J. OPERATOR THEORY
7(1982), 209–218

© Copyright by INCREST, 1982

TWO REMARKS CONCERNING THE THEOREM OF S. AXLER, S.-Y. A. CHANG AND D. SARASON

A. L. VOLBERG

1. INTRODUCTION

This article is devoted to a proof of a conjecture of S. Axler, S.-Y. A. Chang and D. Sarason [2], and to an application of one of their results to bases consisting of rational fractions.

Let $\mathbf{D} = \{\xi \in \mathbf{C}: |\xi| < 1\}$ be the unit disc and $\mathbf{T}$ be its boundary. For $\varphi \in L^\infty(\mathbf{T})$ the Toeplitz operator T_φ on the Hardy space H^2 is defined by the equality:

$$T_\varphi h = \mathbf{P}_+ \varphi h, \quad h \in H^2,$$

where $\mathbf{P}_+$ is the orthogonal projection from L^2 onto H^2. The function φ is called the symbol of this operator. The Hankel operator with the same symbol is defined by the formula:

$$H_\varphi h = \mathbf{P}_- \varphi h, \quad h \in H^2,$$

where $\mathbf{P}_- = I - \mathbf{P}_+$, I being the identity operator.

The following question arises rather naturally: For what symbols f, g is the product of two Toeplitz operators $T_f T_g$ a compact perturbation of some Toeplitz operator? (Then it is well-known that this Toeplitz operator is T_{fg}, [7].) So the question above can be reduced to the following one: For what symbols f, g is the operator $T_{fg} - T_{\bar f} T_g = H_{\bar f}^* H_g$ compact?

Let us also recall that $H^\infty[f]$ for $f \in L^\infty$ denotes the uniformly closed algebra generated by H^∞ and f. One of the most interesting algebras of this type is the algebra $H^\infty[\bar z]$. It is well-known that $H^\infty[\bar z] = H^\infty + C$, where C is the algebra of all continuous functions on the unit circle. Now we are in the position to state the following remarkable theorem proved in [2].

THEOREM A. *If*

$$(*) \qquad H^\infty[f] \cap H^\infty[g] \subset H^\infty + C$$

then the operator $H_{\bar f}^* H_g$ *is compact.*

The necessity of the condition $(*)$ was proved in [2] for a large class of functions f, g but not in the general case.

2. THE PROOF OF THE NECESSITY

Suppose that the operator $H_f^* H_g$ is compact. We have to prove (*). Without loss of generality we may suppose that $\|f\| < 1$, $\|g\| < 1$. Then it is well-known that there are unimodular functions $u \in f + H^\infty$, $v \in g + H^\infty$ such that

$$\text{(1)} \qquad \operatorname{dist}(\bar{z}u, H^\infty) = 1, \quad \operatorname{dist}(\bar{z}v, H^\infty) = 1$$

(see [8], [14]). But $\operatorname{dist}(u, H^\infty) < 1$, $\operatorname{dist}(v, H^\infty) < 1$ and we conclude that the operators T_u, T_v are invertible (see [14], Ch. VIII). Now we recall a well-known definition.

DEFINITION. An outer function $h \in H^2$ is called a *Helson-Szegö function* ($h \in (HS)$) if $|h^2| = e^{u+\tilde{v}}$, $u \in L^\infty$, $\|v\|_\infty < \frac{\pi}{2}$; here $\tilde{v}$ is the harmonic conjugate function of v.

Thus the functions u, v are unimodular and the operators T_u, T_v are invertible, and so there are two Helson-Szegö functions h, h_1 such that

$$\text{(2)} \qquad u = \frac{\bar{h}_1}{h_1}, \quad v = \frac{\bar{h}}{h}$$

(see [1], [14] Ch. VIII).

To continue the proof we need four easy lemmas and some additional notations to state them. If $f \in L^2 = L^2(\mathbf{T})$ we denote $\mathbf{P}_+ f$ by f_+ and $\mathbf{P}_- f$ by f_-. We shall not distinguish notationally between f and its harmonic extension to the unit disc (so for $a \in \mathbf{D}$ we have

$$f(a) = \int_{\mathbf{T}} f(\xi) P_a(\xi) \, dm(\xi),$$

$P_a(\xi)$ being the Poisson kernel).

By S^* we denote the operator on the space $H^2_- = \overline{zH^2}$ (the bar denotes complex conjugation) which is defined by the equality $S^* h_- \stackrel{\text{def}}{=} \mathbf{P}_- \bar{z} h_-$ for any function h_- in H^2_-. It is obvious that for $a \in \mathbf{D}$ the operator $(I - \bar{a}S^*)$ is invertible in H^2_-, and

$$\text{(3)} \qquad ((I - \bar{a}S^*)^{-1} h_-)(\xi) = \frac{h_-(\xi) - h_-(a)}{1 - \bar{a}\xi}, \quad \xi \in \mathbf{T}.$$

LEMMA 1. *For any $f \in L^2$ we have*

$$\text{(4)} \qquad \mathbf{P}_- \frac{f}{1 - \bar{a}\xi} = \frac{f_-(\xi) - f_-(a)}{1 - \bar{a}\xi}.$$

Proof.

$$\mathbf{P}_- \frac{f}{1 - \bar{a}\xi} = \sum_{j=0}^\infty \bar{a}^j \mathbf{P}_- \xi^j f = \sum_{j=0}^\infty \bar{a}^j \mathbf{P}_- \xi^j f_- =$$

$$= \sum_{j=0}^\infty \bar{a}^j S^{*j} f_- = (I - \bar{a}S^*)^{-1} f_- = \frac{f_-(\xi) - f_-(a)}{1 - \bar{a}\xi}.$$

LEMMA 2. *For any* $u, v \in L^\infty$ *we have* $\left(k_a \overset{\text{def}}{=} \dfrac{(1-|a|^2)^{1/2}}{1-\bar{a}\xi}\right)$:

$$(v_-(\xi) - v_-(a))(\bar{u}_-(\xi) - \bar{u}_-(a))P_a(\xi) = \tag{5}$$
$$= \bar{k}_a(\xi)(H_u^* H_v k_a)(\xi) + k_a(\xi)\overline{(H_v^* H_u k_a)(\xi)} - \langle H_v k_a, H_u k_a\rangle P_a(\xi);$$

here $\langle \cdot, \cdot \rangle$ *denotes the usual scalar product in* L^2.

Proof. For any analytic polynomial f the following chain of the equalities is obvious:

$$\langle \bar{k}_a H_u^* H_v k_a, f\rangle = \langle H_u^* H_v k_a, k_a f\rangle = \langle H_v k_a, H_u f k_a\rangle =$$
$$= \langle H_v k_a, \mathbf{P}_-(f \cdot H_u k_a)\rangle = \langle H_v k_a, f \cdot H_u k_a\rangle = \langle H_v k_a \cdot \overline{H_u k_a}, f\rangle.$$

Let $F \overset{\text{def}}{=} (v_-(\cdot) - v_-(a))(\bar{u}_-(\cdot) - \bar{u}_-(a))P_a(\cdot)$. Then using Lemma 1 and the fact that f is arbitrary we conclude that

$$\mathbf{P}_+ F = \mathbf{P}_+(\bar{k}_a H_u^* H_v k_a). \tag{6}$$

Interchanging the roles of u and v we obtain

$$\mathbf{P}_+ \bar{F} = \mathbf{P}_+(\bar{k}_a H_v^* H_u k_a). \tag{7}$$

It is easy to compute the right-hand parts in (6) and (7). Namely

$$\mathbf{P}_+(\bar{k}_a H_u^* H_v k_a)(\xi) = (1-|a|^2)^{1/2} \frac{\xi (H_u^* H_v k_a)(\xi) - a(H_u^* H_v k_a)(a)}{\xi - a} = \tag{8}$$
$$= \bar{k}_a(\xi)(H_u^* H_v k_a)(\xi) - \frac{a}{\xi - a}\langle H_u^* H_v k_a, k_a\rangle.$$

Now using the equalities (6), (7), (8) and the fact that $F(0) = \langle H_v k_a, H_u k_a\rangle$ we obtain the following chain of equalities:

$$F = \mathbf{P}_+ F + \mathbf{P}_- F = \mathbf{P}_+ F + \overline{\mathbf{P}_+ \bar{F}} - F(0) = \bar{k}_a (H_u^* H_v k_a) +$$
$$+ k_a \overline{(H_v^* H_u k_a)} - \frac{a}{\xi - a}\langle H_v k_a, H_u k_a\rangle - \frac{\bar{a}}{\bar{\xi} - \bar{a}}\langle H_v k_a, H_u k_a\rangle -$$
$$- \langle H_v k_a, H_u k_a\rangle = \bar{k}_a H_u^* H_v k_a + k_a \overline{H_v^* H_u k_a} - \langle H_v k_a, H_u k_a\rangle P_a(\cdot).$$

LEMMA 3. *If the operator $H_u^* H_v$ is compact then*

$$\lim_{|a|\to 1} \int_{\mathbf{T}} |u_-(\xi) - u_-(a)| \cdot |v_-(\xi) - v_-(a)| P_a(\xi)\, \mathrm{d}m(\xi) = 0.$$

Proof. This follows immediately from (5) and the fact that $k_a \to 0$ in the weak topology of L^2 if $|a| \to 1$.

The following lemma was proved in a particular case by S. V. Hruschev and V. V. Peller (see [13]).

LEMMA 4. *Let the operator $H_u^* H_v$ be compact and let $v = \frac{\bar h}{h}$, h being a Helson-Szegö function. Then the operator $H_u^* H_{\bar v}$ is also compact.*

Proof. It is a well-known fact (see [11]) that if h is a Helson-Szegö function then the operator $\mathbf{P}_+$ (defined on trigonometric polynomials by $\mathbf{P}_+(\sum_n a_n z^n) = \sum_{n \geq 0} a_n z^n$) can be extended to a bounded linear operator on $L^2(|h|^2)$. By $H^2(|h|^2)$ we denote the closure of the analytical polynomials in $L^2(|h|^2)$. It is obvious that $H^2(|h|^2) = \mathbf{P}_+ L^2(|h|^2)$. If $L(H_1, H_2)$ denotes the space of bounded operators from the Hilbert space H_1 to the Hilbert space H_2, then it is clear that

$$T_{\frac{1}{\bar h}} \in L(H^2, H^2(|h|^2)),$$

$$T_h T_{\frac{1}{h}} T_h \in L(H^2(|h|^2), H^2),$$

$$L \stackrel{\mathrm{def}}{=} T_{u\frac{h}{\bar h}} - T_u T_{\frac{h}{\bar h}} \in L(H^2(|h|^2), H^2(|h|^2)).$$

So using the hypothesis of the lemma we may conclude that the operator $T_{\frac{1}{\bar h}} H_u^* H_v T_h T_{\frac{1}{\bar h}} T_h$ is compact from the space $H^2(|h|^2)$ to itself. But for every analytical polynomial p it is easy to check that

$$Lp = - T_{\frac{1}{\bar h}} H_u^* H_v T_h T_{\frac{1}{\bar h}} T_h p$$

so L is a compact operator in $H^2(|h|^2)$. From this it is clear that

$$\lim_{N\to\infty} \| L z^N \mathbf{P}_+ \|_{L^2(|h|^2) \to L^2(|h|^2)} = 0.$$

Here we use once more the fact that $\mathbf{P}_+ \in L(L^2(|h|^2), H^2(|h|^2))$. But if h is a Helson-Szegö function then $\frac{1}{h}$ is also a Helson-Szegö function. So $L \in L\left(H^2\left(\frac{1}{|h|^2}\right), H^2\left(\frac{1}{|h|^2}\right)\right)$ and we have:

$$\sup_N \| L z^N \mathbf{P}_+ \|_{L^2\left(\frac{1}{|h|^2}\right) \to L^2\left(\frac{1}{|h|^2}\right)} = c < \infty.$$

Now by a well-known interpolation theorem ([3], Ch. V), we have

$$\lim_{N\to\infty} \| Lz^N \mathbf{P}_+ \|_{L^2 \to L^2} = 0. \tag{9}$$

From (9) we deduce immediately that L is compact on H^2; but it is obvious that $L = H_u^* H_{\bar{v}}$.

Corollary. *If the symbols* u, v *are of the form* $u = \dfrac{\bar{h}_1}{h_1}$, $v = \dfrac{\bar{h}}{h}$, *where* $h, h_1 \in (HS)$ *and if the operator* $H_u^* H_v$ *is compact then the operators* $H_u^* H_{\bar{v}}$, $H_{\bar{u}}^* H_v$, $H_{\bar{u}}^* H_{\bar{v}}$ *are compact too.*

Theorem 1. *Suppose that the operator* $H_u^* H_v$ *is compact. Then*

$$H^\infty[u] \cap H^\infty[v] \subset H^\infty + C.$$

Proof. We know already that without loss of generality we can assume the functions u, v are unimodular and moreover that $u = \dfrac{\bar{h}_1}{h_1}$, $v = \dfrac{\bar{h}}{h}$, $h,\ h_1 \in (HS)$. Using the corollary above and Lemma 3 we deduce that

$$\lim_{|a|\to 1} \int_{\mathbf{T}} |u(\xi) - u(a)| \cdot |v(\xi) - v(a)| P_a(\xi) \, dm(\xi) = 0. \tag{10}$$

For a uniform algebra A, $H^\infty \subseteq A \subseteq L^\infty$ (i.e. a Douglas algebra) let $M(A)$ denote the maximal ideal space of A. For every $t \in M(H^\infty)$ let μ_t denote the (unique) representing measure of the homomorphism t; the (compact) support of this measure will be denoted by S_t, $S_t \subset M(L^\infty)$. Now we apply the following four well-known facts:

i) There is a homeomorphic embedding of the unit disc $\mathbf{D}$ into $M(H^\infty)$ and the disc is a dense subset of $M(H^\infty)$ (see [4]).

ii) For every function $f \in L^\infty$ let $\hat{f}(s)$, $s \in M(L^\infty)$, be the Gelfand transform of f. Then the function $t \mapsto \int_{S_t} \hat{f}(s) \, d\mu_t(s)$, defined on $M(H^\infty)$ is continuous. We denote $\hat{f}(t) \overset{\text{def}}{=} \int_{S_t} \hat{f}(s) \, d\mu_t(s)$, $t \in M(H^\infty)$. On the unit disc $\mathbf{D}$ the function $\hat{f}$ defined above is the harmonic continuation of the function f to the unit disc. (See [8]).

iii) $M(L^\infty) \subseteq M(A) \subseteq M(H^\infty)$ and $t \in M(A)$ if and only if the measure μ_t is multiplicative on A. The algebra A is uniquely determined by its maximal ideal space ([5]).

iv) $M(H^\infty + C) = M(H^\infty) \setminus \mathbf{D}$.

Let $t \in M(H^\infty)\setminus \mathbf{D}$. We have to prove that either $|\hat{u}(t)|=1$ or $|\hat{v}(t)|=1$. Suppose that $|\hat{u}(t)|, |\hat{v}(t)| < \lambda < 1$. Then using i), ii) we deduce that there is a sequence of points a_n, $|a_n| \to 1$ such that $|\hat{u}(a_n)|, |\hat{v}(a_n)| < \lambda$. But this contradicts (10) because the functions u, v are unimodular. Thus we have

$$M(H^\infty)\setminus \mathbf{D} = \{t \in M(H^\infty)\setminus \mathbf{D} : |\hat{u}(t)| = 1\} \cup \{t \in M(H^\infty)\setminus \mathbf{D} : |\hat{v}(t)| = 1\} \overset{\text{def}}{=} M_u \cup M_v .$$

Let $t \in M_u$, then

$$1 = |\hat{u}(t)| = \left|\int_{S_t} \hat{u}(s)\, d\mu_t(s)\right| \leqslant \int_{S_t} |\hat{u}(s)|\, d\mu_t(s) \leqslant 1,$$

and we deduce that $\hat{u}(s) \equiv$ const for $s \in S_t$. This means that the measure μ_t is multiplicative on the algebra $H^\infty[u]$ and using iii) and iv) we see that

$$M(H^\infty)\setminus \mathbf{D} \subset M(H^\infty[u]) \cup M(H^\infty[v]) \subset M(H^\infty[u] \cap H^\infty[v]).$$

Now from iii), iv) the conclusion of Theorem 1 follows.

3. BASES OF RATIONAL FRACTIONS WHICH ARE NEAR TO ORTHOGONAL BASES

It is known that the rational fractions $\{k_{\lambda_n}\}_{n\geqslant 1}$ form a Riesz basis in their linear span if and only if the Blaschke product $B = \prod_{n\geqslant 1} \frac{\bar{\lambda}_n}{|\lambda_n|} \frac{\lambda_n - \xi}{1 - \bar{\lambda}_n \xi}$ satisfies the following condition:

(C) $$\delta = \inf_n \prod_{k \neq n} \left| \frac{\lambda_k - \lambda_n}{1 - \bar{\lambda}_k \lambda_n} \right| > 0.$$

For every set $\sigma \subset \mathbf{N}$ let $B_\sigma = \prod_{n\in\sigma} \frac{\bar{\lambda}_n}{|\lambda_n|} \frac{\lambda_n - \xi}{1 - \bar{\lambda}_n \xi}$ and let $K_{B_\sigma} = H^2 \ominus B_\sigma H^2$ be the closed linear span of $\{k_{\lambda_n}\}_{n\in\sigma}$ in H^2. Let P_{B_σ} denote the orthogonal projection of H^2 onto K_{B_σ}, and let $\mathscr{P}_\sigma$ be the projection onto K_{B_σ} along $K_{B_{\mathbf{N}\setminus\sigma}}$. By the fact mentioned above, the condition (C) is equivalent to the following one:

$$\| P_{B_{\mathbf{N}\setminus\sigma}} P_{B_\sigma} \| \leqslant 1 - \varepsilon$$

for some $\varepsilon > 0$ and every $\sigma \subset \mathbf{N}$.

Now our aim is to find the condition ensuring that $\{k_{\lambda_n}\}$ is near an orthogonal basis in the following sense:

$$k_{\lambda_n} = (U + K)e_n$$

where $\{e_n\}_{n\geqslant 1}$ is the standard orthogonal basis of ℓ^2, U is unitary and K is a compact (or Hilbert-Schmidt) operator. Such bases will be called $U + \mathfrak{S}_\infty$ (resp. $U + \mathfrak{S}_2$) bases.

The two theorems below were conjectured by N. K. Nikolskiĭ.

THEOREM 2. *The following statements are equivalent:*

a) $\{k_{\lambda_n}\}_{n\geqslant 1}$ *is a* $U+\mathfrak{S}_\infty$ *basis;*

b) $\lim_{n\to\infty}\delta_n = 1$, *where* $\delta_n \overset{\text{def}}{=} |B_{\mathbf{N}\setminus\{n\}}(\lambda_n)|$;

c) *for every* $\sigma\subset\mathbf{N}$, $P_{B_{\mathbf{N}\setminus\sigma}}P_{B_\sigma}\in\mathfrak{S}_\infty$.

THEOREM 3. *The following statements are equivalent:*

a) $\{k_{\lambda_n}\}$ *is a* $U+\mathfrak{S}_2$ *basis;*

b) $\prod_{n\geqslant 1}\delta_n$ *converges;*

c) *for every* $\sigma\subset\mathbf{N}$, $P_{B_{\mathbf{N}\setminus\sigma}}P_{B_\sigma}\in\mathfrak{S}_2$.

Proof of Theorem 2. a) $\Rightarrow$ b). If $\{y_n\}$ is the biorthogonal system for $\{k_{\lambda_n}\}$ and $\{k_{\lambda_n}\}$ is a $U+\mathfrak{S}_\infty$ basis then $\|y_n\|\to 1$ and so the projection $\mathscr{P}_{\{n\}}$ satisfies $\|\mathscr{P}_{\{n\}}\| = \|y_n\|\cdot\|k_{\lambda_n}\|\to 1$. But it is easy to see that $\|\mathscr{P}_{\{n\}}\| = |B_{\mathbf{N}\setminus\{n\}}(\lambda_n)|^{-1}$.

b) $\Rightarrow$ a). We know that $\{k_{\lambda_n}\}$ is a Riesz basis and so there is an isomorphism $A:\ell^2\to K_B$, $k_{\lambda_n} = Ae_n$. We have to show that $A = U+K$. To prove this we need the main lemma from [2] and also some additional lemmas.

LEMMA 5. [2]. *There are two absolute constants C_1 and C_2 so that if B_1, B_2 are two Blaschke products with* $\inf_{\mathbf{D}}\max(|B_1|,|B_2|)\geqslant\delta$ *then* $\|H^*_{\bar{B}_1}H^*_{B_2}\|\leqslant C_1(1-\delta)^{C_2}$.

LEMMA 6. *Let B be a Blaschke product with zero sequence $\{\lambda_n\}$ and* $\delta\overset{\text{def}}{=}\inf_n|B_{\mathbf{N}\setminus\{n\}}(\lambda_n)|$. *Then for every two Blaschke products B_1, B_2 such that $B=B_1B_2$ we have*

$$\inf_{\mathbf{D}}\max(|B_1|,|B_2|)\geqslant\eta^2$$

where $\eta\overset{\text{def}}{=}\dfrac{1-\sqrt{1-\delta^2}}{\delta}$.

Proof. Let $\Delta(\lambda,\eta)\overset{\text{def}}{=}\left\{\xi\in\mathbf{C}:\left|\dfrac{\xi-\lambda}{1-\bar{\lambda}\xi}\right|<\eta\right\}$. Applying the Schwarz lemma to the functions $g_n(\xi)\overset{\text{def}}{=}B_{\mathbf{N}\setminus\{n\}}\left(\dfrac{z+\lambda_n}{1+\bar{\lambda}_n z}\right)$ we conclude immediately that $|B_{\mathbf{N}\setminus\{n\}}(\xi)|\geqslant\eta$ for $\xi\in\Delta(\lambda_n,\eta)$, $\eta=\dfrac{1-\sqrt{1-\delta^2}}{\delta}$. It is easy to see also that $\Delta(\lambda_n,\eta)\cap\Delta(\lambda_m,\eta)=\emptyset$, $n\neq m$.

Now let $B=B_1B_2$. If $z\notin\bigcup_{n\geqslant 1}\Delta(\lambda_n,\eta)$ then it is clear that $|B(z)|\geqslant\eta^2$ (as $|B(\xi)|\geqslant\eta^2$ for $\xi\in\bigcup_{n\geqslant 1}\partial\Delta(\lambda_n,\eta)$) and thus $\max(|B_1(z)|,|B_2(z)|)\geqslant\eta^2$. Now we suppose that $z\in\Delta(\lambda_n,\eta)$ and λ_n is a zero of B_1. Then $|B_2(z)|\geqslant|B_{\mathbf{N}\setminus\{n\}}(z)|\geqslant\eta$.

I am grateful to S. V. Kisljakov, who directed my attention to the following lemma.

LEMMA 7. *Let $x_n = Ae_n$ be a Riesz basis in a Hilbert space $\mathscr{H}$ (here A denotes an isomorphism between ℓ^2 and $\mathscr{H}$). If for every set $\sigma \subset \mathrm{N}$, $\|P_{\mathrm{N}\setminus\sigma}P_\sigma\| < \varepsilon$ then*

$$(1 - 4\varepsilon\|A\|^2) \leqslant A^*A \leqslant (1 + 4\varepsilon\|A\|^2).$$

Proof. Let $e = \sum_{k=1}^{N} c_k e_k$, $x \stackrel{\text{def}}{=} Ae$, $K \stackrel{\text{def}}{=} (A^*Ae, e) = \left\| \sum_1^N c_k x_k \right\|_H^2$, $\mathscr{L}(t) \stackrel{\text{def}}{=}$ $\stackrel{\text{def}}{=} \left\| \sum_1^N c_k \tau_k(t) x_k \right\|_H^2$, where $\{\tau_k(t)\}$ are Rademacher functions. For every fixed number $t \in [0, 1]$ let $\sigma_1 = \{k \in \mathrm{N}: \tau_k(t) = 1\}$, $\sigma_2 = \{k \in \mathrm{N}, \tau_k(t) = -1\}$. From the hypothesis of the lemma we know that the angle between $H_1 \stackrel{\text{def}}{=} V(x_k, k \in \sigma_1)$ and $H_2 \stackrel{\text{def}}{=} V(x_k, k \in \sigma_2)$ is such that $\cos(H_1, H_2) \leqslant \varepsilon$. Now if we denote by $\mathscr{P}_1$ the projection onto H_1 along H_2 and $\mathscr{P}_2 = I - \mathscr{P}_1$ then it is clear that

$$|K - \mathscr{L}(t)| = \left| \|\mathscr{P}_1 x + \mathscr{P}_2 x\|_H^2 - \|\mathscr{P}_1 x - \mathscr{P}_2 x\|_H^2 \right| \leqslant 4|(\mathscr{P}_1 x, \mathscr{P}_2 x)| \leqslant$$

$$\leqslant 4\cos(H_1, H_2)\|A\|^2 \sum_1^N |c_k|^2.$$

So we have

$$\left| K - \int_0^1 \mathscr{L}(t)\,\mathrm{d}t \right| \leqslant 4\cos(H_1, H_2)\|A\|^2 \sum_1^N |c_k|^2 \leqslant 4\varepsilon\|A\|^2 \sum_1^N |c_k|^2.$$

But it is easy to see that $\int_0^1 \mathscr{L}(t)\,\mathrm{d}t = \sum_1^N |c_k|^2$.

It is obvious that to finish the proof of condition a) it is sufficient (and necessary) to show that $A^*A = I + \mathfrak{S}_\infty$. Let Q_n denote the orthogonal projection of ℓ^2 on $V(e_k, k > n)$. It remains to show that $1 - \varepsilon_n \leqslant Q_n A^* A Q_n \leqslant 1 + \varepsilon_n$, $\varepsilon_n \to 0$. Now we note that $P_{B_1}P_{B_2} = UH_{\bar{B}_1}H^*_{\bar{B}_2}V$, where U is isometric and V is a partial isometry. The required estimate for $Q_n A^* A Q_n$ is an easy consequence of these lemmas and the fact that $\|P_{B_1}P_{B_2}\| = \|H^*_{\bar{B}_1}H_{\bar{B}_2}\|$ for every pair of Blaschke products B_1, B_2.

b) $\Rightarrow$ c). It is clear that for every set σ, $\sigma \subset \mathrm{N}$ we have

$$\lim_{|z|\to 1} \max(|B_{\mathrm{N}\setminus\{\sigma\}}(z)|, |B_\sigma(z)|) = 1$$

if B satisfies the condition b). Then by the main theorem of [2] we may conclude that $H_{\bar{B}_{\mathrm{N}\setminus\sigma}} H^*_{\bar{B}_\sigma} \in \mathfrak{S}_\infty$.

c) $\Rightarrow$ b). Here we use a very refined theorem, proved by K. Hoffman in [12].

THEOREM. (see [12]). *Every Blaschke product (with distinct zeroes) has a factorization as a product of two Blaschke products B_1, B_2, $B = B_1B_2$, such that:*

(i) *if α is a zero of B_1 then*

$$(1 - |\alpha|^2)|B_1^\nabla(\alpha)| \geqslant |B_2(\alpha)|;$$

(ii) *if α is a zero of B_2 then*

$$(1 - |\alpha|^2)|B_2^\nabla(\alpha)| \geqslant |B_1(\alpha)|.$$

Now let $\sigma = \{n \in \mathbf{N}: B_2(\lambda_n) = 0\}$. By the condition c), $P_{B_{\mathbf{N}\setminus\sigma}} P_{B_\sigma} \in \mathfrak{S}_\infty$, and so $P_{B_{\mathbf{N}\setminus\sigma}}|K_{B_\sigma} \in \mathfrak{S}_\infty$. But for every λ_m, $m \in \sigma$, we have

$$\|P_{B_{\mathbf{N}\setminus\sigma}} k_{\lambda_m}\|^2 = 1 - |B_{\mathbf{N}\setminus\sigma}(\lambda_m)|^2$$

and so $\lim\limits_{m\in\sigma, m\to\infty} |B_{\mathbf{N}\setminus\sigma}(\lambda_m)| = 1$. Using now the part (ii) of the above theorem we deduce that

$$\lim_{m\in\sigma, m\to\infty} |B_{\mathbf{N}\setminus\{m\}}(\lambda_m)| = 1.$$

Similarly we have $\lim\limits_{m\in\mathbf{N}\setminus\sigma, m\to\infty} |B_{\mathbf{N}\setminus\{m\}}(\lambda_m)| = 1.$

Proof of Theorem 3. a) $\Rightarrow$ b). Let Γ be the operator such that $\Gamma M_{\lambda_n} = k_{\lambda_n}$ where $M_{\lambda_n} = k_{\lambda_n} \prod\limits_{j<n} \frac{\bar{\lambda}_j}{|\lambda_j|} \frac{\lambda_j - \xi}{1 - \bar{\lambda}_j \xi}$. Then $\{k_{\lambda_n}\}$ is a $U + \mathfrak{S}_2$ basis if and only if $\Gamma = U + K$, $K \in \mathfrak{S}_2$. It is clear that $\Gamma = U + K$, $K \in \mathfrak{S}_2$, if and only if $\Gamma^*\Gamma = I + K_1$, $K_1 \in \mathfrak{S}_2$. But the matrix of this operator in the basis $\{M_{\lambda_n}\}$ is $\left(\frac{(1-|\lambda_i|^2)^{1/2}(1-|\lambda_j|^2)^{1/2}}{1 - \bar{\lambda}_i \lambda_j}\right)_{i,j=1}^\infty$. Thus this part of the theorem is proved.

b) $\Rightarrow$ c). Let $B = B_1B_2$, $\{\xi_n\}$ denote the zero sequence of B_2. We have to prove that $P_{B_1}|K_{B_2} \in \mathfrak{S}_2$. Let

$$b_n \stackrel{\text{def}}{=} \prod_{j<n} \frac{\bar{\xi}_n}{|\xi_n|} \frac{\xi_n - \xi}{1 - \bar{\xi}_n \xi},$$

$M_n \stackrel{\text{def}}{=} k_{\xi_n} b_n$. $\{M_n\}$ is an orthogonal basis in K_{B_2}.

$$\|P_{B_1} M_n\|^2 = \|P_- b_n \bar{B}_1 k_{\xi_n}\| = \|P_- b_n P_- \bar{B}_1 k_{\xi_n}\| \leqslant \|P_- \bar{B}_1 k_{\xi_n}\| =$$

$$= 1 - |B_1(\xi_n)|^2 \leqslant 1 - \delta_n^2 .$$

c) $\Rightarrow$ b). The proof of this part repeats the proof of the last part of Theorem 2.

REMARK. From the theorem of Earl (see [9] or [10]) or from one of the results of S. A. Vinogradov it follows that, if for an interpolating Blaschke product B with zero sequence $\{\lambda_n\}$ the quantity $\delta = \inf_n |B_{\mathbb{N}\setminus\{n\}}(\lambda_n)|$ is close to one, then the constant of interpolation is close to one, too. It is easy to see that here we have obtained another proof of this fact.

In conclusion the author wishes to express his deep gratitude to N. K. Nikolskiĭ and S. V. Hruschev for valuable discussion and constant interest to this work. The author thanks also S. V. Kisljakov for reading the manuscript.

REFERENCES

1. ADAMJAN, V. M.; AROV, D. Z.; KREĬN, M. G., Infinite Hankel matrices and generalized problems of Carathéodory-Fejér and I. Schur (Russian), *Funckional. Anal. i Priložen*, 2:4(1968), 1–17.
2. AXLER, S.; CHANG, S.-Y.A.; SARASON, D., Product of Toeplitz operators, *Integral Equations and Operator Theory*, **1**: 3(1978), 285–309.
3. BERGH, J.; LÖFSTRÖM, J., *Interpolation spaces*, Springer-Verlag, 1976.
4. CARLESON, L., The corona problem, *Lect. Notes in Math.*, Springer-Verlag, **118**(1972), 121–132.
5. CHANG, S.-Y.A., A characterisation of Douglas subalgebras, *Acta Math.*, **137**(1976), 81–89.
6. CHANG, S.-Y.A.; MARSHALL, D. E., Some algebras of bounded analytic functions containing the disc algebra, *Lect. Notes in Math.*, Springer-Verlag, **604**(1977), 12–20.
7. DOUGLAS, R. G., *Banach algebra technique in the theory of Toeplitz operators*, Reg. Conf. Ser. in Math. no. 15, Amer. Math. Soc., 1973.
8. GLEASON, A.; WHITNEY, H., The extension of linear functionals defined on H^∞, *Pacific J. Math.*, **12**(1962), 163–182.
9. EARL, J. P., On the interpolating of bounded sequences by bounded functions, *J. London Math. Soc.*, **2**(1970), 544–548.
10. HAVIN, V. P.; VINOGRADOV, S. A., Free interpolation in H^∞ and some other function classes (Russian), *Zap. Naučn. Sem. (LOMI)*, **47**(1974), 15–54.
11. HELSON, H.; SZEGÖ, G., A problem in prediction theory, *Ann. Mat. Pura Appl.*, **51**(1960), 107–138.
12. HOFFMAN, K., Bounded analytic functions and Gleason parts, *Ann. of Math.*, **86**(1967), 74–111.
13. HRUSCHEV, S. V.; PELLER, V. V., Hankel operators, best approximations and stationary Gaussian processes (Russian), *Uspehi Mat. Nauk.*, to appear.
14. NIKOLSKII, N. K., *Lectures on the shift operator* (Russian), Nauka, Moscow, 1980.

A. L. VOLBERG
Kosmonavtov 21-2-45,
196211 Leningrad,
U.S.S.R.

Received January 20, 1981.