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ON M-SPECTRAL SETS AND RATIONALLY INVARIANT SUBSPACES

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1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $\mathscr{A}$ is a subalgebra of $\mathscr{L}(\mathscr{H})$ a nontrivial $\mathscr{A}$-invariant subspace is a subspace $\mathscr{M}$ of $\mathscr{H}$ such that $(0) \neq \mathscr{M} \neq \mathscr{H}$ and such that $A\mathscr{M} \subset \mathscr{M}$ for each A in $\mathscr{A}$. If $\mathscr{A}$ is the algebra of all polynomials in a fixed operator A an $\mathscr{A}$-invariant subspace is exactly an invariant subspace for A and if $\mathscr{A}$ is the commutant of A an $\mathscr{A}$-invariant subspace is exactly an *hyperinvariant subspace* for A.

In 1978 S. Brown proved that every subnormal operator has a nontrivial invariant subspace (see [8]). The method he used was immediately adapted by various authors to obtain invariant subspaces for other classes of operators (see [1], [2], [3], [4], [10], [15]). The present paper is another one in the same area and can be regarded as a sequel to [10]. In that paper the authors first established a theorem that gives sufficient conditions for a representation of $H^\infty(G)$ into $\mathscr{L}(\mathscr{H})$ to have a nontrivial invariant subspace. (If G is a bounded open set in $\mathbf{C}$, $H^\infty(G)$ denotes the Banach algebra of functions bounded and analytic in G, equipped with the supremum norm.) This result paved the way for a systematic investigation of the existence of invariant subspaces for operators A for which there exists a bounded open set G such that

(1) G^- is an M-spectral set for A (i.e. $\|r(A)\| \leqslant M \sup_{\lambda\in G^-} |r(\lambda)|$ for every rational function r with poles off G^-), and

(2) the intersection of the spectrum of A, $\sigma(A)$, with G is dominating in G (i.e. $\|h\|_\infty = \sup_{\lambda\in\sigma(A)\cap G} |h(\lambda)|$ for any function h in $H^\infty(G)$). Eventually it was proved that if the boundary of G consists of finitely many disjoint Jordan loops then the operator A has a nontrivial subspace invariant under the rational functions of A with poles off G^-.

Our main result (Theorem 4.1) says that the same conclusion holds if, in addition to conditions (1) and (2), G is such that $R(G^-)$ is pointwise boundedly dense in $H^\infty(G)$ and $R(\partial G) = C(\partial G)$. (For a compact set X in **C** we denote as usual by $C(X)$ the Banach algebra of complex continuous functions on X equipped with the supremum norm and by $R(X)$ the closure in $C(X)$ of the algebra of rational functions with poles off X.) A corollary of this result (Theorem 8.1) takes care of the case in which G is finitely connected thus generalizing the above result of [10]. Our second application refers to the case when $\sigma(A)$ is an M-spectral set for A. We improve Stampfli's result [15] by showing (Theorem 8.2) that for any finite set of holes in $\sigma(A)$ there is a nontrivial subspace of $\mathscr{H}$ invariant under any rational function of A with poles in the union of these holes.

The paper is organized as follows. The first four sections are devoted to the proof of the basic result (Theorem 4.1). To apply Theorem 3.2 of [10] to our situation we need first to extend the representation of $R(G^-)$ into $\mathscr{L}(\mathscr{H})$ (obtained from the fact that G^- is M-spectral for A) into a representation of $H^\infty(G)$ and next to show that this extension satisfies the proper hypotheses. These two problems reduce essentially to establishing the w*-S.O.T. sequential continuity of the corresponding representations. (We say that a representation is w*-S.O.T. sequential continuous if it maps a sequence of functions that converges to 0 in the weak* topology into a sequence of operators that converges to 0 in the strong operator topology.) Of course these continuity difficulties appear more or less explicitly in any application of the S. Brown technique. So far they have been solved by transfering the problem to the unit disk where a result of Sz. Nagy-Foiaş (Theorem II 5.4 of [16]) enables one to assume that either A^n or A^{*n} tends to 0 in the strong operator topology. This transfer to the unit disk is precisely the basic limitation of most of the previous applications of the S. Brown technique (especially with regard to the type of invariant subspace produced). Our main innovation is to deal directly with the continuity difficulties via a result (Theorem 3.2) which, roughly speaking, generalizes the above theorem of Sz. Nagy-Foiaş and enables us to exhibit a nontrivial hyperinvariant subspace for A whenever the w*-S.O.T. sequential continuity property is not satisfied.

A consequence of this result is that the w*-S.O.T. sequential continuity of the representation can be removed from the hypotheses of Theorem 3.2 of [10] without affecting its conclusion, as was conjectured in that paper. The proof of our Theorem 3.2 is broken into two steps. In § 2 we show how the w*-S.O.T. sequential discontinuity implies the existence of nontrivial intertwinings between A and the operator M_ζ of multiplication by ζ on $R(\partial G)$, on one hand, and between A^* and M_ζ on $R(\partial G^*)$, on the other hand. In § 3 we show how these intertwinings lead to a nontrivial hyperinvariant subspace for A. Here the hypothesis $R(\partial G) = C(\partial G)$ plays a crucial role via the characterization of closed ideals in $C(X)$. The first author was initially lead to Theorem 3.2

via local spectral theory techniques; we briefly sketch this approach at the end of § 3. Section 4 completes the proof of Theorem 4.1. The applications of this theorem that we give rely heavily on the results of [13] on pointwise bounded approximation and Dirichlet algebras. The basic definitions and results that we need are presented in § 5 in a form suitable to our purposes; as a tool for generalizing Stampfli's result and in connection with Dirichlet algebras, we develop in § 6 a natural (and, we believe, interesting in its own right) partition of the set of holes of a connected compact set in the plane. Before concluding the proof of the applications in § 8 we need a few additional results on $H^\infty(G)$ which we present in § 7.

2. w*-S.O.T. SEQUENTIAL DISCONTINUITY OF REPRESENTATIONS AND INTERTWININGS

Throughout this section G is an arbitrary bounded open set in $\mathbf{C}$. Let Q be a subalgebra of $H^\infty(G)$ which contains $\mathrm{R}(G^-)$. A representation of Q is a Banach algebra homomorphism, Φ, of Q into $\mathscr{L}(\mathscr{H})$. We say that Φ is unital if $\Phi(1_G) = 1$. Naturally associated with Φ is an "adjoint" representation $\Phi^\sim$ defined as follows. Let $G^* = \{\bar{\lambda} : \lambda \in G\}$ and, for any function h defined on G, let h^* be the function defined on G^* by $h^*(\lambda) = \overline{h(\bar{\lambda})}$. Set $Q^\sim = \{h^* : h \in Q\}$ and define $\Phi^\sim$ on $Q^\sim$ by $\Phi^\sim(h) = \Phi(h^*)^*$. It is well-known (see, for example, Proposition 2.1 of [9]) that for any h in $H^\infty(G)$ and any μ in G there exists a unique function h_μ in $H^\infty(G)$ such that $h(\lambda) - h(\mu) = (\lambda - \mu)\, h_\mu(\lambda)$, $\lambda \in G$. We will say that Q satisfies condition (*) if h_μ belongs to Q whenever h belongs to Q. (Thus $H^\infty(G)$ itself and $\mathrm{R}(G^-)$ satisfy (*).) The following is the key result of this section.

THEOREM 2.1. *Let Q be a subalgebra of $H^\infty(G)$ which contains $\mathrm{R}(G^-)$ and satisfies (*) and let Φ be a norm-continuous unital representation of Q with the following properties:*

(a) *If f_0 ($f_0(\lambda) = \lambda$) is the position function and $\Phi(f_0) = A$ then A has no eigenvalues,*

(b) *the adjoint representation $\Phi^\sim$ is not* w*-S.O.T. *sequentially continuous.*

Then there exists a nonzero bounded linear map T from $\mathrm{R}(\partial G)$ into $\mathscr{L}(\mathscr{H})$ such that $TM_\zeta = AT$ where M_ζ denotes the operator of multiplication by ζ on $\mathrm{R}(\partial G)$.

Before proving this theorem we establish two lemmas.

LEMMA 2.2. *If $\Phi^\sim$ is not* w*-S.O.T. *sequentially continuous then there exists a sequence of functions f_n in Q converging pointwise boundedly to* 0 *and a sequence of unit vectors x_n in $\mathscr{H}$ such that $\Phi(f_n)x_n$ converges weakly to a nonzero vector y.*

Proof. If $\Phi^{\sim}$ is not w*-S.O.T. sequentially continuous then there exists a sequence f_n in Q converging pointwise boundedly to 0 and a vector x in $\mathscr{H}$ such that $(\Phi(f_n))^* x$ does not tend to 0. (We use here the obvious fact that the map $f \to f^*$ is an (isometric) weak* homeomorphism of Q onto $Q^{\sim}$; also recall that for sequences in $H^\infty(G)$ weak* convergence to 0 is equivalent to pointwise bounded convergence to 0.)

By selecting a subsequence we may assume that $\lim \|(\Phi(f_n))^* x\| =: a > 0$. Let $x_n = (\Phi(f_n))^* x_n \| (\Phi(f_n))^* x_n \|$; the sequence $\Phi(f_n) x_n$ is bounded and, again by dropping to a subsequence, we may assume that it is weakly convergent to some vector y. The equalities

$$(y, x) = \lim(\Phi(f_n)x_n, x) = \lim(x_n, \Phi(f_n)^* x) = \lim \| \Phi(f_n)^* x \|$$

show that $(y, x) \neq 0$; consequently y is nonzero as desired. ▨

The operator T of Theorem 2.1. will represent a sort of $\mathrm{R}(\partial G)$-functional calculus for A but localized on y. The following approximation lemma will enable us to "remove" the undesirable poles of rational functions in $\mathrm{R}(\partial G)$ (that is the poles that are in G).

LEMMA 2.3. *Let f_n be a sequence in Q converging pointwise boundedly to* 0 *and let φ be a rational function with poles off ∂G. Then there exists a sequence of polynomials P_n such that:*

1) $\|P_n\|_\infty$ *tends to* 0 *as n tends to ∞, and*

2) $\varphi(f_n - P_n)$ *belongs to Q and converges pointwise boundedly to* 0 *as n tends to ∞.*

Proof. Let $\lambda_1, \ldots, \lambda_k$ denote the poles of φ which are in G and let $\alpha_1, \ldots, \alpha_k$ be their multiplicity. (Of course if φ has no poles in G we set $P_n = 0$.) By ([14], Ch. V, § 2) we can find a system of polynomials $L_{j,i}$ such that $L_{j,i}^{(l)}(\lambda_m) = \delta_{j,l}\delta_{i,m}$ for $1 \leqslant i, m \leqslant k$ and $0 \leqslant l, j \leqslant \alpha_i$ (here δ is the Kronecker symbol and $h^{(l)}$ denotes the l-th derivative of h). Let P_n be the sequence of polynomials defined by

$$P_n = \sum_{i=1}^{k} \sum_{j=0}^{\alpha_i - 1} f_n^{(j)}(\lambda_i)\, L_{j,i}\,.$$

For each fixed pair j, i the sequence $f_n^{(j)}(\lambda_i)$ tends to 0 as $n \to \infty$ (Cauchy integral formula and Lebesgue dominated convergence theorem); this proves that $\{P_n\}$ satisfies 1). From the definition of the $L_{j,i}$'s it is clear that each λ_i is a zero of order $\geqslant \alpha_i$ for $f_n - P_n$. Thus we can write

$$(f_n - P_n)(\lambda) = \prod_{1 \leqslant i \leqslant k} (\lambda - \lambda_i)^{\alpha_i}\, g_n(\lambda).$$

By a repeated application of the property that f_μ is in Q whenever f is in Q and μ in G we see that g_n belongs to Q. From the inequality $\|f_\mu\|_\infty \leqslant 2\|f\|_\infty (\text{dist}(\mu, \partial G))^{-1}$ we see that $\{g_n\}$ is a bounded sequence. If λ is none of the λ_i's then $g_n(\lambda)$ clearly converges to 0. The convergence to 0 at the λ_i's now follows from the Cauchy integral formula for $g_n(\lambda_i)$. We have $\varphi(\lambda) = \prod_{1 \leqslant i \leqslant k} (\lambda - \lambda_i)^{-\alpha_i} \psi(\lambda)$ where ψ is a rational function with poles off G^-; therefore $\varphi(f_n - P_n)\ (= \psi g_n)$ converges pointwise boundedly to 0. ▨

Proof of Theorem 2.1. Since $\Phi^\sim$ is not w*-S.O.T. sequentially continuous, by Lemma 2.2 there exist a sequence f_n in Q converging weak* to 0 and a sequence of unit vectors x_n such that $\Phi(f_n)x_n$ converges weakly to a nonzero vector y. Let now $\varphi = R/S$ (R, S polynomials) be a rational function with poles off ∂G and let P_n be a sequence of polynomials given by Lemma 2.3 (with respect to φ and f_n). Since Φ is norm-continuous and $\varphi(f_n - P_m)$ converges weak* to 0, the sequence $\Phi(\varphi(f_n - P_n))x_n$ is bounded. Let u be the weak limit of any of its weakly convergent subsequences. We have

$$\begin{aligned} S(A)u &= \text{weak-lim}\, \Phi(S)\, \Phi(\varphi(f_{n_k} - P_{n_k}))x_{n_k} = \\ &= \text{weak-lim}\, \Phi(S\varphi(f_{n_k} - P_{n_k}))x_{n_k} = \\ &= \text{weak-lim}\, \Phi(R(f_{n_k} - P_{n_k}))x_{n_k} = \\ &= R(A)\, \text{weak-lim}\, \Phi(f_{n_k} - P_{n_k})x_{n_k} = \\ &= R(A)\, \text{weak-limit}\, \Phi(f_{n_k})\, x_{n_k} = \\ &= R(A)y. \end{aligned}$$

(The next-to-last equality follows from the fact that $\|P_n\|_\infty$ converges to 0 together with the norm-continuity of Φ. We have also used repeatedly the fact (easily deduced from the hypothesis that Φ is a unital representation) that $\Phi(r) = r(A)$ for any rational function, r, with poles off G^-, thus in particular for any polynomial.)

It follows from the equality $S(A)u = R(A)y$ that all weakly convergent subsequences have the same limit (otherwise by taking differences we would have a nonzero vector w such that $S(A)w = 0$, so A would have an eigenvalue). This result together with the metrizability of the weak topology on bounded subsets of $\mathcal{H}$ implies that in fact $\Phi(\varphi(f_n - P_n))x_n$ is weakly convergent; it also implies that the limit depends only on φ. Let then $T(\varphi)$ denote the limit. The linearity of T is immediate. To extend T to all of $\mathrm{R}(\partial G)$ we need a bound on $T(\varphi)$. We claim first that

$$\|\varphi(f_n - P_n)\|_\infty \leqslant \|\varphi\|_{\partial G}\|f_n - P_n\|_\infty$$

(where $\|\varphi\|_{\partial G} = \sup_{\lambda \in \partial G} |\varphi(\lambda)|$). Indeed let ε be any positive number; there is an open neighborhood Ω of ∂G such that the rational function φ is defined on Ω and such that for λ in Ω $|\varphi(\lambda)| \leqslant \|\varphi\|_{\partial G} + \varepsilon$; the maximum modulus principle implies that

$$\|\varphi(f_n - P_n)\|_\infty = \sup_{\lambda \in \Omega \cap G} |\varphi(\lambda)(f_n - P_n)(\lambda)|;$$

therefore we have

$$\|\varphi(f_n - P_n)\|_\infty \leqslant (\|\varphi\|_{\partial G} + \varepsilon)\|f_n - P_n\|_\infty,$$

and the desired result follows since ε is arbitrary. As a consequence we have

$$\|T(\varphi)\| \leqslant \|\Phi\| \|\varphi\|_{\partial K} \limsup \|f_n - P_n\|_\infty = C\|\varphi\|_{\partial K}$$

with $C = \|\Phi\| \limsup \|f_n\|$. Thus T has a unique (linear bounded) extension to $\mathrm{R}(\partial G)$. We still denote this extension by T.

To prove that T satisfies $TM_\zeta = AT$ let again $\varphi = R/S$ with R, S polynomials, the zeros of S lie off ∂G. By previous considerations we have

$$S(A)T(\varphi) = R(A)y.$$

The same equality with $f_0\varphi$ instead of φ leads to

$$S(A)T(f_0\varphi) = AR(A)y.$$

Combining both equalities we obtain

$$S(A)TM_\zeta(\varphi) = S(A)T(f_0\varphi) = AS(A)T(\varphi) = S(A)AT(\varphi).$$

Since $S(A)$ is one-to-one (otherwise A would have an eigenvalue) we conclude that

$$TM_\zeta(\varphi) = AT(\varphi).$$

By continuity the equality extends to any φ in $\mathrm{R}(\partial G)$ as was to be proved. ▨

3. INTERTWININGS WITH M_ζ AND HYPERINVARIANT SUBSPACES

The basic result of this section is the following theorem.

THEOREM 3.1. *Let X be a compact subset of the complex plane such that $\mathrm{R}(X) = \mathrm{C}(X)$ and let A be an operator in $\mathscr{L}(\mathscr{H})$ for which there exist nonzero bounded linear operators T and V from, respectively,* $\mathrm{R}(X)$ *and* $\mathrm{R}(X^*)$ *into* $\mathscr{H}$

such that

(1) $TM_\zeta = AT$ *and*

(2) $VM_\zeta^\sim = A^*V$

where M_ζ *and* $M_\zeta^\sim$ *denote the operator of multiplication by* ζ *on* R(X) *and on* R(X^*), *respectively. Then* A *has a nontrivial hyperinvariant subspace.*

Before proving this theorem we give two corollaries. The first one says that when $\mathrm{R}(\partial G) = \mathrm{C}(\partial G)$ one may assume that one of the two representations Φ and $\Phi^\sim$ (as considered in § 2) is w*-S.O.T. sequentially continuous without loss of generality with respect to the existence of hyperinvariant subspace for the operator $\Phi(f_0)$.

THEOREM 3.2. *Let* G *be a bounded open set in* $\mathbf{C}$ *such that* $\mathrm{R}(\partial G) = \mathrm{C}(\partial G)$, *and let* Q *be a subalgebra of* $H^\infty(G)$ *which contains* $\mathrm{R}(G^-)$ *and satisfies condition* (*). *Let* Φ *be a norm-continuous unital representation of* Q *into* $\mathcal{L}(\mathcal{H})$ *such that* $\Phi(f_0) = A$. *Then, if neither* Φ *nor* $\Phi^\sim$ *is* w*-S.O.T. *sequentially continuous,* A *has a nontrivial hyperinvariant subspace.*

Proof. By Theorem 2.1 applied to Φ and $\Phi^\sim$ we obtain non-zero bounded linear operators $T : \mathrm{R}(\partial G) \to \mathcal{H}$ and $V : \mathrm{R}(\partial G^*) \to \mathcal{H}$ such that $TM_\zeta = AT$ and $VM_\zeta^\sim = A^*V$. Since $\mathrm{R}(\partial G) = \mathrm{C}(\partial G)$ the conclusion now follows from Theorem 3.1. ▨

The second corollary shows that the conjecture raised in [10] at the end of § 3 and proved to be true in the special case in which G is a finitely connected domain ([10], Theorem 4.1) is true for any domain G.

THEOREM 3.3. *Let* G *be a bounded domain in* $\mathbf{C}$ *and let* Φ *be a representation of* $H^\infty(G)$ *into* $\mathcal{L}(\mathcal{H})$ *with the following properties:*

(a) Φ *is norm-continuous;*

(b) *If* $f_0(\zeta) = \zeta$ *is the position function in* $H^\infty(G)$ *and* $\Phi(f_0) = A$, *then* $\sigma(A) \cap G$ *is dominating in* G.

Then Φ *has a nontrivial invariant subspace, i.e. there is a nontrivial* $\Phi(H^\infty(G))$-*invariant subspace.*

Proof. We begin by observing that $\mathrm{R}(\partial G) = \mathrm{C}(\partial G)$. Indeed the complement of G in the Riemann sphere S^2 is a compact set K. By a result recalled at the beginning of § 6 it follows from the connectedness of G that the algebra R(K) is Dirichlet (see § 5 for definitions) and therefore $\mathrm{R}(\partial K) = \mathrm{C}(\partial K)$. Since $\partial K = \partial G$ the observation is proved.

If neither Φ nor $\Phi^\sim$ is w*-S.O.T. sequentially continuous, A has a nontrivial hyperinvariant subspace by Theorem 3.2 and, since the range of Φ is contained in the commutant of A, Φ has a nontrivial invariant subspace. On the other hand if either Φ or $\Phi^\sim$ is w*-S.O.T. sequentially continuous the conclusion follows from Theorem 3.2 of [10] (applied to either Φ or $\Phi^\sim$) together with the fact that the orthocomplement of an invariant subspace of $\Phi^\sim$ is invariant for Φ. ▨

The proof of Theorem 3.1 will be broken into a few lemmas. Of course there is no loss of generality in assuming that neither A nor A^* has eigenvalues and we do so throughout the remainder of this section. We will denote by $\mathcal{S}$ the set of operators T satisfying (1) and by $\mathcal{S}^\sim$ the set of operators V satisfying (2). It is clear that any result about $\mathcal{S}$ has a dual version about $\mathcal{S}^\sim$. Though we do not state these dual versions we will use them freely. The first lemma summarizes some elementary properties of $\mathcal{S}$. We omit the proof.

LEMMA 3.3. *The set* $\mathcal{S}$ *is a submanifold of* $\mathcal{L}(\mathrm{R}(X), \mathcal{H})$. *Moreover for any* φ *in* $\mathrm{R}(X)$, *any* B *in the commutant of* A, *and any* T *in* $\mathcal{S}$ *the operators* TM_φ *and* BT *belong to* $\mathcal{S}$. *(*M_φ *denotes the operator of multiplication by* φ *on* $\mathrm{R}(X)$.*)*

The key properties of an intertwining T in $\mathcal{S}$ are gathered in the following lemma.

LEMMA 3.4. *Let* T *belong to* $\mathcal{S}$. *Then we have the following properties :*

a) $\mathrm{Ker}T$ *is a closed ideal of* $\mathrm{R}(X)$.

b) *Let* $s(T) = \{\lambda \in X : f(\lambda) = 0$ *for all* f *in* $\mathrm{Ker}T\}$; *then* $\mathrm{Ker}T = \{f \in \mathrm{C}(X) : f|s(T) = 0\}$.

c) *There exists a bounded linear operator* $\hat{T} : \mathrm{C}(s(T)) \to \mathcal{H}$ *such that* $\hat{T} M_\zeta = A\hat{T}$ *where* M_ζ *is now multiplication by* ζ *on* $\mathrm{C}(s(T))$.

d) *Suppose* $T \neq 0$ *and let* λ_0 *belong to* $s(T)$; *then there exists a nonzero operator* T_1 *in* $\mathcal{S}$ *such that*

$$s(T_1) \subset \{\lambda \in X : |\lambda - \lambda_0| \leqslant \varepsilon\}.$$

Proof. a) Since $\mathrm{Ker}T$ is a closed subspace of $\mathrm{R}(X)$ we just have to prove that $f\varphi$ belongs to $\mathrm{Ker}T$ whenever f is in $\mathrm{R}(X)$ and φ in $\mathrm{Ker}T$. Using the density of the rational functions with poles off X in $\mathrm{R}(X)$ we see that it is enough to prove this result when f is one of these rational functions, say p/q. From $AT = TM_\zeta$ we obtain $p(A)T = TM_p$ for any polynomial p. Thus

$$q(A)T((p/q)\varphi) = TM_q((p/q)\,\varphi) = T(p\varphi) = p(A)T(\varphi) = 0.$$

Since $q(A)$ is one-to-one (otherwise A would have an eigenvalue) we have $T((p/q)\,\varphi) = 0$ as was to be proved.

b) Since $\mathrm{R}(X) = \mathrm{C}(X)$ the result follows immediately from the well-known characterization of the closed ideals of $\mathrm{C}(X)$.

c) Let r denote the restriction map from $\mathrm{C}(X)$ into $\mathrm{C}(s(T))$. Since r is onto and $\mathrm{Ker}\, r \subset \mathrm{Ker}T$ (in fact $\mathrm{Ker}\, r = \mathrm{Ker}T$) there exists a linear map $\hat{T} : \mathrm{C}(s(T)) \to \mathcal{H}$ such that $T = \hat{T}r$. The verification of the equality $\hat{T}M_\zeta = A\hat{T}$ is straightforward. To prove that $\hat{T}$ is bounded let g_n be a sequence converging to 0 in $\mathrm{C}(s(T))$. It follows from the classical Tietze's extension theorem that for each n there exists a function f_n in $C(X)$ such that $r(f_n) = g_n$ and $\|g_n\| = \|f_n\|$. Thus $\|\hat{T}(g_n)\| (= \|Tf_n\| \leqslant \leqslant \|T\| \, \|f_n\|)$ tends to zero and $\hat{T}$ is bounded.

d) From the definition of $s(T)$ we obtain easily that for any T in $\mathscr{S}$ and any φ in $C(X)$ $s(TM_\varphi) = s(T) \cap \operatorname{supp}\varphi$. (Recall that $\operatorname{supp}\varphi = \{\lambda \in X : \varphi(\lambda) \neq 0\}^-$.) Choose φ to be nonzero in a neighborhood of λ_0 but zero for $|\lambda - \lambda_0| \geqslant \varepsilon$. The operator $T_1 = TM_\varphi$ satisfies the desired properties. ▨

The next and last lemma expresses the basic idea which we will use to obtain hyperinvariant subspaces. It ultimately boils down to the well-known result that if two operators are nontrivially intertwined then their spectra must overlap.

LEMMA 3.5. *Let T belong to $\mathscr{S}$ and V to $\mathscr{S}^\sim$. Then:*

a) *$s(T)$ cannot be a singleton, and*

b) *if $s(T) \cap s(V)^* = \emptyset$ then* $\operatorname{Ran}(T) \perp \operatorname{Ran}(V)$.

Proof. a) If $s(T)$ is a singleton $\{\lambda_0\}$ then $C(s(T))$ is one-dimensional and it follows from c) of Lemma 3.4 that λ_0 is an eigenvalue for A. This contradicts the assumption that A has empty point spectrum.

b) By c) of Lemma 3.4 and its dual version for $\mathscr{S}^\sim$ we may see T and V as operators defined on $C(s(T))$ and $C(s(V))$ respectively. We want to "dualize" the intertwining $A^*V = VM_\zeta^\sim$; to avoid the difficulty created by the mixture of Banach space and Hilbert space dualities we proceed directly as follows. For x in $\mathscr{H}$ we define a continuous linear functional $V'(x)$ on $C(s(V)^*)$ by

$$(3) \qquad \langle V'(x), f \rangle = \langle x, V(f^*) \rangle, \quad f \in C(s(V)^*).$$

It is easy to check that V' is a bounded linear map from $\mathscr{H}$ into $\mathscr{M}(s(V)^*)$ (the Banach space of complex Borel measures on $s(V)^*$) and that $V'A = M'_\zeta V'$ where M'_ζ is multiplication by ζ on $\mathscr{M}(s(V)^*)$ (i.e. $M'_\zeta(\mu) = \nu$ with $d\nu = \zeta d\mu$). Finally the definition of V' makes it clear that $\operatorname{Ker}V' = (\operatorname{Ran}V)^\perp$. Thus we have to prove that $V'T = 0$.

Let $W = V'T$; from the intertwinings $TM_\zeta = AT$ and $V'A = M'_\zeta V'$ we obtain $V'TM_\zeta = V'AT = M'_\zeta V'T$ that is $M'_\zeta W = WM_\zeta$. But the spectra, $s(V)^*$ and $s(T)$, of M'_ζ and M_ζ are disjoint; therefore by a well-known result we see that $W = 0$, as was to be shown. ▨

With these tools at hand it is now easy to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let T and V be nonzero operators in $\mathscr{S}$ and $\mathscr{S}^\sim$ respectively. Since neither $s(T)$ nor $s(V)$ can be a singleton we can find λ in $s(T)$ and μ in $s(V)$ such that $\lambda \neq \bar{\mu}$. Applying d) of Lemma 3.4 (and its dual version for $\mathscr{S}^\sim$) with $\varepsilon < |\lambda - \bar{\mu}|/2$ we obtain nonzero operators T_1 and V_1 respectively in $\mathscr{S}$ and $\mathscr{S}^*$ such that $s(T_1) \cap s(V_1)^* = \emptyset$. For any operator B commuting with A, BT_1 is in $\mathscr{S}$ and satisfies $s(BT_1) \cap s(V_1)^* = \emptyset$. By b) of Lemma 3.5 we have $\operatorname{Ran}(BT_1) \perp \operatorname{Ran} V_1$ and the closed linear span of $\bigcup_{BA=AB} B \operatorname{Ran} T_1$ is a nontrivial hyperinvariant subspace for A. ▨

REMARK. As mentioned in the introduction, Corollary 3.2 can be proved using local spectral techniques (see [6], for basic definitions and results). In that context Theorem 2.1 is replaced by a result saying that (under the same hypotheses) there exists a nonzero vector y whose local spectrum with respect to A (notation $\sigma_A(y)$) is contained in ∂G. Instead of Lemma 3.4 we have a proposition stating that once we have a nonzero vector y such that $\sigma_A(y) \subset \partial G$ we can find (under the assumption $\mathrm{R}(X) = \mathrm{C}(X)$) another vector $y_1 \neq 0$ with $\sigma_A(y_1) \subset X \cap \cap \Delta(\lambda_0, \varepsilon)$ ($\lambda_0 \in \sigma_A(y)$, ε arbitrary) and such that the local resolvent satisfies a certain growth condition. Finally Lemma 3.5 is replaced by the following two results of local spectral theory.

1. *If $\sigma_A(y) = \{\lambda_0\}$ and the local resolvent of y, $\rho_{A,y}(\lambda)$ satisfies $\|\rho_{A,y}(\lambda)\| \leqslant \leqslant M|\lambda - \lambda_0|^{-k}$ for some k then λ_0 is an eigenvalue of A.*

2. *If $\sigma_A(x) \cap (\sigma_{A^*}(y))^* = \emptyset$ (for nonzero vectors x and y) then A has a nontrivial hyperinvariant subspace.*

4. THE MAIN THEOREM

We are now ready to prove the central result of the paper which is the following.

THEOREM 4.1. *Let A be an operator in $\mathscr{L}(\mathscr{H})$ and let G be a bounded open set in $\mathbf{C}$ such that:*

(i) *G^- is an M-spectral set for A,*

(ii) *$\sigma(A) \cap G$ is dominating in G,*

(iii) *$\mathrm{R}(G^-)$ is pointwise boundedly dense in $H^\infty(G)$,*

and

(iv) $\mathrm{R}(\partial G) = \mathrm{C}(\partial G)$.

Then there exists a nontrivial subspace invariant under any rational function of A with poles off G^-.

Proof. First we claim that we may assume that $G = (G^-)^0$. Indeed let $G_1 = = (G^-)^0$. Since $G_1^- = G^-$, Hypothesis (i) and the conclusion are unaffected when we replace G by G_1. The inclusions $G \subset G_1$, $\partial G_1 \subset \partial G$ together with the maximum principle imply that the natural embedding $H^\infty(G_1) \subset H^\infty(G)$ is an isometry; thus $\sigma(A) \cap G$ is dominating in G_1 and consequently so is $\sigma(A) \cap G_1$. Therefore G_1 satisfies (ii). That G_1 satisfies (iii) follows immediately from the inclusion $H^\infty(G_1) \subset \subset H^\infty(G)$. Finally an easy argument based on the norm decreasing inclusion $\mathrm{R}(\partial G) \subset \subset \mathrm{R}(\partial G_1)$ and the fact that any f in $\mathrm{C}(\partial G_1)$ has a continuous extension to ∂G, of norm no greater than $\|f\|$, shows that $\mathrm{R}(\partial G_1) = \mathrm{C}(\partial G_1)$; that is G_1 satisfies (iv). Thus we may pursue the proof with the additional hypothesis that $G = (G^-)^0$.

Let ψ denote the representation from $\mathrm{R}(G^-)$ defined by $\psi(r) = r(A)$ (ψ is a priori defined only for a r rational with poles off G^- but, since G^- is an M-spectral set for A, it extends by continuity to all of $\mathrm{R}(G^-)$; for details on this extension see Proposition 5.1 of [10]). By Theorem 3.2 there is no loss of generality in assuming that one — say ψ — of the two representations ψ and $\psi^\sim$ is w*-S.O.T. sequentially continuous. Suppose that we can extend ψ into a norm-continuous, w*-S.O.T. sequentially continuous representation Φ of $H^\infty(G)$; then by Theorem 3.2 of [10] there is a nontrivial invariant subspace for $\Phi(H^\infty(G))$. Thus, since the range of Φ contains all rational functions with poles off G^-, it is sufficient to extend ψ as indicated above to complete the proof. The argument is essentially the same as in ([10], Theorem 7.3) with additional details to obtain directly the w*-S.O.T. sequential continuity of the extension.

We denote by R_ρ, H_ρ, and $\mathscr{B}_\rho$ the balls of radius ρ about the origin in $\mathrm{R}(G^-)$, $H^\infty(G)$ and $\mathscr{L}(\mathscr{H})$. Also, as usual, if E is a set and d a metric on E we denote by (E, d) the topological space (E, τ) where τ is the topology provided by the metric d. We observe first that the weak* closure of R_ρ in $H^\infty(G)$ is H_ρ. Indeed H_ρ is weak* closed (this is true for any ball in the dual of a Banach space) and by Theorem 6.9 of [13] $\mathrm{R}(G^-)$ is strongly pointwise boundedly dense in $H^\infty(G)$; that is for any g in $H^\infty(G)$ there exists a sequence r_n in $\mathrm{R}(G^-)$ which is weak* convergent to g and satisfies the additional condition $\|r_n\| \leqslant \|g\|$ for all n (note that it is precisely to have this strongly pointwise bounded density that we need the assumption $(G^-)^0 = G$). Our observation follows easily from these two facts. Next recall (fromq, for example, [7, Problem 15 N]) that there exists a translation invariant metric d on $H^\infty(G)$ such that

$$(H_\rho, d) = (H_\rho, \mathrm{w}^*) \quad \text{for all } \rho.$$

Similarly by [12, p. 33] there exists a translation invariant metric d' on $\mathscr{L}(\mathscr{H})$ such that

$$(\mathscr{B}_\rho, d') = (\mathscr{B}_\rho, \mathrm{S.O.T.}) \quad \text{for all } \rho.$$

Therefore the sequentially continuous map $\psi : (R_\rho, \mathrm{w}^*) \to (\mathscr{B}_{M\rho}, \mathrm{S.O.T.})$ is actually continuous. An elementary argument based on the continuity at 0 of $\psi : (R_{2\rho}, \mathrm{w}^*)$ $(= (R_{2\rho}, d)) \to (\mathscr{B}_{2M\rho}, \mathrm{S.O.T.})$ $(= (\mathscr{B}_{2M\rho}, d'))$ shows that in fact ψ is uniformly continuous from (R_ρ, w^*) into $(\mathscr{B}_{M\rho}, \mathrm{S.O.T.})$. Since the latter is complete ψ has a unique uniformly continuous extension $\psi_\rho : (H_\rho, \mathrm{w}^*) \to (\mathscr{B}_{M\rho}, \mathrm{S.O.T.})$. The uniqueness of ψ_ρ implies that ψ_σ is an extension of ψ_ρ whenever $\sigma \geqslant \rho$. Consequently by setting $\Phi(g) = \psi_\rho(g)$ where $g \in H^\infty(G)$ and $\|g\| \leqslant \rho$ we obtain a well-defined extension of ψ to $H^\infty(G)$. The above considerations show clearly that Φ satisfies $\|\Phi(g)\| \leqslant M\|g\|$ for any g in $H^\infty(G)$ and that Φ is w*-S.O.T. sequentially continuous.

In particular we have

$$\Phi(g) = (\text{S.O.T.})\text{-lim}\, r_n(A)$$

whenever the sequence $\{r_n\} \subset \mathrm{R}(G^-)$ converges weak* to g. From this equality we easily deduce the linearity of Φ as well as its multiplicativity (for the latter recall that the product in $\mathscr{L}(\mathscr{H})$ is jointly sequentially strongly continuous). Therefore Φ is the desired extension of ψ and the proof is complete. ▨

REMARKS. 1. We could have used the somewhat less involved argument. of [10] to extend ψ into a norm-continuous representation of $H^\infty(G)$ into $\mathscr{L}(\mathscr{H})$ and then conclude by Theorem 3.3. However it seems of interest to show that if ψ is w*-S.O.T. sequentially continuous (from $\mathrm{R}(G^-)$ into $\mathscr{L}(\mathscr{H})$) then it has a norm--continuous, w*-S.O.T. sequentially continuous extension to $H^\infty(G)$.

2. The above remark suggests the following question:

Let ψ be a norm-continuous, w*-S.O.T. sequentially continuous representation from $\mathrm{R}(G^-)$ into $\mathscr{L}(\mathscr{H})$ and assume that $\mathrm{R}(G^-)$ is strongly pointwise boundedly dense in $H^\infty(G)$. Does ψ have a unique norm-continuous extension to $H^\infty(G)$?

Since the above proof actually shows that ψ has a unique norm-continuous, w*-S.O.T. sequential continuous extension the question is equivalent to whether a norm-continuous representation of $H^\infty(G)$ whose restriction to $\mathrm{R}(G^-)$ is w*-S.O.T. sequentially continuous is itself w*-S.O.T. sequentially continuous.

We note that Theorem 7.5 of [10] answers these questions in the affirmative in the case in which G is a circular domain.

5. C-SETS AND D-SETS IN S^2

In this section we develop some material on Dirichlet algebras necessary for our applications of Theorem 4.1. Most of it is contained in [13] where characterizations of conditions (3) and (4) (in Theorem 4.1) and of Dirichlicity of $\mathrm{R}(K)$ are given in terms of analytic capacity. Since we need to extend some of these results to compact subsets of the Riemann sphere we recall the basic definitions. We deal with the usual model of the Riemann sphere: $\mathrm{S}^2 = \mathbf{C} \cup \{\infty\}$. Though most of what we say applies to any compact subset of S^2 we are interested only in ordinary compact subsets of $\mathbf{C}$ or in complements of bounded open subsets of $\mathbf{C}$ (in other words we never consider compact subsets K of S^2 such that $\infty \in \partial K$). The usual definitions of $\mathrm{C}(K)$, $\mathrm{R}(K)$ and $H^\infty(\mathring{K})$ extend obviously to the case $\infty \in \mathring{K}$ (considering that ∞ is a pole of a rational function f if 0 is a pole of $f(1/\zeta)$ and that f is analytic at ∞ if $\zeta \to f(1/\zeta)$ is analytic at 0). We say that K is a *Dirichlet set* (or briefly a D-*set*) if K is a compact set such that $\mathrm{R}(K)$ is a Dirichlet algebra (i.e. $\mathrm{Re}(\mathrm{R}(K))$ is dense in $\mathrm{C_R}(\partial K)$). We say that the compact set K is a C-*set* if $\mathrm{R}(\partial K) = \mathrm{C}(\partial K)$

and $R(K)$ is pointwise boundedly dense in $H^\infty(\mathring{K})$. A *circular transformation* of S^2 is a map $\varphi : S^2 \to S^2$ of the form $\varphi(\zeta) = (a\zeta + b)/(c\zeta + d)$ with $ad - bc \neq 0$. If φ is a circular transformation such that $\varphi(K) = K_1$ then the map $f \to f \circ \varphi$ maps $R(K_1)$, $R(\partial K_1)$, $H^\infty(\partial K_1)$ and $C(\partial K_1)$ isometrically onto, respectively, $R(K)$, $R(\partial K)$, $H^\infty(K)$ and $C(\partial K)$, and preserves pointwise convergence. Consequently $\varphi(K)$ is a C-set (resp. a D-set) if and only if K is a C-set (resp. a D-set). Using a circular transformation of the type $\varphi(\zeta) = 1/\zeta - \zeta_0$ ($\zeta_0 \notin K$) we see that the following results (known to be true in the case of compact subsets of the plane) remain valid in the case when $\infty \in \mathring{K}$.

PROPOSITION 5.1. ([13], Theorem 5.1). *Let K be a compact subset of S^2 ($\infty \notin \partial K$). Then the following are equivalent:*

(i) *K is a D-set.*

(ii) *K is a C-set and each component of $\mathring{K}$ is simply connected.*

PROPOSITION 5.2. ([13], Corollary 9.6). *The intersection of countably many, decreasing D-sets is a D-set.*

We now turn our attention to the characterization of C-sets and D-sets which involve analytic capacity (denoted by γ). Again we merely adjust results of [13] to our needs. (Here and elsewhere $\Delta(\zeta; \delta)$ is the open disk of radius δ centered at ζ.)

PROPOSITION 5.3. ([13], Theorem 8.9). *Let K be a compact subset of S^2 ($\infty \notin \partial K$). Then the following are equivalent*

(i) *K is a C-set,*

(ii) *There exists $\delta_0 > 0$ such that $\gamma(\Delta(\zeta; \delta) \setminus K) = \gamma(\Delta(\zeta; \delta) \setminus \mathring{K})$ for each $\zeta \in \partial K$ and $0 < \delta < \delta_0$.*

(iii) *There exists a σ-curvilinear null set E such that for each $\zeta \in (\partial K) \setminus E$ there exists $r \geqslant 1$ satisfying*

$$\liminf_{\delta \to 0} \frac{\gamma(\Delta(\zeta; r\delta) \setminus K)}{\gamma(\Delta(\zeta; \delta) \cap \partial \mathring{K})} > 0.$$

Proof. When $K \subset \mathbf{C}$ this is Theorem 8.9 of [13] (the "localized" version of (ii) clearly implies (iii)).

Suppose now that ∞ belongs to $\mathring{K}$. Let Δ be an open disk large enough to contain the closure of $S^2 \setminus K$. The transformation $\varphi : \varphi(\zeta) = 1/(\zeta - a)$ (where a is a fixed point in $S^2 \setminus K$) maps $K \cap \Delta^-$ into $K_1 \setminus \Delta_1$ where $K_1 = \varphi(K)$ and $\Delta_1 = \varphi(S^2 \setminus \Delta^-)$; Δ_1 is an open disk whose closure is contained in K_1. Observing that conditions (ii) and (iii) are automatically satisfied on $\partial \Delta_1$ we see that K_1 is a C-set if and only if $K_1 \setminus \Delta_1$ is a C-set. Consequently K is a C-set if and only if $K \cap \Delta^-$ is a C-set. The desired equivalences for K now follow from their counterparts for $K \cap \Delta^-$ (observe again that the latter is a compact subset of $\mathbf{C}$ for which (ii) and (iii) are automatically satisfied on $\partial \Delta$). ▨

An immediate consequence of this result is that the characterization of D-sets (Theorem 9.3 of [13]) can be extended to compact sets K such that $\infty \in \mathring{K}$.

The following lemma will lead us to a somewhat more convenient version of this characterization when K is connected.

LEMMA 5.4. *Let K be a compact, connected subset of* S^2. *Then the following are equivalent:*

(i) *∂K is connected;*

(ii) *Each component of $\mathring{K}$ is simply connected.*

Proof. Suppose that a component G of $\mathring{K}$ is not simply connected and let L_1 and L_2 be two nonempty compact disjoint subsets of S^2 such that $S^2 \setminus G = L_1 \cup L_2$. From the inclusion $S^2 \setminus K \subset S^2 \setminus G$ we obtain $\partial K = (\partial K \cap L_1) \cup (\partial K \cap L_2)$. Since ∂K contains each ∂L_i (because $\partial L_i \subset \partial G \subset \partial K$) we have a nontrivial splitting of ∂K into two disjoint compact sets. This proves that (i) implies (ii).

To prove the converse we use the following observation whose proof we omit. The boundary of a simply connected domain is connected. Suppose now that each component of K is simply connected and let $\partial K = L_1 \cup L_2$ be a splitting of ∂K into two disjoint compact sets. By the above observation for each component G of K, ∂G is a connected set; hence we have either $\partial G \subset L_1$ or $\partial G \subset L_2$. Let V_i denote the union of the components G of K such that $\partial G \subset L_i$ $(i = 1, 2)$ and let $K_i = L_i \cup V_i$. Since $K_1 \cap K_2 = \emptyset$ and $K = K_1 \cup K_2$ the proof will be completed once we show that each K_i is closed. Let then λ_n be a sequence in, say, K_1 that converges to λ. We can assume that $\lambda \notin V_1$ (otherwise we are done). We define a sequence μ_n in L_1 as follows: $\mu_n = \lambda_n$ whenever $\lambda_n \in L_1$; if $\lambda_n \notin L_1$ then λ_n belongs to a component G of $\mathring{K}$ satisfying $\partial G \subset L_1$; in that case we choose μ_n to be the point of ∂G nearest to λ on the segment with endpoints λ_n and λ. In all cases $\mu_n \in L_1$ and $|\lambda - \mu_n| \leqslant |\lambda - \lambda_n|$; therefore $\lambda = \lim \mu_n$ and λ belongs to L_1, hence to K_1 as desired. ▨

We now state a convenient characterization of connected D-sets.

THEOREM 5.5. *Let K be a connected compact subset of* S^2 $(\infty \notin \partial K)$. *Then the following are equivalent:*

(i) *K is a* D-*set.*

(ii) *∂K is connected and there exists $\delta_0 > 0$ such that, for ζ in ∂K and* $0 < \delta < \delta_0$, $\gamma(\Delta(\zeta; \delta) \setminus K) \geqslant \delta/4$.

Proof. That (i) implies (ii) follows from the previous lemma together with the equivalence of (i) and (iii) in Theorem 9.3 of [13].

Now suppose that condition (ii) holds; since

$$\gamma(\Delta(\zeta; \delta) \cap \partial \mathring{K}) \leqslant \gamma(\Delta(\zeta; \delta)) = \delta$$

we get at each ζ of ∂K

$$\gamma(\Delta(\zeta;\delta)\setminus K)/\gamma(\Delta(\zeta;\delta)\cap\partial\mathring{K}) \geqslant 1/4;$$

therefore K is a C-set by Proposition 5.3; by Lemma 5.4 each component of $\mathring{K}$ is simply connected; thus K is a D-set (Proposition 5.1). ▨

6. DIRICHLET CHAINS FOR CONNECTED COMPACT SETS

In this section K denotes a fixed connected compact subset in the complex plane. (It will be convenient to consider K as embedded in S^2; the unbounded component of the complement of K will be identified as the hole that contains ∞.) Let $\mathscr{G}$ denote the set of holes in K. For each hole H the set $K_H = S^2\setminus H$ is a D-set. (This is a well-known result if $\infty\in H$; a suitable circular transformation transfers the result to any hole — recall that, here, since K is connected each hole is simply connected). Now if H' is another hole "touching" H (i.e. $\partial H\cap\partial H'\neq\emptyset$) Theorem 5.5 (or Corollary 9.7 of [13]) shows that $S^2\setminus(H\cup H')$ is still a D-set. This process can be repeated and motivates the following definition. A set $\mathscr{C}$ of holes (i.e. a subset of $\mathscr{G}$) is called a *Dirichlet chain* (or shortly a D-*chain*) for K if the (compact) set $K_{\mathscr{C}} = S^2\setminus\bigcup_{H\in\mathscr{C}} H$ is a D-set. We denote by $\mathscr{F}$ the set of D-chains for K ordered by inclusion. Finally we define the *boundary of a* D-*chain* (notation $\partial\mathscr{C}$) to be the boundary of the corresponding D-set $K_{\mathscr{C}}$. Since $K_{\mathscr{C}}$ is connected (it is the union of a connected compact set with some of its holes) and Dirichlet its boundary is connected. Note that $\partial\mathscr{C}\,(=\partial K_{\mathscr{C}}) = (\bigcup_{H\in\mathscr{C}}\partial H)^-$. We begin with an elementary but useful result.

LEMMA 6.1. *The union of two Dirichlet chains whose boundaries overlap is a Dirichlet chain.*

Proof. Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two D-chains and let $\mathscr{C}=\mathscr{C}_1\cup\mathscr{C}_2$; it follows easily from the above observation that $\partial\mathscr{C}=\partial\mathscr{C}_1\cup\partial\mathscr{C}_2$; therefore $\partial\mathscr{C}$ is connected. The analytic capacity condition (ii) of Theorem 5.5 is satisfied at any point of the boundary with respect to $K_{\mathscr{C}_1}$ or $K_{\mathscr{C}_2}$; it is therefore also satisfied with respect to $K_{\mathscr{C}_1\cup\mathscr{C}_2} = K_{\mathscr{C}_1}\cap K_{\mathscr{C}_2}$ because γ is a monotone increasing set function. This concludes the proof. ▨

We can now prove the main result on Dirichlet chains.

THEOREM 6.2. *Let K be a connected compact set in* $\mathbf{C}$ *and let $\mathscr{G}$ denote its set of holes. Then*

(i) *any hole in K belongs to a unique maximal* D-*chain (consequently these maximal* D-*chains determine an (at most countable) partition of $\mathscr{G}$),*

(ii) *the boundaries of these maximal* D*-chains are pairwise disjoint,*
and

(iii) *if $\mathscr{C}$ is a maximal* D*-chain then $K \cap \mathring{K}_{\mathscr{C}}$ is dominating in $\mathring{K}_{\mathscr{C}}$.*

Before proving Theorem 6.2 we observe that the existence of maximal D--chains was already implicitly established in [3], offering a convenient substitute for the transfinite induction argument used in [1] and [15]; we repeat the proof for completeness.

Proof. (ii) as well as the uniqueness part of (i) follows from Lemma 6.1 and from (i). To finish the proof of (i), let H be a hole in K and let $\mathscr{L}$ be the set of D-chains containing H. $\mathscr{L}$ is nonempty since $\{H\} \in \mathscr{L}$.

Let now $\{\mathscr{C}_i\}_{i\in I}$ be a totally ordered set of D-chains in $\mathscr{L}$ and let $\mathscr{C} = \bigcup_{i\in I} \mathscr{C}_i$. Since $\mathscr{G}$ is countable we can write $\mathscr{C} = \bigcup_{n\in \mathbf{N}} \mathscr{C}_{i_n}$ where the $\mathscr{C}_{i_n}$ are increasing; we have now $K_{\mathscr{C}} = \bigcap_{n\in \mathbf{N}} K_{\mathscr{C}_{i_n}}$ and $K_{\mathscr{C}}$ is a D-set by Proposition 5.2; thus $\mathscr{C}$ is in $\mathscr{L}$ and $\mathscr{L}$ is an inductive set. Zorn's lemma now concludes the proof of (i). To prove (iii) let f belong to $H^{\infty}(\mathring{K}_{\mathscr{C}})$ where $\mathscr{C}$ is a maximal D-chain and let $s = \sup_{\lambda\in K\cap \mathring{K}_{\mathscr{C}}} |f(\lambda)|$. For any $H \in \mathscr{G}$ we have either $H \in \mathscr{C}$ or else $H^- \subset \mathring{K}_{\mathscr{C}}$ (indeed by (ii) if $H \notin \mathscr{C}$ the maximal D-chain containing H has a boundary disjoint from $\partial K_{\mathscr{C}}$). For any $H \notin \mathscr{C}$ we have

$$\sup_{\lambda\in H} |f(\lambda)| = \sup_{\lambda\in \partial H} |f(\lambda)| \leqslant s$$

(the equality is a consequence of the maximum modulus principle, the inequality follows from the definition of s combined with the inclusion $\partial H \subset K \cap \mathring{K}_{\mathscr{C}}$). Therefore

$$\sup_{\lambda\in (\mathbf{C}\setminus K)\cap \mathring{K}_{\mathscr{C}}} |f(\lambda)| \ (= \sup_{H\notin\mathscr{C}} (\sup_{\lambda\in H}|f(\lambda)|)) \leqslant s$$

and

$$s = \sup_{\lambda\in \mathring{K}_{\mathscr{C}}} |f(\lambda)|$$

as desired. ▨

7. SPLITTING $H^{\infty}(G)$ WHEN $G = G_1 \cap G_2$

Throughout this section G_1 is a bounded open set in $\mathbf{C}$, G_2 is an open set of $\mathbf{S}^2$ such that $\mathbf{S}^2 \setminus G_2 \subset G_1$, and $G = G_1 \cap G_2$. A version of the following decomposition theorem was already given in [10] for circular domains.

THEOREM 7.1. *Let G_1, G_2, and G as above and let $H_0^{\infty}(G_2)$ denote the subalgebra of $H^{\infty}(G_2)$ that consists of those functions in $H^{\infty}(G_2)$ vanishing at ∞. Then there are projections P_i, $i = 1, 2$ defined on $H^{\infty}(G)$ such that:*

1) *The ranges of P_1 and P_2 are respectively $H^{\infty}(G_1)$ and $H_0^{\infty}(G_2)$, and $P_1 + P_2 = I$,*

2) *P_1 and P_2 are norm-continuous,*

and

3) *P_1 and P_2 are weak*-continuous.*

Proof. 1) We only outline it since it is a standard application of Cauchy integral techniques. Let $0 < \varepsilon < \inf_{\lambda_i \in \partial G_i} |\lambda_1 - \lambda_2|$. By Problem 5K of [7] we can choose (for $i = 1, 2$) a system Γ_i of closed rectifiable Jordan curves in G_i such that:

(a) If $V_i = \{\lambda : I(\Gamma_i, \lambda) = \delta_{i,1}\}$ then $V_i^- \subset G_i$ and $\{\lambda \in G_i : \mathrm{d}(\lambda, \partial G_i) \geqslant \geqslant \varepsilon/4\} \subset V_i$.

(b) The geometrical range of Γ_i is the boundary of V_i.

(c) $I(\Gamma_i, \lambda) = -\delta_{i,2}$ whenever $\lambda \in \mathbf{C} \setminus V_i^-$.

(Here $I(\Gamma, \lambda)$ denotes the winding number of Γ with respect to λ and $\delta_{i,j}$ the usual Kronecker symbol). For f in $H^\infty(G)$ we set $(i = 1, 2)$

$$f_i(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_i} f(\xi)/(\xi - \zeta) \mathrm{d}\xi \qquad (\zeta \in V_i).$$

The following facts (all easy consequences of the definition and of the Cauchy integral formula) conclude the proof of 1). (We let $P_i(f) = f_i$.)

— f_i is analytic in V_i and can be analytically extended on G_i.

— $f = f_1 + f_2$.

— f_1 is bounded on any compact set contained in G_1 in particular on ∂G_2, thus f_2 is also bounded near ∂G_2 and consequently belongs to $H^\infty(G_2)$; similarly f_1 belongs to $H^\infty(G_1)$.

— $f_2(\infty) = 0$ (thus, $f_2 \in H_0^\infty(G_2)$).

— for f in $H^\infty(G_1)$ $f_1 = f$ and, for f in $H_0^\infty(G_2)$, $f_2 = f$.

2) It follows from the maximum modulus principle that the embeddings $H^\infty(G_1) \subset H^\infty(G)$, $H_0^\infty(G_2) \subset H^\infty(G)$ are isometries. Thus the ranges of the projections P_1 and P_2 are norm-closed and P_1 and P_2 are norm-continuous.

3) Since $H^\infty(G_1)$ is the dual of a separable Banach space it is enough to prove the sequential w* continuity of P_1 ([9], Theorem 2.3). Let then f_n converging pointwise boundedly to 0 in $H^\infty(G)$. Then $P_1(f_n)$ is norm bounded and the pointwise convergence to 0 on G_1 follows from the definition together with the uniform convergence to 0 of f_n on compact sets of G. ▨

Next we wish to show that the notion of dominating set behaves well with respect to that decomposition of $H^\infty(G)$.

Though the result holds without restriction on G_2 we give the proof only in the case that G_2 is connected, sufficient for our applications.

THEOREM 7.2. *Let G_1, G_2 and G be as above and suppose G_2 connected. A subset S of G that is dominating in both G_1 and G_2 is also dominating in G.*

First we establish the following lemma.

LEMMA 7.3. *Let S and G as in Theorem 7.2 (that is, S is dominating in G_1 and G_2). Let f in $H^\infty(G)$ such that $f|S = 0$. Then $f = 0$.*

Proof. Let $f = f_1 + f_2$ be the decomposition of f given by Theorem 7.1. We will show that f_1 and f_2 are identically 0. Let $\varepsilon = \inf_{\lambda_i \in \partial G_i} |\lambda_1 - \lambda_2|$; by the maximum modulus principle the set $S_i = \{\lambda \in S : d(\lambda, \partial G_i) \leqslant \varepsilon/4\}$ is still dominating in G_i. Clearly we have $S_1 \cup S_2 \subset S$, $S_1^- \subset G_2$, $S_2^- \subset G_1$ and $S_1^- \cap S_2^- = \emptyset$. Let $M_{ij} = \sup_{\lambda \in S_i} |f_j(\lambda)|$. Then we have

$$M_{11} = M_{12} \leqslant M_{22} = M_{21} \leqslant M_{11}$$

(the equalities come from the relation $f_1 = -f_2$ on $S_1 \cup S_2$ and the inequalities from the fact that S_i is dominating in G_i). Now the equality $M_{12} = M_{22}$ implies that f_2 attains its maximum at some point of ∂G_1 which is in G_2; thus f_2 must be constant and equal to $0 = f_2(\infty)$; now $f_1|S = -f_2|S = 0$ and since S is dominating in G_1, $f_1 = 0$. ▨

Proof of Theorem 7.2. Proceeding as in Lemma 7.3. we may assume that $S = S_1 \cup S_2$ with S_i dominating in G_i, $S_1^- \subset G_2$, $S_2^- \subset G_1$ and $S_1^- \cap S_2^- = \emptyset$. Suppose that S is not dominating in G; then there exists a function of norm one in $H^\infty(G)$ such that $\sup_{\lambda \in S} |f(\lambda)| = \alpha < 1$. Any subsequence of $\{f^p\}_{p \in \mathbf{N}}$ converges uniformly to 0 on S. Since $\|f^p\| = 1$ for all p we can choose one (denote it f_n) which is weak* convergent to say g. Of course $g|S = 0$. By the previous theorem $f_{1,n}$ and $f_{2,n}$ are weak* convergent to respectively g_1 and g_2. It follows from Lemma 7.1 that $g_1 = g_2 = 0$. Now the weak*-convergence of $f_{2,n}$ to 0 implies its uniform convergence (to 0) on the compact set S_1. By taking differences ($f_{1,n} = f - f_{2,n}$) we get that $f_{1,n}$ converges uniformly to 0 on S_1 and consequently $f_{1,n}$ converges to 0 in norm (recall that S_1 is dominating in G_1). Similarly $\|f_{2,n}\|_\infty$ tends to zero. Thus $\|f_n\|$ tends to zero in contradiction to the fact that $\|f_n\| = 1$ for all n. ▨

8. APPLICATIONS

We are now ready to prove the announced applications of Theorem 4.1.

THEOREM 8.1. *Let A be an operator in $\mathscr{L}(\mathscr{H})$ and let G be a bounded open set in $\mathbf{C}$ such that:*

a) *G^- is a connected, M-spectral set for A,*

b) *$\sigma(A) \cap G$ is dominating in G,*

and

c) *G^- has only a finite number of maximal* D*-chains;*

then there exists a nontrivial subspace invariant under any rational function of A with poles off G^-.

Before proving this theorem we make two remarks. First the assumption that G^- is connected is necessary if one is to talk about D-chains of G^- but is in fact nonrestrictive; indeed if G^- is disconnected then an easy argument using b) shows that $\sigma(A)$ itself is disconnected (with the consequence that A has a nontrivial hyperinvariant subspace). The other observation is that condition c) is obviously satisfied in the case when G^- has only a finite number of holes (without any restriction on their boundaries). Thus Theorem 8.1 generalizes Theorem 5.2 of [10].

Proof of Theorem 8.1. Let $K = G^-$; as in the proof of Theorem 4.1 there is no loss of generality in assuming that $\hat{K} = G$. It is sufficient to show that K is a C-set: once this is done the conclusion follows from Theorem 4.1. The boundary of K is the union of the boundaries of the maximal D-chains. Thus for each $\zeta \in \partial K$ and δ small enough (recall that the boundaries of the maximal D-chains are disjoint and there are only a finite number of them) we have $\gamma(\Delta(\zeta; \delta)\setminus K) \geqslant \delta/4$. Since the inequality $\gamma(\Delta(\zeta, \delta) \cap \partial\mathring{K}) \leqslant \delta$ is always satisfied we obtain that K is a C-set via (iii) of Proposition 5.3. ▨

We now turn our attention to the case of an operator having its spectrum as an M-spectral set and give the following generalization of the main result of [15].

THEOREM 8.2. *Let A be an operator in $\mathscr{L}(\mathscr{H})$ such that $\sigma(A)$ is a connected M-spectral set for A, let $\mathscr{C}_1, \ldots, \mathscr{C}_n$ be (finitely many) maximal* D-*chains for $\sigma(A)$. Then there exists a nontrivial subspace invariant under any rational function of A with poles in the set $\bigcup_{\substack{H\in\mathscr{C}_i\\ 1\leqslant i\leqslant n}} H$.*

Proof. Let $K = \sigma(A)$, $K_i = K_{\mathscr{C}_i}$ (i.e. $K_i = \mathrm{S}^2 \setminus \bigcup_{H\in\mathscr{C}_i} H$), $i = 1, \ldots, n$, $L = \bigcap_{1\leqslant i\leqslant n} K_i$, and $G = \mathring{L}\,(= \bigcap_{1\leqslant i\leqslant n} \mathring{K}_i)$. We may assume without loss of generality that $\mathscr{C}_1$ contains the unbounded component of $\sigma(A)$. Suppose first that $\sigma(A) \not\subset G^-$ (that will happen for instance if part of the boundary of $\sigma(A)$ lies in the interior of the closure of one hole). It follows from the proof of Theorem 1 of [11] that for each $\zeta_0 \in \sigma(A)\setminus G^-$ and each ε sufficiently small there exists a function f_ε in $\mathrm{R}(L)$ such that $f_\varepsilon(\lambda) = 0$ for $|\lambda - \zeta_0| \geqslant \varepsilon$ and $f_\varepsilon(\lambda) = 1$ for $|\lambda - \zeta_0| \leqslant \varepsilon/2$. Choose ε and δ small enough and such that $\varepsilon < \delta/2$ and let $A_0 = f_\varepsilon(A)$ and $A_1 = (1 - f_\delta)(A)$. Approximating f_ε by rational functions with poles off L and using the spectral mapping theorem for these rational functions we obtain that $f_\varepsilon(\sigma(A)) \subset \sigma(f_\varepsilon(A))$. Therefore $A_0 \neq 0$; similarly $A_1 \neq 0$. Let $\mathscr{M}$ denote the closure of the range of A_0. Since $A_0 \neq 0$, $\mathscr{M} \neq (0)$. On the other hand since $(1 - f_\delta)f_\varepsilon = 0$ we have $A_1A_0 = 0$; thus $\mathscr{M} \subset \mathrm{Ker}A_1$ and $\mathscr{M} \neq H$. It is easily seen that any operator that commutes with A also commutes with A_0. Thus $\mathscr{M}$ is invariant for any operator that commutes with A and in particular for any rational function of A with poles in $\bigcup_{\substack{H\in\mathscr{C}_i\\ 1\leqslant i\leqslant n}} H$.

To finish the proof we consider now the case $\sigma(A) \subset G^-$. An induction argument based on Theorem 7.2 and Part (iii) of Theorem 6.2 shows that $\sigma(A) \cap G$ is dominating in G. A similar argument to the one used in the proof of Theorem 8.1 shows that L is a C-set. Therefore $\mathrm{R}(G^-)$ which contains $\mathrm{R}(L)$ is pointwise boundedly dense in $H^\infty(G)$; the equality $\mathrm{R}(\partial G) = \mathrm{C}(\partial G)$ follows from the similar equality for ∂L and the inclusion $\partial G \subset \partial L$. (In other words G^- itself is a C-set, a fact which could also have been proved using the more sophisticated characterization of C-sets given by (ii) of Proposition 5.3). Thus all the hypotheses of Theorem 4.1 are met and the desired conclusion follows from that theorem. ▨

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