for every d_i in $\mathfrak{G}$, $i > 2$. Since $L_\Gamma|_{BH^2}$ is bounded, we have

$$\|x\| = \|Bx\| \geqslant c\|L_2 Bx\| = c\|\rho_2 B(\bar{d}_2(\tau(e^{it}))) x(\bar{d}_2(\tau(e^{it})))\| \geqslant c' \|\rho x(\bar{d}_2(\tau(e^{it})))\|$$

which implies L_2 is bounded (on all of H^2).

For our applications of Lemmas 1.1 and 2.3, we will actually need to consider L_Γ operators from H^2 to $L^2(|\tau'| dt)$, the latter space having the norm

$$\|x\|^2 = \int_0^{2\pi} |x(e^{it})|^2 |\tau'(e^{it})| \, dt.$$

Evidently, the same lemmas can be applied to L_Γ^* with ρ replaced by $|\tau'|^{\frac{1}{2}}\rho$ to yield results about this situation. For example, (5) is to be replaced by

$$\int\limits_{a+\frac{h}{2}\geq \arg \bar{d}_i(\tau(e^{it}))\geq a-\frac{h}{2}} |\rho_i(e^{it})|^2 |t-w_j|^{\alpha_j-1}\,dt = O(h), \tag{11}$$

and (6) by

$$|\rho(e^{it})| = O(t-w_0)^{-\frac{\alpha_0}{2}+[2(p+q)-1]\alpha_0/(2\beta)}. \tag{12}$$

3. PROOF OF THEOREM 2

This proof is identical to that of Theorem 1 (as given in [3, §2]) except for modifications in the definition and boundedness proofs for the similarity operators, and we shall just indicate those modifications.

Write

$$f(z)-\lambda =$$

$$= a(\lambda)\prod(1-d_i(\lambda)z)\prod(1-e_i(\lambda)z)\prod(1-g_i(\lambda)z) \Big/ \prod_{i=1}^{m}(z-\gamma_i)\prod_{i=1}^{n}(z-\delta_i)$$

where $|d_i| < 1 = |g_i| < |e_i|$ and $|\gamma_i| < 1 < |\delta_i|$. On a branch γ of some $\partial\ell_j$ (an arc on $\partial\ell_j$ terminating at a multiple point λ_0) suppose the d_i and g_i are divided into groupings as in §1. For any grouping let β_0 be the order of vanishing $F(z)-\lambda_0$ at $z=z_0$, and let p_0 be the number of d_i in the grouping. For $x\in H^2$, $\lambda\in\ell_i$, define

$$(L_{0f}x)(\lambda) = \rho(\lambda)\sum_{k\leq v_i}\rho_k(\lambda)(x, h_\lambda^{(k)})u_k$$

where $u_1, \ldots, u_v$ are $v=\max\{-v_i\}$ vectors in some auxiliary Hilbert space, $h_\lambda^{(k)}$ $(\lambda\in\ell_i)$ is the eigenvector of T_f defined by

$$h_\lambda^{(k)}(z) = \prod_{\substack{j\leq v_i\\ j\neq k}}(z-z_j)\prod(1-\delta_i^{-1}z)/\prod(1-d_i(\lambda)z),$$

where $z_1, \ldots, z_v$ are fixed (distinct) complex numbers of modulus <1, $\rho_k(\lambda) = \overline{h}_\lambda^{(k)}(z_k)^{-1}$ and ρ is to be defined. If τ_j is the Riemann map from $|z|<1$ into one of the ℓ_j, we must choose ρ so that $x\to(L_{0f}x)(\tau_j(z))$ is bounded in L^2 norm, for each j. Let $\rho(\tau_j(z))\in H^2$ be an outer function with modulus

$$|\rho(\tau_j(e^{it}))| = |t-w_0|^{-\alpha_0/2+[2p_0+1]\alpha_0/(2\beta_0)} \tag{13}$$

for $e^{it} \in \tau_j^{-1}(\gamma)$ ($|\rho| = 1$ at points $e^{it} \notin \tau_j^{-1}(\gamma)$ for any γ), where the γ's are neighborhoods of the multiple points $\{\lambda_i\}$ on the $\partial \ell_i$.

To prove $x \to (L_{0f}x)(\tau_j)$ is bounded in L^2 norm, we have only to check (6) (i.e. (12)) in a neighborhood of τ_j^{-1} of a multiple point (the methods of [2] apply at all other points). That is, if $\mathfrak{G} = \{d_2, \ldots, d_p, \Gamma \cap \mathfrak{G}\}$ is any grouping at λ_0 with β satisfying property 2. (in which case β must be the β-order of (λ_0, z_0, γ) by [2, Lemma 1.3]), and if $\pi\alpha_0$ is the inner angle of ℓ_i at λ_0, (so that $\tau_i(\lambda) - \tau_i(\lambda_0) \sim c|\lambda - \lambda_0|^{\alpha_0}$; Warschawski [6]) we must check that

$$- \alpha_0/2 + [2(p+q) - 1]\alpha_0/(2\beta) \leqslant - \alpha_0/2 + (2p_0 + 1)\alpha_0/(2\beta_0).$$

But this is equivalent to

$$[2(p+q) - 1]/\beta \leqslant (2p_0 + 1)/\beta_0$$

which is the first inequality in (3) if $q = 0$ and (3) with equality replacing the second inequality if $q = 1$.

Now write

$$F(z) - \lambda =$$

$$= A(\lambda) \prod (1 - D_i(\lambda)z) \prod (1 - E_i(\lambda)z) \prod (1 - G_i(\lambda)z)/[\prod (z - \Gamma_i) \prod (z - \Delta_i)]$$

where $|D_i| < 1 = |G_i| < |E_i|$ and $|\Gamma_i| < 1 < |\Delta_i|$. We want to prove that for each (j, m), the operator $L_{j,m}$ defined for polynomials $p(\lambda)$ by

$$L_{j,m}p = \int A^{-1} \frac{\prod (z - \Delta_i) \prod (1 - \Gamma_i \bar{z}_m)(1 - \bar{z}_m z)^{-1} p(\tau_j) d\tau_j}{\prod (1 - D_i(\tau_j)z) \prod (1 - G_i(\tau_j)z) \prod (z_m - E_i(\tau_j))}$$

is bounded in L^2 norm. As in [3], it suffices to prove boundedness of the L_Γ operator with $\tau(z) = \tau_j(z)$, $\Gamma = \{G_i\}$ and $\rho(e^{it})$ the reciprocal of the $\rho(e^{it})$ chosen above. By Lemma 1.1(e) (i.e. (12)) and (13), we must verify that

$$- \alpha_0/2 + [2(P + Q) - 1]\alpha_0/(2\beta) \leqslant \alpha_0/2 - (2p_0 + 1)\alpha_0/(2\beta_0).$$

But note that $D_i(\lambda) = \bar{e}_i(\bar{\lambda})^{-1}$ and $G_i(\lambda) = g_i(\bar{\lambda})$ [2; §5] so that (number of i such that $D_i(\lambda_0) = \bar{z}_0) + p + q = \beta$. Thus we must prove

$$\alpha_0/2 - (2p_0 + 1)\alpha_0/(2\beta_0) \geqslant - \alpha_0/2 + [2(\beta - p) - 1]\alpha_0/(2\beta) =$$

$$= \alpha_0/2 - (2p + 1)\alpha_0/(2\beta),$$

or

$$(2p_0 + 1)/\beta_0 \leqslant (2p + 1)/\beta$$

which is the second inequality in (3). The proof of Theorem 2 now follows the lines of that of Theorem 1 [3].

REMARK. If all β_i are odd, we must have $2p_0 + 1 = \beta_0$ in (III') so the middle term in (3) is $\equiv 1$. Inequality (3) follows.

4. THE EXAMPLE

From now on, we confine ourselves to the case

$$F_0(z) = (1 + z^2)^2/z.$$

First we will analyze T_F with $F = F_0$, and then with $F = F_0 \circ \varphi$, for a certain rational, orientation preserving homeomorphism φ of $\mathbf{T}$.

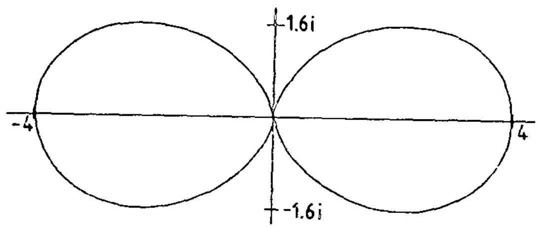

Fig. 1.

a. *The function* F_0. This function maps $\mathbf{T}$ to a "figure 8" as shown in figure 1, with both loops oriented positively. Indeed $\nu_1 = \nu_2 = -1$. The only multiple point of F_0 is $\lambda = 0$; there are two preimages, $z_1 = \mathrm{i}$ and $z_2 = -\mathrm{i}$; F_0 has a zero of order $\beta_1 = \beta_2 = 2$ at both points.

Let γ be an arc lying on one of the four branches of $F(\mathbf{T})$ with endpoint at 0, and consider the functions $\{d_i, g_i\}$ satisfying $F_0(\overline{d}_i(\lambda)) = F_0(\bar{g}_i(\lambda)) = \overline{\lambda}$. There are two groupings $\mathfrak{G}_1$ and $\mathfrak{G}_2$ corresponding to γ (around the two preimages of $\lambda = 0$). Both contain at most two elements (since F_0 is 4-to-1 and vanishes to order 2 at $z = z_1, z_2$) and one, $\mathfrak{G}_1$ say, contains g_1. We claim that they both contain 1 element; i.e. that $|\mathfrak{G}_1| = |\mathfrak{G}_2| = 1$. We prove

i. $|\mathfrak{G}_1| + |\mathfrak{G}_2| \leqslant 2$

ii. $|\mathfrak{G}_2| \geqslant 1$

Since we already know $|\mathfrak{G}_1| \geqslant 1$, this will prove the claim.

To prove i., note that for every $\lambda \in \ell_1 \cup \ell_2$, $F_0(z) = \lambda$ has at most two solutions in $|z| < 1$. (Indeed, as t increases from 0 to 2π, $F_0(\mathrm{e}^{\mathrm{i}t})$ travels around $\partial\ell_1 \cup \partial\ell_2$ counter clockwise, exactly once; since F_0 has a pole at 0, the argument principle applies.) Now to see i., just note that if λ moves from γ inside $\ell_1 \cup \ell_2$, $g_1(\lambda)$ becomes a $d_i(\lambda)$ [2, Lemma 1.3].

To prove ii., let $\mathfrak{D} = \{z : |z| < 1 \text{ and } F_0(z) \in \ell_1 \cup \ell_2\}$. A simple computation shows for z imaginary:

(14) $F_0(z)$ is imaginary, $F_0(z) \downarrow 0$ as $z \downarrow -\mathrm{i}$ and $F_0(z) \uparrow 0$ as $z \uparrow \mathrm{i}$.

(↑ and ↓ refer to going up or down the imaginary axis.) It follows from (14) that $\pm$i belong to the closure of $\mathfrak{D}$ and that the following four arcs: the two arcs of $\partial\mathfrak{D}$ and the two arcs of **T** terminating at i [resp $-$i] are mapped to the four arcs of $F_0(\mathbf{T})$ terminating at 0. ($\partial\mathfrak{D}$ cannot intersect **T** in more than isolated points, by [2, Lemma 1.3].) This proves ii. .

Therefore, for the q_j, p_j and β_j corresponding to $\mathfrak{G}_j$ and the arc γ, we have

$$q_1 = 1, \quad p_1 = 0, \quad \beta_1 = 2$$

$$q_2 = 0, \quad p_2 = 1, \quad \beta_2 = 2.$$

As a result, the inequalities (3) hold and T_{F_0} is similar to $T_{\tau_1} \oplus T_{\tau_2}$, where τ_1 and τ_2 are the Riemann maps from $|z| < 1$ into $\mathcal{C}_1$ and $\mathcal{C}_2$.

b. *The function* $F_0 \circ \varphi$. Here

$$\varphi(z) = z^2(z + 3i)/(3iz - 1).$$

Since φ is the quotient of two finite Blaschke products φ maps **T** to **T**. Furthermore $\varphi(i) = i$, $\varphi(-i) = -i$ and $\varphi'(z) = 0$ on **T** only for $z = -i$, where $\varphi(z) + i = 0$ to order 3. Thus φ never backs up (or φ' would vanish at both end points of an arc where φ backs up) and so φ is an orientation preserving homeomorphism of **T**. $F = F_0 \circ \varphi$ maps **T** to $F_0(\mathbf{T})$ and $F'(z) = 0$ at $z = \pm i$, with $F(z) - i = 0$ to order $\beta_1 = 2$ and $F(z) + i = 0$ to order $\beta_2 = 6$.

Let γ be an arc of $F_0(\mathbf{T})$ terminating at 0, which is the image under F of an arc of **T** terminating at $z_1 = i$. Since φ maps $|z| < 1$ inside $|z| < 1$ and $|z| > 1$ outside $|z| < 1$ in a neighborhood of i, we have

$$\beta_1 = 2, \ q_1 = 1, \ p_1 = 0$$

(as in case a.).

In a neighborhood of $z_2 = -i$, φ is:

2-to-1 from $|z| < 1$ to $|z| < 1$

1-to-1 from $|z| < 1$ to $|z| > 1$

2-to-1 from $|z| > 1$ to $|z| > 1$

1-to-1 from $|z| > 1$ to $|z| < 1$.

Therefore γ has, under $F_0 \circ \varphi$, 3 preimages in $|z| < 1$ and 3 preimages in $|z| > 1$ terminating at $-$i,

$$\beta_2 = 6, \ q_2 = 0, \ p_2 = 3.$$

Finally, we need to note that the preimage of γ under F_0 in $|z| < 1$ terminating at $-\mathrm{i}$ meets $\mathbf{T}$ tangentially. This is true since γ meets one of the arcs from

(15) $$F_0(\{e^{it} \mid 3\pi/4 - \varepsilon < t < 3\pi/4\}), \quad F_0(\{e^{it} \mid 3\pi/4 < t < 3\pi/4 + \varepsilon\})$$

at angle π and so F_0^{-1} (which looks like $\lambda^{\frac{1}{2}} - \mathrm{i}$ near $\lambda = 0$) must map γ and the arc from (15) into arcs meeting at angle 0. Therefore one of the three arcs in $|z| < 1$ that $F_0 \circ \varphi$ sends to γ must meet $\mathbf{T}$ tangentially (and only one since the six preimages of γ meet at equal angles at $-\mathrm{i}$).

Now suppose $T_{F_0 \circ \varphi}$ is similar to $T_{\tau_1} \oplus T_{\tau_2}$:

$$LT_{F_0 \circ \varphi} = [T_{\tau_1} \oplus T_{\tau_2}]L.$$

As L^* must map eigenvectors of $T^*_{\tau_1} \oplus T^*_{\tau_2}$ to eigenvectors of $T^*_{F_0 \circ \varphi}$, and as the eigenvector k_λ for $T^*_{\tau_j}$ with eigenvalue $\bar{\tau}_j(\lambda)$ is the reproducing kernel for the jth component of Lx ($j = 1, 2$), we have

(16) $$(Lx)(\tau_j(z)) = (Lx, k_{\bar{\tau}_j(z)}) = (x, L^* k_{\bar{\tau}_j(z)}) = (x, \rho(z) h_{\bar{\tau}_j(z)}),$$

where h_λ are the eigenvectors of $T_{F_0 \circ \varphi}$ with $h_\lambda(0) = 1$. By (16),

$$(Lx)(\lambda) = \rho(\tau_j^{-1}(\lambda))[\sum \bar{c}_i(\lambda) x(\bar{d}_i(\lambda)) + \sum \bar{\xi}_i(\lambda) x(\bar{g}_i(\lambda))]$$

where

$$h_\lambda(z) = \sum c_i(1 - d_i z)^{-1} + \sum \xi_i(1 - g_i z)^{-1}.$$

L maps H^2 into the Hilbert space of analytic functions in $\ell_1 \cup \ell_2$ with norm

$$\|x\|^2 = \sum_{j=1}^{2} \int |x(\tau_j(e^{it}))|^2 |\tau_j'(e^{it})| \mathrm{d}t.$$

Similarity implies L is bounded and invertible. By Lemma 2.3(b), using the arc γ above and $\alpha_0 = 1, (\beta, p, q) = (\beta_1, p_1, q_1) = (2, 0, 1)$, this implies

$$|\rho(e^{it})| \geqslant c|t|^{-\frac{1}{4}}$$

(where we are assuming γ is an arc of ℓ_1 and $\tau_1(1) = 0$). Now Lemma 2.3(c) implies that the principal term in the grouping around $-\mathrm{i}$ must be bounded. By Lemma 2.3(a), this means

(17) $$\int_{a + \frac{h}{2} > \arg \bar{d}_1(\bar{\tau}_1(e^{it})) \geqslant a - \frac{h}{2}} |\rho_1(e^{it})|^2 \mathrm{d}t = O(h).$$

Since

$$|\rho_1(e^{it})| = |c_1(\bar{\tau}_j(e^{it}))\tau_j'(e^{it})\rho| \geqslant c|t|^{-(p_2+q_2-1)\alpha_0/\beta_2-1/4} = c|t|^{-1/3-1/4}$$

($|\tau_j'(e^{it})|$ is bounded below by [5, Th.IX.8]), the left side of [17] is bounded below by

$$c \int_{a+\frac{h}{2} \geqslant \arg \bar{d}_1(\bar{\tau}_1(e^{it})) \geqslant a-\frac{h}{2}} |t|^{-7/6} dt \geqslant c \int_{a-\frac{h}{2}}^{a+\frac{h}{2}} |t|^{(-7/6)6+5} dt = c \int_{a-\frac{h}{2}}^{a+\frac{h}{2}} |t|^{-2} dt$$

(by the change of variable used with the proof of Lemma 1.1(c)), which cannot be $O(h)$. This contradicts the supposed similarity of $T_{F_0 \circ \varphi}$ and $T_{\tau_1} \oplus T_{\tau_2}$.

REMARK. It is not difficult to show, using the methods of Theorem 2, that $T_{F_0 \circ \varphi}$ is quasi-similar to $T_{\tau_1} \oplus T_{\tau_2}$. Therefore, the conclusions of Corollaries 1 and 2 of [3] are valid for $T_{F_0 \circ \varphi}$.

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