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ON DERIVATION RANGES AND THE IDENTITY OPERATOR

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1. INTRODUCTION

Let $\mathscr{L}(\mathscr{H})$ be the algebra of all (bounded linear) operators acting on the complex separable infinite dimensional Hilbert space $\mathscr{H}$. Given $A \in \mathscr{L}(\mathscr{H})$, let δ_A be the inner derivation induced by A (defined by $\delta_A(X) = AX - XA, X \in \mathscr{L}(\mathscr{H})$) and let $\operatorname{ran}\delta_A = \delta_A[\mathscr{L}(\mathscr{H})]$ be the range of δ_A .

In [1], Joel H. Anderson proved that $\mathscr{L}(\mathscr{H})$ contains a C^*-algebra $C^*(A)$ (generated by A and the identity operator 1) such that its intersection with

$$\mathrm{JA}(\mathscr{H}) = \{B \in \mathscr{L}(\mathscr{H}) : 1 \in (\operatorname{ran}\delta_B)^-\}$$

(the upper bar denotes norm-closure) is a G_δ-dense subset of $C^*(A)$ (in particular, $\mathrm{JA}(\mathscr{H}) \neq \emptyset$).

Furthermore, Anderson's proof strongly suggests that every operator of the form $T \otimes 1 \in \mathscr{L}(\mathscr{H} \otimes \mathscr{H})$ (such an operator will be called *an ampliation*) is the (norm) limit of a sequence of operators in $\mathrm{JA}(\mathscr{H} \otimes \mathscr{H})$. It will be shown here that this is indeed the case:

If $\mathscr{T}(\mathscr{H}) = \{A \in \mathscr{L}(\mathscr{H}) : A$ is unitarily equivalent to an ampliation$\}$, then $\mathscr{T}(\mathscr{H}) \cap \mathrm{JA}(\mathscr{H})$ is a G_δ-dense subset of $\mathscr{T}(\mathscr{H})$.

Let $\sigma_e(T)$ denote the *essential spectrum* of $T \in \mathscr{L}(\mathscr{H})$, i.e., the spectrum of the canonical projection $\pi(T)$ of T in the quotient Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, where $\mathscr{K}(\mathscr{H})$ denotes the ideal of all compact operators; then $\sigma_0(T) = \{\lambda \in \sigma(T)\setminus\sigma_e(T) : \lambda$ is an isolated point of $\sigma(T)\}$ is the set of all *normal eigenvalues* of T, $\sigma_B(T)=\sigma(T)\setminus\sigma_0(T)$ is the *Browder spectrum* of T and $\mathscr{L}(\mathscr{H})$ can be written as the disjoint union of

$$\mathscr{L}(\mathscr{H})_0 = \{T \in \mathscr{L}(\mathscr{H}) : \sigma_0(T) \neq \emptyset\}$$

(which is open and dense in $\mathscr{L}(\mathscr{H})$ [15]) and

$$\mathscr{L}(\mathscr{H})_B = \{T \in \mathscr{L}(\mathscr{H}) : \sigma_0(T) = \emptyset\} = \{T \in \mathscr{L}(\mathscr{H}) : \sigma(T) = \sigma_B(T)\}$$

(which, of course, is closed and nowhere dense).

It follows from [16, Lemma 2] that $\mathrm{JA}(\mathscr{H}) \subset \mathscr{L}(\mathscr{H})_{\mathrm{B}}$, so that $\mathrm{JA}(\mathscr{H})$ is nowhere dense in $\mathscr{L}(\mathscr{H})$. Furthermore, it is well-known that, in a certain sense, 1 "tends to be far from $\operatorname{ran}\delta_T$" for any T in $\mathscr{L}(\mathscr{H})$ (see [9, Chapter 19]), so that $\mathrm{JA}(\mathscr{H})$ is a "very small" subset of $\mathscr{L}(\mathscr{H})$ in many senses.

A straightforward matrix computation shows that if $A = B \oplus C$, then

$$\operatorname{dist}[1, \operatorname{ran}\delta_A] = \max\{\operatorname{dist}[1, \operatorname{ran}\delta_B], \operatorname{dist}[1, \operatorname{ran}\delta_C]\}. \tag{1}$$

Thus, if $\|A - A_n\| \to 0$ $(n \to \infty)$, $A_n = B_n \oplus C_n$ and $\operatorname{dist}[1, \operatorname{ran}\delta_{B_n}] \geqslant \eta$ for all $n = 1, 2, \ldots$, then $\operatorname{dist}[1, \operatorname{ran}\delta_A] \geqslant \eta$. In [24], J. P. Williams introduced and analyzed the class $\mathscr{F}(\mathscr{H}) = \{T \in \mathscr{L}(\mathscr{H}) : \operatorname{dist}[1, \operatorname{ran}\delta_T] = 1\}$ of *finite operators*. (This class includes Halmos' quasitriangular operators [8], as well as those operators studied by C. Pearcy and N. Salinas in [19]. It is completely apparent that if B_n is a finite operator for all $n = 1, 2, \ldots$, then A is finite.) It will be shown that if $A \in \mathrm{JA}(\mathscr{H})$ and $\rho : C^*(A) \to \mathscr{L}(\mathscr{H})$ is a (not necessarily faithful!) unital $*$-representation of $C^*(A)$ and $T = \rho(A)$, then $T \in \mathrm{JA}(\mathscr{H}_\rho)$, so that T cannot be a finite operator. It is easily seen that $\mathrm{JA}(\mathscr{H})$ is invariant under similarities, so that the same result applies to every operator similar to A, and $T = \rho(A)$ cannot be even similar to an operator in $\mathscr{F}(\mathscr{H}_\rho)$.

It has been shown in [24] that $\mathscr{F}(\mathscr{H})$ contains every operator $T \simeq A \oplus B$ ($\simeq$ denotes unitary equivalence and $\oplus$ denotes orthogonal direct sum) such that A acts on a non-zero finite dimensional space, whence it readily follows that

$$\mathscr{L}(\mathscr{H})_0 \subset \mathscr{S}\mathscr{F}(\mathscr{H}) = (\mathrm{def})\,\{WTW^{-1} : T \in \mathscr{F}(\mathscr{H}) \text{ and } W \text{ is invertible}\}.$$

Recall that T is *quasidiagonal (quasitriangular)* if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank orthogonal projections such that $P_n \to 1$ strongly as $n \to \infty$ and $\|TP_n - P_nT\| \to 0$ ($\|(1 - P_n)TP_n\| \to 0$, resp.). T is biquasitriangular if both, T and T^* (T^* denotes the adjoint of T), are quasitriangular. Let (BQT) denote the class of all biquasitriangular operator and let

$$(\mathrm{SQD}) = \{WTW^{-1} : T \text{ is quasidiagonal and } W \text{ is invertible}\}.$$

It is well-known that (SQD) $\subset$ (BQT) [8] and (BQT) $= \{T \in \mathscr{L}(\mathscr{H}) : \text{if } \lambda - T \text{ is semi-Fredholm, then } \operatorname{ind}(\lambda - T) = 0\}$ [4]. (The reader is referred to [17] for definition and properties of the semi-Fredholm operators.)

In fact, (SQD) is dense in (BQT) (observe that (SQD) contains every operator similar to a normal operator) and $(\mathrm{SQD})_{\mathrm{B}} = (\mathrm{SQD}) \cap \mathscr{L}(\mathscr{H})_{\mathrm{B}}$ is dense in $(\mathrm{BQT})_{\mathrm{B}} = (\mathrm{BQT}) \cap \mathscr{L}(\mathscr{H})_{\mathrm{B}}$ [21] and (as remarked above) disjoint from $\mathrm{JA}(\mathscr{H})$. In spite of these facts, $\mathrm{JA}(\mathscr{H})$ contains a large family of biquasitriangular operators. Indeed, $(\mathrm{BQT}) \cap \mathrm{JA}(\mathscr{H})$ is a G_δ-dense subset of $(\mathrm{BQT})_{\mathrm{B}}$ (contained in $(\mathrm{BQT})_{\mathrm{B}} \setminus \mathscr{S}\mathscr{F}(\mathscr{H})$).

It easily follows from our previous observations that $(\mathrm{SQD}) \cap \mathrm{JA}(\mathscr{H}) = \mathscr{S}\mathscr{F}(\mathscr{H}) \cap \mathrm{JA}(\mathscr{H}) = \emptyset$.

CONJECTURE 1.1. *$\mathcal{S}\mathcal{J}(\mathcal{H}) = \mathcal{L}(\mathcal{H})\setminus \mathrm{JA}(\mathcal{H})$, i.e. $A \in \mathrm{JA}(\mathcal{H})$ if and only if $\mathrm{dist}[1, \mathrm{ran}\delta_B] < 1$ for all B similar to A.*

In [12], the author exhibited a concrete example of a biquasitriangular operator T such that its *similarity orbit* $\mathcal{S}(T) = \{WTW^{-1} : W \text{ is invertible}\}$ does not intersect (QD). (Equivalently, $T \notin$ (SQD). It is not hard to infer from [12] that $C^*(B)$ admits a representation ρ in $\mathcal{L}(\mathbf{C}^3)$ for some B similar to T. Hence, $T \notin \mathrm{JA}(\mathcal{H})$.)

By using the previous result, it is possible to show that (BQT) is the disjoint union of (SQD), $[(\mathrm{BQT}) \cap \mathrm{JA}(\mathcal{H})] + \mathcal{K}(\mathcal{H})$ (which is a G_δ-dense subset of (BQT)) and $\mathcal{N} = (\mathrm{BQT})\setminus\{(\mathrm{SQD}) \cup ([(\mathrm{BQT}) \cap \mathrm{JA}(\mathcal{H})] + \mathcal{K}(\mathcal{H}))\}$, which is also dense in (BQT).

Let $\{T\}' = \{B \in \mathcal{L}(\mathcal{H}) : BT = TB\}$ be the commutant of $T \in \mathcal{L}(\mathcal{H})$ and let

$$\mathcal{A}(\mathcal{H}) = \cup \{\{T\}' \cap (\mathrm{ran}\delta_T)^- : T \in \mathcal{L}(\mathcal{H})\}$$

and

$$\mathcal{J}(\mathcal{H}) = \cup \{\{T\}' \cap (\mathrm{ran}\delta_T) : T \in \mathcal{L}(\mathcal{H})\}.$$

Since $\mathrm{JA}(\mathcal{H})$ is clearly contained in $\mathcal{A}(\mathcal{H})$, the above results disprove the author's conjecture that $\mathcal{A}(\mathcal{H})^-$ was contained in (BQT) [14], (observe that if $S \otimes 1$ is a shift of infinite multiplicity, then $S \otimes 1 \in \mathrm{JA}(\mathcal{H})^- \setminus (\mathrm{BQT})$) and provide some extra information about $\mathcal{A}(\mathcal{H})^-$.

By Kleinecke-Shirokov theorem [9], $\mathcal{J}(\mathcal{H}) \subset Q(\mathcal{H}) = \{Q \in \mathcal{L}(\mathcal{H}) : \sigma(Q) = \{0\}\}$. It is not difficult to see that $\mathcal{J}(\mathcal{H})$ is in fact, a dense subset of $Q(\mathcal{H})$. This provides some (very partial) information towards Problem 3 of [23].

2. A SECOND LOOK AT ANDERSON'S PROOF

Let $\mathcal{R}$ be a complex Hilbert space of dimension $h \geq 2^{\mathrm{c}}$; then $\mathcal{J}_h = \{T \in \mathcal{L}(\mathcal{R}) : \dim(\mathrm{ran}T)^- < h\}^-$ is the maximal bilateral ideal of $\mathcal{L}(\mathcal{H})$ [18]. Let π_h be the canonical projection of $\mathcal{L}(\mathcal{R})$ onto $\mathcal{L}(\mathcal{R})/\mathcal{J}_h$.

If $T \in \mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ separable), then the correspondences

$$T \leftrightarrow T \otimes 1 \leftrightarrow \pi(T \otimes 1) \leftrightarrow \pi_h(T \otimes 1_h)$$

(where 1_h denotes the identity on $\mathcal{R}$; $T \otimes 1_h \in \mathcal{L}(\mathcal{R} \otimes \mathcal{R})$ and $\mathcal{H} \otimes \mathcal{R}$ is isomorphic to $\mathcal{R}$) induce (uniquely determined) isometric $*$-isomorphisms of the C^*-algebras with identity $C^*(T)$, $C^*(T \otimes 1)$, $C^*(\pi(T \otimes 1))$ and $C^*(\pi_h(T \otimes 1_h))$, generated by these operators.

In [7], D. W. Hadwin proved that if $\{U_n\}_{n=1}^\infty \subset \mathcal{L}(\mathcal{H})$ is a sequence of unitary operators such that $\{U_n T U_n^*\}_{n=1}$ is a Cauchy sequence in $\mathcal{L}(\mathcal{H})$, then $\mathcal{C}(\{U_n\}) = \{L \in \mathcal{L}(\mathcal{H}) : \{U_n L U_n^*\}_{n=1}^\infty \text{ is Cauchy}\}$ is a C^*-algebra containing $C^*(T)$ and $\rho(A) = \lim_{n\to\infty} U_n A U_n^*$ is a faithful unital $*$-representation of $C^*(T)$ onto $C^*(R)$, where $R = \lim_{n\to\infty} U_n T U_n^*$. The following lemma is the best possible "converse" of that result.

LEMMA 2.1. *If* $T \in \mathscr{L}(\mathscr{H})$ *and* $C^*(T)$ *admits a faithful unital* $*$*-representation* ρ *in* $\mathscr{L}(\mathscr{H}_\rho)$ ($\mathscr{H}, \mathscr{H}_\rho$ *separable spaces) such that* $\rho(T) = R$, *then* $R \otimes 1$ ($T \otimes 1$) *is the norm limit of a sequence of operators unitarily equivalent to* $T \otimes 1$ (*to* $R \otimes 1$, *resp.*).

Proof. As remarked above, $T \to \pi(T \otimes 1)$ induces a unique isometric $*$-isomorphism $\gamma : C^*(T) \to C^*[\pi(T \otimes 1)]$ defined by $\gamma(A) = \pi(A \otimes 1)$, $A \in C^*(T)$. Therefore $\rho \circ \gamma : C^*[\pi(T \otimes 1)] \to C^*(R) \subset \mathscr{L}(\mathscr{H}_\rho)$ defines a faithful unital $*$-isomorphism such that $\rho \circ \gamma[\pi(T \otimes 1)] = R$.

By Voiculescu's theorem [22], the closure of the unitary orbit $\mathscr{U}(T \otimes 1) = \{U(T \otimes 1)U^* : U$ is unitary$\}$ of $T \otimes 1$ contains an operator $L \simeq (T \otimes 1) \oplus (R \otimes 1)$, so that $\mathscr{U}(T \otimes 1)^- = \mathscr{U}(L)^-$.

Since T is the image of R under the faithful unital $*$-isomorphism $\rho^{-1} : C^*(R) \to C^*(T)$, the same arguments applies to R, whence we conclude that $\mathscr{U}(R \otimes 1)^- = \mathscr{U}(L')^-$ for some $L' \in \mathscr{L}(\mathscr{H}_\rho \otimes \mathscr{H})$ unitarily equivalent to L.

Since $L' \simeq L$, up to a suitable unitary identification of $\mathscr{H} \otimes \mathscr{H}$ with $\mathscr{H}_\rho \otimes \mathscr{H}$, we can assume that $\mathscr{U}(T \otimes 1)^- = \mathscr{U}(R \otimes 1)^-$, whence the result follows. ▨

REMARK. It is easily seen that the term "$\otimes 1$" cannot be avoided in the hypotheses of the lemma. Consider $T = 0$ in an infinite dimensional Hilbert space and $R = 0 \in \mathscr{L}(\mathbf{C}^1)$, or $T = S$ (a unilateral shift of multiplicity one) in $\mathscr{L}(\mathscr{H})$ and $R = S \oplus S \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H})$. In both cases, $T \leftrightarrow R$ defines a faithful unital $*$-isomorphism of the corresponding C^*-algebras, but it is completely apparent that R (T, resp.) cannot be the norm limit of operators unitarily equivalent to T (to R, resp.).

COROLLARY 2.2. *Let* T *and* R *be as in Lemma* 2.1. *If* $R \in \mathrm{JA}(\mathscr{H}_\rho)^-$, *then* $T \otimes 1 \in \mathrm{JA}(\mathscr{H} \otimes \mathscr{H})^-$.

Proof. It is completely apparent that, if $A \in \mathrm{JA}(\mathscr{H}_\rho)^-$, then $A \otimes 1 \in \mathrm{JA}(\mathscr{H}_\rho \otimes \mathscr{H})^-$ and, a fortiori, that $\mathscr{U}(A \otimes 1)^- \subset \mathrm{JA}(\mathscr{H} \otimes \mathscr{H})^-$. Now the result follows from Lemma 2.1. ▨

Let $h \geqslant 2^c$. According to [1], $\{a \in \mathscr{L}(\mathscr{R})/\mathscr{I}_h : 1 \in (\mathrm{ran}\delta_a)^-\}$ is a G_δ-dense subset of $\mathscr{L}(\mathscr{R})/\mathscr{I}_h$. Thus, given $T \in \mathscr{L}(\mathscr{H})$, we can find $L \simeq T \otimes 1_h$ in $\mathscr{L}(\mathscr{R})$ and a sequence $\{a_n\}_{n=1}^\infty$ in $\mathscr{L}(\mathscr{R})/\mathscr{I}_h$ such that $1 \in (\mathrm{ran}\delta_{a_n})^-$ for all $n = 1, 2, \ldots$, and $\|\pi_h(L) - a_n\| \to 0$ $(n \to \infty)$.

Let $\gamma : \mathscr{L}(\mathscr{R})/\mathscr{I}_h \to \mathscr{L}(\mathscr{R}_\gamma)$ be an isometric unital $*$-representation of $\mathscr{L}(\mathscr{R})/\mathscr{I}_h$ and let $A_\gamma = \gamma \circ \pi_h(L)$, and $A_n = \gamma(a_n)$, $n = 1, 2, \ldots$. Clearly, there is a non-zero *separable* subspace $\mathscr{H}_\rho$ of $\mathscr{R}_\gamma$ reducing A_γ and A_n (for all $n = 1, 2, \ldots$) such that if $A = A_\gamma|\mathscr{H}_\rho$, $B_n = A_n|\mathscr{H}_\rho$ $(n = 1, 2, \ldots)$, C_γ is any element of the C^*-algebra generated by A_γ and 1_γ, then $\|C_\gamma|\mathscr{H}_\rho\| = \|C_\gamma\|$.

Let ρ be the isometric unital $*$-representation of $C^*[\pi_h(L)]$ in $\mathcal{L}(\mathcal{H}_\rho)$ defined by $\rho \circ \pi_h(L) = A$. Then $1 \in (\operatorname{ran}\delta_{A_n})^-$ for all $n = 1, 2, \ldots$, and therefore $A \in \mathrm{JA}(\mathcal{H}_\rho)^-$. (Indeed, $\|A - B_n\| \leqslant \|\pi_h(L) - a_n\| \to 0$, as $n \to \infty$, and $1 \in (\operatorname{ran}\delta_{B_n})^-$; see Lemma 3.1, below.)

By Corollary 2.2 and our previous observations, $T \otimes 1 \in \mathrm{JA}(\mathcal{H} \otimes \mathcal{H})$. Thus, we have the following

THEOREM 2.3. *If* $T \in \mathcal{L}(\mathcal{H})$, *then* $T \otimes 1 \in \mathrm{JA}(\mathcal{H} \otimes \mathcal{H})^-$.

The (very simple) proof of the following result will be left to the reader.

LEMMA 2.4. (i) *Given* $T \in \mathcal{L}(\mathcal{H})$, *the set*

$$\{A \in \mathcal{L}(\mathcal{H}) : \operatorname{dist}[T, \operatorname{ran}\delta_A] < \eta\}$$

is open in $\mathcal{L}(\mathcal{H})$ *for each* $\eta > 0$.

(ii) $\mathrm{JA}(\mathcal{H}; T) = \{A \in \mathcal{L}(\mathcal{H}) : T \in (\operatorname{ran}\delta_A)^-\}$ *is a* G_δ*-set. In particular* $\mathrm{JA}(\mathcal{H})$ *is a* G_δ*-subset of* $\mathcal{L}(\mathcal{H})$.

CONJECTURE 2.5. *If* $T \neq 0$, *then* $\mathrm{JA}(\mathcal{H}; T)$ *is nowhere dense in* $\mathcal{L}(\mathcal{H})$.

COROLLARY 2.6. *If*

$$\mathcal{T}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) : A \simeq T \otimes 1 \textit{ for some } T \in \mathcal{L}(\mathcal{H})\},$$

then $\mathcal{T}(\mathcal{H}) \cap \mathrm{JA}(\mathcal{H})$ ($\mathcal{T}(\mathcal{H})^- \cap \mathrm{JA}(\mathcal{H})$) *is a* G_δ*-dense subset of* $\mathcal{T}(\mathcal{H})$ (*of* $\mathcal{T}(\mathcal{H})^-$, *resp.*).

Proof. Let $R \in \mathcal{T}(\mathcal{H})^-$ and let $\varepsilon > 0$; then there exists $A \simeq T \otimes 1$ such that $\|R - A\| < \varepsilon/2$.

Observe that $A \otimes 1 \simeq (T \otimes 1) \otimes 1 \simeq T \otimes (1 \otimes 1) \simeq T \otimes 1 \simeq A$. Thus, we can directly assume that $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_0$, $A = B \otimes 1_0$ and $B \simeq T \otimes 1$, for some $B \in \mathcal{L}(\mathcal{H}_0)$. By Theorem 2.3, there exists $C \in \mathrm{JA}(\mathcal{H}_0)$ such that $\|B - C\| < \varepsilon/2$, whence it readily follows that $C \otimes 1_0 \in \mathcal{T}(\mathcal{H}) \cap \mathrm{JA}(\mathcal{H})$ and $\|R - C \otimes 1_0\| < \varepsilon$.

Since, by Lemma 2.4(ii), $\mathrm{JA}(\mathcal{H})$ is a G_δ in $\mathcal{L}(\mathcal{H})$, we conclude that $\mathcal{T}(\mathcal{H})^- \cap \mathrm{JA}(\mathcal{H})$ is a G_δ-dense subset of $\mathcal{T}(\mathcal{H})^-$.

The same proof applies to $\mathcal{T}(\mathcal{H})$. ▨

It is clear that if $A \in \mathrm{JA}(\mathcal{H})$, then the similarity orbit of A is contained in $\mathrm{JA}(\mathcal{H})$.

Following D. W. Hadwin [6], we shall define

$$\mathcal{S}_r(A) = \{WAW^{-1} : W \text{ is invertible, } \|W\| \cdot \|W^{-1}\| \leqslant r\} \quad (r \geqslant 1)$$

and

$$\mathcal{S}_{\mathrm{ap}}(A) = \bigcup_{r \geqslant 1} \mathcal{S}_r(A)^-$$

(the *approximate* similarity orbit).

The proof of the following result is straightforward and will be left to the reader.

PROPOSITION 2.7. (i) *If* $A \in \mathrm{JA}(\mathscr{H})$, *then* $\mathscr{S}_{\mathrm{ap}}(A) \subset \mathrm{JA}(\mathscr{H})$. *In particular,* $\mathscr{U}(A)^- \subset \mathrm{JA}(\mathscr{H})$.

(ii) *If* $A \in \mathrm{JA}(\mathscr{H})^-$, *then* $\mathscr{S}(A)^- \subset \mathrm{JA}(\mathscr{H})^-$.

THEOREM 2.8. *If $C^*(A)$ admits a (not necessarily faithful) unital $*$-representation ρ such that $\rho(A) \notin \mathrm{JA}(\mathscr{H}_\rho)$, (in particular, if $\rho(A) \in \mathscr{S}\mathscr{I}(\mathscr{H}_\rho)$), then $A \notin \mathrm{JA}(\mathscr{H})$.*

Proof. Let $R = \rho(A)$; then $C^*[\pi(A \otimes 1)]$ admits a unital $*$-representation τ in $\mathscr{L}(\mathscr{H}_\rho)$ defined by $\tau \circ \pi(A \otimes 1) = R$ and therefore, by Voiculescu's theorem [22], $\mathscr{U}(A \otimes 1)^-$ contains an operator $L \simeq (A \otimes 1) \oplus R$. By formula (1), $\mathrm{dist}[1, \mathrm{ran}\delta_L] \geqslant \mathrm{dist}[1, \mathrm{ran}\delta_R] > 0$. Hence $L \notin \mathrm{JA}(\mathscr{H})$. By Proposition 2.7 (i), $A \notin \mathrm{JA}(\mathscr{H})$. ▨

3. BIQUASITRIANGULAR OPERATORS IN JA($\mathscr{H}$)

It is not clear a priori, that $\mathrm{JA}(\mathscr{H})$ should contain any biquasitriangular operator. It will be shown that, on the contrary, $(\mathrm{BQT}) \cap \mathrm{JA}(\mathscr{H})$ is rather large.

LEMMA 3.1. *There exists $B \in (\mathrm{BQT}) \cap \mathrm{JA}(\mathscr{H})$ such that $\sigma(B) = \sigma_{\mathrm{e}}(B) = \{\lambda : |\lambda| \leqslant 1\}$.*

Proof. Let $D = \{\lambda : |\lambda| \leqslant 1\}$ and let $\{\lambda_n\}_{n=1}^{\infty}$ be a denumerable dense subset of D. Let A be any element of $\mathrm{JA}(\mathscr{H})$. We can assume, without loss of generality, that $0 \in \sigma(A) \subset D$. Define

$$C = \bigoplus_{n=1}^{\infty} (\lambda_n + A) \in \mathscr{L}(\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} \oplus \ldots).$$

If $\|\delta_A(X_j) - 1\| \to 0$ $(j \to \infty)$ and $Y_j = X_j \oplus X_j \oplus \ldots$, then

$$\|\delta_C(Y_j) - 1\| = \left\| \left\{ \bigoplus_{n=1}^{\infty} [(\lambda_n + A)X_j - X_j(\lambda_n + A)] \right\} - 1 \right\| =$$

$$= \left\| \bigoplus_{n=1}^{\infty} \{[(\lambda_n + A)X_j - X_j(\lambda_n + A)] - 1\} \right\| =$$

$$= \left\| \bigoplus_{n=1}^{\infty} [(AX_j - X_jA) - 1] \right\| = \|\delta_A(X_j) - 1\| \to 0 \ (j \to \infty).$$

Hence $C \in \mathrm{JA}(\mathscr{H})$. Since $\sigma(A) \subset D$, it is clear that

$$D^- = \{\lambda_n\}^- \subset \sigma(C) = \left[\bigcup_{n=1}^{\infty} (\lambda_n + A)\right]^- = \bigcup_{\lambda \in D^-} [\lambda + \sigma(A)] = \bigcup_{\lambda \in D^-} [\lambda + \partial\sigma(A)],$$

where $\partial\sigma(A)$ denotes the boundary of $\sigma(A)$, $\sigma(C)$ has no isolated points and $0 \in \sigma(C)$.

Given $\mu \in \sigma(C)$, there exists a sequence $\{\mu_k\}_{k=1}^{\infty}$ such that $\mu_k \in \partial\sigma(A)$ (for all $k = 1, 2, \ldots$) and a subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ such that $\mu_k + \lambda_{n_k} \to \mu$ $(k \to \infty)$. It is easily seen that $\mu_k + \lambda_{n_k} \in \partial\sigma(\lambda_{n_k} + A)$ and therefore $\mu_k + \lambda_{n_k}$ belongs to the intersection of the left spectrum of C and the right spectrum of C. Moreover, if either $\mu_k + \lambda_{n_k} = \mu$ for infinitely many k's or $\mu_k + \lambda_{n_k} \neq \mu$ for all but finitely many k's, the conclusion is the same: μ belongs to the intersection of the left essential spectrum of C and the right essential spectrum of C. Hence by [4], $C \in (\mathrm{BQT})$ and $\sigma(C) = \sigma_e(C)$.

Define

$$B = \mathrm{sp}(C)^{-1}\left\{\bigoplus_{m=1}^{\infty} \mathrm{e}^{im}C\right\}, \quad Z_j = \mathrm{sp}(C)\left\{\bigoplus_{m=1}^{\infty} \mathrm{e}^{-im}Y_j\right\},$$

then

$$\|\delta_B(Z_j) - 1\| = \|\delta_C(Y_j) - 1\| \to 0 \ (j \to \infty),$$

$$B \in (\mathrm{BQT}) \text{ and } \sigma(B) = D^-. \ ▨$$

THEOREM 3.2. *$\mathscr{U} = (\mathrm{BQT}) \cap \mathrm{JA}(\mathscr{H})$ is a G_δ-dense subset of $(\mathrm{BQT})_\mathrm{B}$.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{C}$, let $\varepsilon > 0$ and let B be the operator of Lemma 3.1. It is not difficult to see that $\bigoplus_{j=1}^{n} (\lambda_j + \varepsilon B) \in \mathscr{U}$. Letting $\varepsilon \to 0$, we see that the closure of $\mathscr{U}$ contains every normal operator N such that $\sigma(N) = \sigma_e(N)$ is finite and, by taking suitable limits, if follows that $\mathscr{U}^-$ actually contains every normal operator whose spectrum is a perfect set.

Let $T \in (\mathrm{BQT})_\mathrm{B}$. According to [2], there exists $K \in \mathscr{K}(\mathscr{H})$ such that $\|K\| < \varepsilon$ and $\sigma(T - K) = \sigma_e(T)$ and, by using the results of [5], we can also find an $A \in (\mathrm{BQT})$ such that $\|(T - K) - A\| < \varepsilon$ and $\sigma(A) = \sigma_e(A)$ is a perfect set, so that $\|T - A\| < 2\varepsilon$.

Let M be a normal operator such that $\sigma(M) = \sigma(A)$; then $A \in \mathscr{S}(M)^-$ [10] and therefore $A \in \mathscr{U}^-$. Since ε can be chosen arbitrarily small, we conclude that $T \in \mathscr{U}^-$.

On the other hand, as remarked in the Introduction, $(\mathrm{BQT}) \cap \mathrm{JA}(\mathscr{H}) \subset \subset (\mathrm{BQT})_\mathrm{B}$. Hence, $\mathscr{U}^- = (\mathrm{BQT})_\mathrm{B}$. ▨

COROLLARY 3.3. $\{(\mathrm{BQT})_\mathrm{B} \setminus [(\mathrm{SQD}) \cup \mathrm{JA}(\mathscr{H})]\}^- = (\mathrm{BQT})_\mathrm{B}$.

Proof. We proceed exactly as above. Observe that M can be chosen so that $M \simeq M \oplus \lambda$ (where λ is an arbitrary point of $\sigma(M)$). We conclude that there exists $C \in \mathscr{U}$ and F similar to $C \oplus \lambda$ such that $\|T - F\| < 3\varepsilon$.

Clearly, $F \notin \mathrm{JA}(\mathscr{H})$ (because $\lambda \in (\mathrm{QD})$) and $F \notin (\mathrm{SQD})$ (because $C \notin (\mathrm{SQD})$). Since ε can be chosen arbitrarily small, we conclude that

$$T \in \{(\mathrm{BQT})_\mathrm{B} \setminus [(\mathrm{SQD}) \cup \mathrm{JA}(\mathscr{H})]\}^-. \ ▨$$

THEOREM 3.4. (BQT) *is the disjoint union of* (SQD), $\mathcal{M} + \mathcal{K}(\mathcal{H})$ *and* $\mathcal{N} = (\text{BQT}) \setminus [(\text{SQD}) \cup \{\mathcal{M} + \mathcal{K}(\mathcal{H})\}]$.

(SQD) *and* $\mathcal{N}$ *are dense in* (BQT) *and* $\mathcal{M} + \mathcal{K}(\mathcal{H})$ *is a* G_δ*-dense subset of* (BQT). *The three subsets are invariant under compact perturbations.*

Proof. It was remarked in the Introduction that (SQD) is dense in (BQT) [21] and that $(\text{SQD}) + \mathcal{K}(\mathcal{H}) = (\text{SQD})$. On the other hand, it is completely apparent that $\mathcal{M} + \mathcal{K}(\mathcal{H})$ is also invariant under compact perturbations. A fortiori so is $\mathcal{N}$.

Given $T \in (\text{BQT})$, it follows from [2], [10] that $B = T - K \in (\text{BQT})_{\text{B}}$ for a suitable chosen K in $\mathcal{K}(\mathcal{H})$. By Theorem 3.2 and Corollary 3.3, $B \in \mathcal{M}^-$ and $B \in [\mathcal{N} \cap \mathcal{L}(\mathcal{H})_{\text{B}}]^-$ whence it readily follows that $T \in \mathcal{M}^- + \mathcal{K}(\mathcal{H}) \subset [\mathcal{M} + \mathcal{K}(\mathcal{H})]^-$ and $B \in [\mathcal{N} \cap \mathcal{L}(\mathcal{H})_{\text{B}}]^- + \mathcal{K}(\mathcal{H}) \subset \mathcal{N}^-$.

On the other hand, $\mathcal{M}$ is a G_δ in $\mathcal{L}(\mathcal{H})$. (Use Lemma 2.4(ii) and the fact that $(\text{BQT})_{\text{B}}$ is closed in $\mathcal{L}(\mathcal{H})$.) Hence $\pi(\mathcal{M})$ is a G_δ in the Calkin algebra and, a fortiori, $\mathcal{M} + \mathcal{K}(\mathcal{H}) = \pi^{-1}[\pi(\mathcal{M})]$ is a G_δ-dense subset of (BQT).

4. SOME RESULTS ON THE NORM-CLOSURE OF $\mathcal{A}(\mathcal{H})$

Combining the above results with the results of [9], it is not difficult to see that $\mathcal{A}(\mathcal{H})^-$ contains all those operators T in $\mathcal{L}(\mathcal{H})$ such that T is similar to $A \oplus B$, where A is a nilpotent acting on a space of dimension d ($0 \leqslant d < \infty$) and B is unitarily equivalent to an ampliation.

Since $T \in \mathcal{A}(\mathcal{H})^-$ ($T \in \text{JA}(\mathcal{H})^-$) implies that $\mathcal{S}(T)^- \subset \mathcal{A}(\mathcal{H})^-$ ($\subset \text{JA}(\mathcal{H})^-$, resp.), we can use the results of [5] and conclude that

$$\text{JA}(\mathcal{H})^- \supset \{T \in \mathcal{L}(\mathcal{H}) : 1)\ \sigma_0(T) = \emptyset;\ 2) \text{ if } \lambda - T \text{ is semi-Fredholm, then } \text{ind}(\lambda - T) = 0,\ +\infty \text{ or } -\infty\}$$

and

$$\mathcal{A}(\mathcal{H})^- \supset \{T \in \mathcal{L}(\mathcal{H}) : 1)\ \sigma_0(T) \subset \{0\};\ 2) \text{ if } \lambda - T \text{ is semi-Fredholm, then } \text{ind}(\lambda - T) = 0,\ +\infty \text{ or } -\infty\}.$$

CONJECTURE 4.1. *The first inclusion is actually an equality. Furthermore, if* $A \in \text{JA}(\mathcal{H})$ *is semi-Fredholm of positive index, then* $\text{ind}A = +\infty$ *and* $\text{Ker}A^* = \{0\}$.

Since $\text{JA}(\mathcal{H})$ contains a large family of operators of index $-\infty$ (and $+\infty$) and $\{T\}' \cap (\text{ran}\delta_T)^-$ is an ideal in $\{T\}'$, comparison with the case when T is a unilateral shift of infinite multiplicity might suggest that $\mathcal{A}(\mathcal{H})$ also contains operators of index $-n$ (for all $n = 1, 2, \ldots$). More precisely: Assume that $T \in \text{JA}(\mathcal{H})$ and $\text{ind}T = -\infty$. Does T commute with a Fredholm operator A with $\text{ind}A = -1$?

The answer could be negative: $\mathcal{L}(\mathcal{H})$ contains a large class of semi-Fredholm operators T such that $\text{ind}T = -\infty$ and $\text{ind}A = -\infty$ for every non-invertible semi-Fredholm operator A in $\{T\}'$. Concrete examples of these operators can be obtained by using, e.g., Lemma 3 of [11] (pick any L as in that lemma, with $\operatorname{ind} L = -\infty$).

Finally, observe that if $q_k \in \mathscr{L}(\mathbf{C}^k)$ is the nilpotent Jordan cell of order k, i.e., $q_k e_{1k} = 0$ and $q_k e_{jk} = e_{j-1,k}$ for $j = 2, 3, \ldots, k$, with respect to the canonical ONB of $\mathbf{C}^k$ $(k = 1, 2, \ldots)$, $Q = \bigoplus_{k=1}^{\infty} \left(\frac{1}{k}\right) q_k$, $R = \bigoplus_{k=1} \frac{1}{k^2} q_k$ and $He_{jk} = \frac{j}{k} e_{jk}$, $j = 1, 2, \ldots, k$, $k = 1, 2, \ldots$, then $R = QH - HQ \in \{Q\}' \cap (\operatorname{ran}\delta_Q)$, so that

$$T = R \otimes 1 = \delta_{Q\otimes 1}(H \otimes 1) \in \mathscr{J}(\mathscr{H}).$$

Hence $\mathscr{S}(T) \subset \mathscr{J}(\mathscr{H})$ and, a fortiori, $\mathscr{S}(T)^- \subset \mathscr{J}(\mathscr{H})^- \subset \{L \in \mathscr{L}(\mathscr{H}) : \sigma(L) = \{0\}\}^-$. Since T is a quasinilpotent and T^k is a compact operator for no value of k $(k = 1, 2, \ldots)$, it follows from [13] that $\mathscr{S}(T)^-$ concides with the closure of the set of all quasinilpotent operators and, a fortiori, that $\mathscr{J}(\mathscr{H})$ is dense in the set of all quasinilpotent operators.

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