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A CARLESON MEASURE THEOREM FOR THE BERGMAN SPACE ON THE BALL

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Let **B** be the closed unit disc in the complex plane **C** and let μ be a finite measure on **B**. For $\theta \in [0, 2\pi]$, $h \in (0, 1)$ let

$$\Omega(\theta, h) \equiv \{z \in \mathbf{B} \mid 1 - h < |z| < 1, \ \theta < \arg z < \theta + h\} .$$

The measure μ is called a Carleson measure if there is a constant $C > 0$ such that

$$\mu(\Omega(\theta, h)) \leqslant Ch \quad \forall \ \theta, h .$$

In [1] it is shown that μ is a Carleson measure if and only if the injection mapping from the Hardy space H^2 into $H^2(\mathrm{d}\mu)$ is bounded (i.e. there is a $C' > 0$ so that for $f \in H^2$,

$$\int_{\mathbf{B}} |f(z)|^2 \mathrm{d}\mu(z) \leqslant C' \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|^2 \frac{\mathrm{d}\theta}{2\pi} .$$

Several analogues of the Carleson theorem have subsequently been obtained. Hormander [6] proved a version of this result for the Hardy space of the unit ball $\mathbf{B}_n$ in $\mathbf{C}^n$. Then Carleson [2] produced an example to show that the result is false for the Hardy space of the polydisc. In a more current paper Hastings [5] has proven a Carleson type theorem for the Bergman space of the polydisc. Further, Stegenga 9] has obtained analogous results for certain weighted Bergman spaces.

In this paper we prove a Carleson theorem for the Bergman space of the unit ball in $\mathbf{C}^n$. We also outline a short proof of Hormander's result [6]. As an application we consider the question of compactness of Toeplitz operators on Bergman spaces. After the results of this paper were obtained Professor Peter Duren pointed out the Hormander reference [6]. We wish to thank him for this reference.

1. THE PRINCIPAL THEOREM

Let $\mathbf{B}_n$ be the closed unit ball in $\mathbf{C}^n$ ($\mathbf{B}_1 = \mathbf{B}$) and let Σ_n be the boundary of $\mathbf{B}_n$. If μ is a measure on $\mathbf{B}_n$ and $p \geqslant 1$, then $H^p(\mathrm{d}\mu)$ is the $L^p(\mathrm{d}\mu)$ closure of the polynomials in $z = (z_1, \ldots, z_n)$. Further, let $m_n(z)$ denote volume measure on C^n.

Fix $n \geqslant 1$ and $\beta > -n$. Consider measures defined by the equations

$$\mathrm{d}\nu_n(z) = (1 - |z|^2)^\beta \mathrm{d}m_n(z), \quad z \in \mathbf{B}_n,$$

$$\mathrm{d}\lambda(z_1) = (1 - |z_1|^2)^{\beta + n - 1} \mathrm{d}m_1(z_1), \quad z_1 \in \mathbf{B}_1.$$

For $0 < t < 1$ and $\eta \in \Sigma_n$, let

$$\mathscr{S}(t) \equiv \{z \in \mathbf{B}_n : |1 - \langle z, \eta \rangle| < t\}.$$

The notation $\langle \cdot, \cdot \rangle$ denotes complex inner product in $\mathbf{C}^n$. A measure μ on $\mathbf{B}_n$ is a ν_n-Carleson measure if there is a constant $C > 0$ so that

$$\mu(\mathscr{S}(t)) \leqslant C \nu_n(\mathscr{S}(t)) \quad \forall\ \eta \in \Sigma_n, \ \forall\ t > 0.$$

THEOREM 1. *A measure μ on $\mathbf{B}_n$ is a ν_n-Carleson measure if and only if there is a constant C' so that*

$$\int_{\mathbf{B}_n} |f(z)|^2 \mathrm{d}\mu(z) \leqslant C' \int_{\mathbf{B}_n} |f(z)|^2 \mathrm{d}\nu_n(z) \quad \forall f \in H^2(\mathrm{d}\nu_n).$$

Proof. Our proof follows the pattern of the proof in [4] of Carleson's theorem. We construct a maximal function, establish a pointwise estimate, and appeal to the Marcinkiewicz interpolation theorem. For $n = 1$, this theorem is due to Stegenga [9]. (See [5] for the $\beta = 0$ case.)

We begin with a lemma (see also [7]) which gives integration with respect to ν_n as an integrated integral. For A a measurable subset of $\mathbf{B}$, let

$$\hat{A} \equiv \{z \in \mathbf{B}_n : z_1 \in A\}.$$

Further, for $z \in \mathbf{C}^n$, write $z = (z_1, u)$, where $u = (z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$.

LEMMA 1. *If $f \in L^1(\mathrm{d}\nu_n)$ and $A \subset \mathbf{B}$, then*

$$\int_{\hat{A}} |f(z)| \mathrm{d}\nu_n(z) = \int_A \mathrm{d}\lambda(z_1) \int_{\mathbf{B}_{n-1}} |f(z_1, \sqrt{1 - |z_1|^2}\, u')| \mathrm{d}\nu_{n-1}(u'),$$

where $u' = u / \sqrt{1 - |z_1|^2}$.

Proof.

$$\int_{\hat{A}} |f(z)|\mathrm{d}\nu_n(z) = \int_A \mathrm{d}m_1(z_1) \int_{|u|^2 \leqslant 1-|z_1|^2} |f(z_1, u)|(1-|z|^2)^\beta \mathrm{d}m_{n-1}(u) =$$

$$= \int_A (1-|z_1|^2)^{\beta+n-1}\mathrm{d}m_1(z_1) \int_{\mathbf{B}_{n-1}} |f(z_1, \sqrt{1-|z_1|^2}\,u')|\,(1-|u'|^2)^\beta \mathrm{d}m_{n-1}(u') =$$

$$= \int_A \mathrm{d}\lambda(z_1) \int_{\mathbf{B}_{n-1}} |f(z_1, \sqrt{1-|z_1|^2}\,u')|\mathrm{d}\,\nu_{n-1}(u') \ .$$

Now suppose μ is ν_n Carleson. Fix $\eta \in \Sigma_n$. By an orthonormal change of basis we can assume $\eta = (1, 0, \ldots, 0) \equiv \mathbf{1}$. Let $f \in L^1(\mathrm{d}\,\nu_n)$, and define a function $\tilde{f}$ on $\mathbf{B}$ by

$$\tilde{f}(z_1) \equiv \int_{\mathbf{B}_{n-1}} |f(z_1, \sqrt{1-|z_1|^2}\,u')|\mathrm{d}\nu_{n-1}(u') \ .$$

By Lemma 1, we have that

(1) $$\int_{\mathbf{B}_n} |f(z)|\mathrm{d}\,\nu_n(z) = \int_{\mathbf{B}} \tilde{f}(z_1)\mathrm{d}\lambda(z_1) \ .$$

If $S(t) \equiv \{z_1 \in \mathbf{B} : |z_1 - 1| < t\}$, then again by Lemma 1, we have

(2) $$\nu_n(\mathscr{S}(t)) = \lambda(S(t)) \ .$$

Let

$$(Mf)(r\mathbf{1}) \equiv \sup_{1>t>1-r} \left(\frac{1}{\nu_n(\mathscr{S}(t))} \int_{\mathscr{S}(t)} |f(z)|\mathrm{d}\nu_n(z) \right) .$$

Also, if $g \in L^1(\mathrm{d}\lambda)$, let

$$(Ng)(r) = \sup_{1>t>1-r} \left(\frac{1}{\lambda(S(t))} \int_{S(t)} |g(z_1)|\mathrm{d}\lambda(z_1) \right) .$$

Suppose now that $f \in H^1(\mathrm{d}\nu_n)$. Then for z_1 fixed, $|f(z_1, \sqrt{1-|z_1|^2}u')|$ is plurisubharmonic in u'. Using this fact and the radial symmetry of the measure $\mathrm{d}\,\nu_{n-1}$ one can show

$$\tilde{f}(z_1) = \int_{\mathbf{B}_{n-1}} |f(z_1, \sqrt{1-|z_1|^2}\,u')|\mathrm{d}\nu_{n-1}(u') \geqslant |f(z_1, 0)|\,\nu_{n-1}(\mathbf{B}_{n-1}) .$$

Thus $f(z_1, 0) \in H^1(\mathrm{d}\lambda)$ and by Stegenga's proof [9, page 117],

$$|f(r, 0)| \leqslant K(Nf)(r, 0)$$

for some $K > 0$. By (1) and (2)

$$(Mf)(r\mathbf{1}) = (N\tilde{f})(r) ,$$

so

$$|f(r, 0)| \leqslant \frac{K}{v_{n-1}(\mathbf{B}_{n-1})} (Mf)(r\mathbf{1}) = K'(Mf)(r\mathbf{1}).$$

Now K' is independent of η (see [9]) so that

$$(3) \qquad |f(z)| \leqslant K'(Mf)(z) \quad \forall\ z \in \mathbf{B}_n .$$

The remainder of this part of the proof is standard. A covering argument shows that the sublinear map M is weak type (1,1) from $L^1(\mathrm{d}v_n)$ to $L^1(\mathrm{d}\mu)$. M is clearly of type (∞, ∞). Hence, by the Marcinkiewicz interpolation theorem, M is bounded from $L^2(\mathrm{d}v_n)$ to $L^2(\mathrm{d}\mu)$. Now the pointwise estimate (3) shows that the inclusion map from $H^2(\mathrm{d}\mu)$ to $H^2(\mathrm{d}v_n)$ is bounded.

For the proof of the converse, suppose that

$$\int_{\tilde{\mathbf{B}}_n} |f(z)|^2 \mathrm{d}\mu(z) \leqslant C' \int_{\mathbf{B}_n} |f(z)|^2 \mathrm{d}v_n(z) \quad \forall\ f \in H^2(\mathrm{d}v_n) .$$

It suffices to check the Carleson condition at $\eta = \mathbf{1}$. For $w \in \mathbf{B}_n$, let $g_w(z) = (1 - \langle z, w \rangle)^{-(\beta+2n)}$. If $r = |w|$, by radial symmetry and Lemma 1,

$$\|g_w\|^2_{L^2(\mathrm{d}v_n)} = \|g_{r\mathbf{1}}\|^2_{L^2(\mathrm{d}v_n)} =$$

$$= v_{n-1}(\mathbf{B}_{n-1}) \cdot \left\| \frac{1}{(1 - rz_1)^{\beta+2n}} \right\|_{L^2(\mathrm{d}\lambda)} \leqslant \frac{C_1}{(1 - r^2)^{\beta+2n}} .$$

(See [9, p. 118].) Thus if $w(t) = (1 - t)\mathbf{1}$, then

$$\|g_{w(t)}\|^2_{L^2(\mathrm{d}v_n)} \leqslant \frac{C_2}{t^{\beta+2n}} .$$

Finally,

$$v_n(\mathscr{S}(t)) = \lambda(S(t)) \geqslant C_4 t^{\beta+2n}$$

and for $z \in \mathscr{S}(t)$,

$$|g_{w(t)}(z)|^2 \geqslant \frac{C_3}{t^{2\beta+4n}} .$$

Thus

$$\int_{\mathbf{B}_n} |g_{w(t)}(z)|^2 \mathrm{d}\mu(z) \geqslant \int_{\mathscr{S}(t)} |g_{w(t)}(z)|^2 \mathrm{d}\mu(z) \geqslant \frac{C_3}{t^{2\beta+4n}} \mu(\mathscr{S}(t)) ,$$

and

$$\int_{\mathbf{B}_n} |g_{w(t)}(z)|^2 \mathrm{d}\mu(z) \leqslant C' \int_{\mathbf{B}_n} |g_{w(t)}(z)|^2 \mathrm{d}\nu_n(z) \leqslant \frac{C'C_2}{t^{\beta+2n}} .$$

This implies

$$\mu(\mathscr{S}(t)) \leqslant \frac{C'C_2}{C_3} t^{\beta+2n} \leqslant C\nu_n(\mathscr{S}(t)) .$$

We complete this section with a few remarks.

REMARK 1. There are other sets which can replace the $\mathscr{S}(t)$ sets to give an equivalent definition of Carleson measure. For instance, we can use the sets $\mathscr{D}(t)$ or $\mathscr{E}(t)$ defined as follows. For $\eta \in \Sigma_n$, let

$$\mathscr{B}(t) = \{z \in \Sigma_n : |1 - \langle z, \eta \rangle| < t\} ,$$

$$\mathscr{D}(t) = \{z \in \mathbf{B}_n : |z| > 1 - t,\ z/|z| \in \mathscr{B}(t)\} ,$$

$$\mathscr{E}(t) = \{z \in \mathbf{B}_n : z = \rho - \lambda\eta, \text{ where } \rho \in \mathscr{B}(t) \text{ and } 0 \leqslant \lambda \leqslant t\} .$$

Note that $\mathscr{D}(t/2) \subseteq \mathscr{S}(t) \subseteq \mathscr{D}(t)$ and $\mathscr{E}(t/2) \subseteq \mathscr{S}(t) \subseteq \mathscr{E}(t)$.

REMARK 2. A careful check of the constants appearing in the proof of Theorem 1 shows that if C is small, then C' can be chosen small.

REMARK 3. Hormander's theorem [6] is valid for surface measure on the boundary of a strictly pseudoconvex domain in $\mathbf{C}^n$. The method of proof of Theorem 1 provides an elementary proof of his theorem for the ball.

Let σ_n denote surface measure of Σ_n. A measure μ on $\mathbf{B}_n$ is σ_n-Carleson if there is a constant $C > 0$ so that

$$\mu(\mathscr{S}(t)) \leqslant C\sigma_n(\mathscr{B}(t)) \quad \forall\, \eta \in \Sigma_n ,\, t > 0 .$$

THEOREM 2 (Hörmander). *A finite measure μ is a σ_n-Carleson measure if and only if there is a constant $C' > 0$ so that*

$$\int_{\mathbf{B}_n} |f(z)|^2 \mathrm{d}\mu(z) \leqslant C' \int_{\Sigma_n} |f(z)|^2 \mathrm{d}\sigma_n(z) \quad \forall\, f \in H^2(\mathrm{d}\sigma_n) .$$

Proof. We begin with a modification of Lemma 1. For $A \subseteq \mathbf{B}$, let $\tilde{A} = \{z \in \Sigma_n;\ z_1 \in A\}$.

LEMMA 2. *If $f \in L^1(\mathrm{d}\sigma_n)$, then*

$$\int_{\tilde{A}} |f(z)| \mathrm{d}\sigma_n(z) = 2 \int_{A} (1 - |z_1|^2)^{n-1} \mathrm{d}m_1(z_1) \cdot \int_{\Sigma_{n-1}} |f(z_1, \sqrt{1 - |z_1|^2}\, u')| \mathrm{d}\sigma_{n-1}(u') .$$

The proof of Lemma 2 is routine. One can use the parametrization

$$z = (r_1 \mathrm{e}^{\mathrm{i}\theta_1}, \ldots, r_{n-1} \mathrm{e}^{\mathrm{i}\theta_{n-1}}, (1 - r_1^2 - \ldots - r_{n-1}^2)^{1/2} \mathrm{e}^{\mathrm{i}\theta_n}) ,$$

$$\mathrm{d}\sigma = 2^{n-2}(1 - r_1^2)^{n-2}(1 - r_2^2)^{n-3} \ldots (1 - r_{n-2}^2)^2 r_1 r_2 \ldots r_{n-1} \mathrm{d}r_1 \ldots \mathrm{d}r_{n-1} \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n .$$

As in the proof of Theorem 1, fix $\eta \in \Sigma_n$ and assume without loss of generality that $\eta = (1, 0, \ldots, 0) = \mathbf{1}$. If $f \in H^1(\mathrm{d}\sigma_n)$ let

$$\tilde{f}(z_1) = \int_{\Sigma_{n-1}} |f(z_1, \sqrt{1 - |z_1|^2}\, u')| \, \mathrm{d}\sigma_{n-1}(u') .$$

The rest of the proof parallels the proof of Theorem 1, and we omit the details.

2. APPLICATIONS

As an application of Theorem 1, we consider the problem of characterizing the compact Toeplitz operators on the Bergman space. (See [3] and [8].)

Let P be the orthogonal projection of $L^2(\mathrm{d}m_n)$ onto $H^2(\mathrm{d}m_n)$. If $\varphi \in L^\infty(\mathrm{d}m_n)$, define

$$T_\varphi : H^2(\mathrm{d}m_n) \to H^2(\mathrm{d}m_n)$$

by $T_\varphi f = P\varphi f$. T_φ is the Toeplitz operator with symbol φ. More generally Voas has observed [10] that an unbounded function φ may induce a bounded Toeplitz operator. Following [10] call a function φ on $\mathbf{B}_n$ admissible if there is a constant $C > 0$ so that

$$\left| \int_{\mathbf{B}_n} f(z)\overline{g(z)}\varphi(z) \, \mathrm{d}m_n(z) \right| \leqslant C \|f\| \cdot \|g\| \quad f, g \in H^2(\mathrm{d}m_n) .$$

Clearly an admissible φ will induce a bounded Toeplitz operator, and conversely.

THEOREM 3. *If $\varphi \geqslant 0$, then φ is admissible if and only if the measure $\varphi \mathrm{d}m_n$ is m_n-Carleson.*

THEOREM 4. *If $\varphi \geqslant 0$ is admissible, then T_φ is compact if and only if*

$$\int_{\mathscr{S}(t)} \varphi(z)\mathrm{d}m_n(z) = \mathrm{o}(m_n(\mathscr{S}(t)))$$

uniformly in $\eta \in \Sigma_n$.

The proof of Theorem 3 is essentially Voas' proof for $n = 1$, and is straightforward. Further, Theorem 4 is proved using Theorem 1 and Remark 2 in exactly the same way that McDonald and Sundberg prove the $n = 1$ case [8]. Of course the above theorems hold in the weighted Bergman spaces $H^2(\mathrm{d}\nu_n)$ of the previous section.

We close with some observations and examples concerning Toeplitz operators T_φ, where φ need not be positive.

1. If φ is any function in $L^1(\mathrm{d}m_n)$ whose support is bounded away from Σ_n, then T_φ is compact. (A weakly convergent sequence in $H^2(\mathrm{d}m_n)$ converges uniformly on compacta.)

2. For $\varphi \geqslant 0$, compactness of T_φ is equivalent to $\lim \langle \varphi f_n, f_n\rangle = 0$ for every weakly convergent sequence $\{f_n\}$. Hence, if $0 \leqslant \psi \leqslant \varphi$ and T_φ is compact, we must have T_ψ being compact. Thus $T_{|\varphi|}$ compact implies T_φ is compact.

3. We construct a real function $\varphi \in L^1(\mathrm{d}m_1)$ such that T_φ is compact, but $T_{|\varphi|}$ is bounded but not compact. Let

$$S = \{re^{i\theta} : r \geqslant 0,\ |\theta| < (1-r)^2\}\,.$$

On S, let $\varphi(re^{i\theta}) = (1-r)^{-1}$ if $\theta > 0$ and $\varphi(re^{i\theta}) = -(1-r)^{-1}$ if $\theta < 0$. Let $\varphi \equiv 0$ off S. As usual φ_+ and φ_- denote the positive and negative parts of φ. Since

$$\int_{1-t}^{1} \varphi_+(re^{i\theta})\, r\mathrm{d}r \int_0^{(1-r)^2} \mathrm{d}\theta = \frac{t^2}{6} \leqslant t^2\,,$$

it follows (Theorems 3 and 4) that T_{φ_+} is bounded but not compact.

The same is true for T_{φ_-}, and hence for $T_{|\varphi|}$.

We show that T_φ is Hilbert-Schmidt. Let $e_k(z) = \sqrt{k+1}\, z^k$. The set $\{e_n\}_0^\infty$ is an orthonormal basis for $H^2(\mathrm{d}m_1)$. First,

$$\langle T_\varphi z^k, z^k\rangle = 0$$

and if $k \neq l$,

$$\langle T_\varphi z^k, z^l\rangle = \int_0^1 \frac{r^{k+l+1}}{(1-r)}\,\mathrm{d}r \int_0^{(1-r)^2} (\mathrm{e}^{\mathrm{i}(k-l)\theta} - \mathrm{e}^{-\mathrm{i}(k-l)\theta})\mathrm{d}\theta =$$

$$= \int_0^1 \frac{r^{k+l+1}}{1-r}\,\mathrm{d}r \int_0^{(1-r)^2} 2\mathrm{i}\sin(k-l)\theta\,\mathrm{d}\theta\ .$$

Thus

$$|\langle T_\varphi z^k, z^l\rangle| \leqslant \int_0^1 \frac{r^{k+l+1}}{(1-r)} |k-l|\,(1-r)^4\mathrm{d}r \leqslant \frac{|k-l|3!}{(k+l+2)^4}\,.$$

Hence,

$$|\langle T_\varphi e_k\,, e_l\rangle| \leqslant \frac{3!\,|k-l|\,\sqrt{k+1}\,\sqrt{l+1}}{(k+l+2)^4}$$

and

$$\sum_k \sum_l |\langle T_\varphi e_k\,, e_l\rangle|^2 < +\infty\,.$$

We can modify this example to produce a real function φ so that T_φ is compact, but T_φ is not even bounded. Let $\varphi(r\mathrm{e}^{\mathrm{i}\theta}) = (1-r)^{-3}$ if $0 < \theta < (1-r)^3$, $\varphi(r\mathrm{e}^{\mathrm{i}\theta}) = -(1-r)^3$ if $0 > \theta > -(1-r)^3$ and $\varphi \equiv 0$ elsewhere. One can check that φ_+ and φ_- are not admissible, but T_φ is Hilbert-Schmidt.

C. Sundberg has informed us that he and G. McDonald have constructed similar examples.

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