J. OPERATOR THEORY
7(1982), 267–277

T-THEOREM FOR L^p-SPACES ASSOCIATED WITH A VON NEUMANN ALGEBRA

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0. INTRODUCTION

Following the development of the Tomita-Takesaki theory [13], Araki [1], introduced a family of positive cones P^α, $0 \leqslant \alpha \leqslant \frac{1}{2}$, associated with a von Neumann algebra $\mathcal{M}$ admitting a cyclic and separating vector (See § 1.). In [9], [10], we observed that the cones are closely related to (the positive parts of) associated non-commutative L^p-spaces, $1 \leqslant p \leqslant \infty$, [2], [8], [10], [11]. In fact, the surjectivity of the map: $\xi \in P^\alpha \mapsto \omega_\xi \in \mathcal{M}_*^+$ is exactly the validity of a "T-theorem" for the L^p-space with $\alpha = \frac{1}{2} - \frac{1}{2p}$ which will be made precise in § 2. Also, for finite von Neumann algebras, this T-theorem holds always. The main purpose of the paper, however, is to prove the converse. Namely, we shall prove that the validity of this "T-theorem" for the L^p-space, $2 < p \leqslant \infty$, implies the finiteness of the algebra $\mathcal{M}$ in question. (For $p = \infty$, this is known as the T-theorem due to Dye, Murray-von Neumann [5].) For a factor of type either I_∞ or III_λ, $0 < \lambda < 1$, we shall prove a slightly stronger result.

We shall freely use the basic facts and notations of (relative) modular theory [3], [13], and non-commutative L^p-spaces [2], [8], [10], [11].

The author is indebted to Dr. Christian Skau for some ideas in the paper. Also, the present research was done while he was at the University of California, Los Angeles, and he thanks Professor Masamichi Takesaki for constant encouragement.

1. PRELIMINARIES

In this section, we collect some basic definitions and properties partially to fix our notations.

Let $(\mathcal{M}, \mathcal{H}, J, \mathcal{P}^{\natural})$ be a standard form [6]. For each $\psi \in \mathcal{M}_*^+$, there corresponds a unique implementing vector in $\mathcal{P}^{\natural}$, which will be denoted by ξ_ψ, that is, $\psi(x) = \omega_{\xi_\psi}(x) = (x\xi_\psi | \xi_\psi)$, $x \in \mathcal{M}$. For a faithful φ and an arbitrary ψ, the closure of the densely-defined closable operator $S_{\psi\varphi} : x\xi_\varphi \in \mathcal{M}\xi_\varphi \to x^*\xi_\psi \in \mathcal{M}\xi_\psi$ admits the polar decomposition $J\Delta_{\psi\varphi}^{\frac{1}{2}}$. Here the positive self-adjoint operator $\Delta_{\psi\varphi}$ is known as the relative modular operator. In case $\varphi = \psi$, $\Delta_{\varphi\varphi} = \Delta_\varphi$ is exactly the usual modular operator determined by the pair $(\mathcal{M}, \varphi)$ in the sense of [13]. The partial isometry $(D\psi : D\varphi)_t = \Delta_{\psi\varphi}^{it}\Delta_\varphi^{-it}$, $t \in \mathbf{R}$, in $\mathcal{M}$ is known as the Radon-Nikodym cocycle (of ψ with respect to φ), [3].

DEFINITION 1.1. (Araki [1]). For $0 \leqslant \alpha \leqslant \frac{1}{2}$, and a faithful $\varphi \in \mathcal{M}_*^+$, P_φ^α is the closure of the positive cone $\Delta_\varphi^\alpha \mathcal{M}_+ \xi_\varphi$ in $\mathcal{H}$.

In the literature, P_φ^0 and $P_\varphi^{\frac{1}{2}}$ are denoted by $P_\varphi^\#$ and $P_\varphi^\flat$ respectively. We also note that $P_\varphi^{\frac{1}{4}}$ is exactly the natural cone $\mathcal{P}^{\natural}$ (for any φ). In [9], we showed the following Radon-Nikodym type theorem:

THEOREM 1.2. *The map: $\xi \in P_\varphi^\alpha \mapsto \omega_\xi \in \mathcal{M}_*^+$ is bijective for $0 \leqslant \alpha \leqslant \frac{1}{4}$. This is also the case for $0 \leqslant \alpha \leqslant \frac{1}{2}$ provided that $\mathcal{M}$ is finite.*

In [10] we also observed that the cones are closely related to Connes-Hilsum's L^p-spaces, $1 \leqslant p \leqslant \infty$, [2], [8], which are isomorphic to other L^p-spaces, [7], [11]. In our set up, their L^p-spaces are described as follows:

DEFINITION 1.3. For a faithful $\varphi \in \mathcal{M}_*^+$, we set $\varphi'(x') = (x'\xi_\varphi | \xi_\varphi)$, $x' \in \mathcal{M}'$ so that φ' belongs to $\mathcal{M}'^+_*$. For each $1 \leqslant p < \infty$, Connes-Hilsum's L^p-space $L^p(\mathcal{M}; \varphi')$ consists of all densely-defined closed operators T on $\mathcal{H}$, with the polar decompositions $T = u|T|$, satisfying the following three conditions:

(i) u belongs to $\mathcal{M}$;

(ii) (homogeneity) $|T|^{it}x' = \sigma^{\varphi'}_{-t/p}(x')\,|T|^{it}$; $t \in \mathbf{R}$, $x' \in \mathcal{M}'$, where $\sigma^{\varphi'}$ is the modular automorphism group on $\mathcal{M}'$ associated with φ';

(iii) (integrability) ξ_φ belongs to $\mathcal{D}(|T|^{\frac{p}{2}})$.

(Also, we set $L^\infty(\mathcal{M}; \varphi') = \mathcal{M}$.) If the above operator T satisfies (i) and (ii), T is said to be *affiliated with* $L^p(\mathcal{M}; \varphi')$ (or T is $\left(-\frac{1}{p}\right)$*-homogeneous*).

Notice that T being affiliated with $L^\infty(\mathcal{M}; \varphi')$ in the above sense is equivalent to T being affiliated with $\mathcal{M}$ in the classical sense. We also notice that the positive part (as operators) of $L^p(\mathcal{M}, \varphi')$ is exactly the set of all $\Delta_{\psi\varphi}^{\frac{1}{p}}$, $\psi \in \mathcal{M}_*^+$.

2. T-THEOREM FOR L^p-SPACES

Setting up our "T-theorem", we state our main results, which are the converses of the T-theorem.

We recall the next result in [10].

THEOREM 2.1. (Theorem 4.1, [10]). *Let φ be a fixed faithful functional in $\mathcal{M}_*^+$ and $2 < p \leqslant \infty$. For each $\psi \in \mathcal{M}_*^+$, the following four conditions are all equivalent:*

(i) *the (densely-defined) operator $\Delta_{\psi\varphi}^{\frac{1}{2}} \Delta_\varphi^{\frac{1}{p} - \frac{1}{2}}$ is closable,*

(ii) *there exists an operator T affiliated with $L^p(\mathcal{M}; \varphi')$ such that $T\xi_\varphi = \xi_\psi$ (with $\xi_\varphi \in \mathcal{D}(T)$),*

(iii) *(resp.* (iv)*) there exists a vector in $P^{\frac{1}{2} - \frac{1}{2p}}$ (resp. $\mathcal{D}(\Delta_\varphi^{\frac{1}{p} - \frac{1}{2}})$) satisfying $\psi(x) = (x\xi \mid \xi)$, $x \in \mathcal{M}$.*

Before going further, we remark that in the above theorem the condition (i) can also be mentioned differently. Namely, (i) can be replaced by (i)$'$: there exists an operator A affiliated with $L^p(\mathcal{M}; \varphi')$ such that $\xi_\varphi \in \mathcal{D}(A)$ and $A\Delta_\varphi^{\frac{1}{2} - \frac{1}{p}}$ is a densely-defined closable operator whose closure is $\Delta_{\psi\varphi}^{\frac{1}{2}}$. In fact, (i)$'$ immediately implies (ii). Also, if one assumes (i), $A = (\Delta_{\psi\varphi}^{\frac{1}{2}} \Delta_\varphi^{\frac{1}{p} - \frac{1}{2}})^-$ is affiliated with $L^p(\mathcal{M}; \varphi')$ and $\xi_\varphi \in \mathcal{D}(A)$. Furthermore, it is easy to prove that $A\Delta_\varphi^{\frac{1}{2} - \frac{1}{p}}$ is densely-defined and closable. (See the proof of Theorem 4.1, [10].) Noticing that $(A\Delta_\varphi^{\frac{1}{2} - \frac{1}{p}})^-$ is $\left(-\frac{1}{2}\right)$-homogeneous ([2]) and that $(A\Delta_\varphi^{\frac{1}{2} - \frac{1}{p}})^- \xi_\varphi = \xi_\psi$, one concludes that $(A\Delta_\varphi^{\frac{1}{2} - \frac{1}{p}})^- = \Delta_{\psi\varphi}^{\frac{1}{2}}$.

COROLLARY 2.2. *For φ and p as in the above theorem, the following four conditions are equivalent:*

(i) *for* **any** *$\psi \in \mathcal{M}_*^+$, the operator $\Delta_{\psi\varphi}^{\frac{1}{2}} \Delta_\varphi^{\frac{1}{p} - \frac{1}{2}}$ is closable,*

(ii) *for* **any** $\xi \in \mathscr{H}$, *there exists an operator* T *affiliated with* $L^p(\mathscr{M};\varphi')$ *such that* $\xi = T\xi_\varphi$,

(iii) (*resp.* (iv)) *the map* : $\xi \in P_\varphi^{\frac{1}{2}-\frac{1}{2p}}$ $(\xi \in \mathscr{D}(\Delta_\varphi^{\frac{1}{p}-\frac{1}{2}})) \to \omega_\xi \in \mathscr{M}_*^+$ *is surjective.*

Proof. If one had $\mathscr{P}^\natural$ instead of $\mathscr{H}$ in (ii), the corollary would be a trivial consequence of the theorem. We thus have to prove (ii) by assuming that (ii) with $\mathscr{P}^\natural$ instead of $\mathscr{H}$ is valid.

Let ξ be an arbitrary vector in $\mathscr{H}$ with the "polar decomposition" $\xi = u|\xi|$ in the sense of [1] and [6] so that u is a partial isometry in $\mathscr{M}$ with $u^*u = [\mathscr{M}'|\xi|]$ and $|\xi| \in \mathscr{P}^\natural$. The vector $|\xi|$ belongs to $\mathscr{P}^\natural$ so that there exists an operator $\tilde{T}$ affiliated with $L^p(\mathscr{M};\varphi')$ such that $\tilde{T}\xi_\varphi = |\xi|$. (Notice that we do not know if $u\tilde{T}$ is closable because $\tilde{T}$ may or may not belong to $L^p(\mathscr{M};\varphi')$.)

Let $\mathscr{M}'_0$ be the set of "smooth" elements $x' \in \mathscr{M}'$ with respect to $\sigma^{\varphi'}$, that is, $z \mapsto \sigma_z^{\varphi'}(x')$ is an $\mathscr{M}'$-valued entire function. We claim that the restriction of $u\tilde{T}$ to $\mathscr{M}'_0\xi_\varphi$ is closable. To prove it, let us assume that a sequence $\{x'_n\xi_\varphi\}$ in $\mathscr{M}'_0\xi_\varphi$ tends to 0 and that $\{u\tilde{T}x'_n\xi_\varphi\}_n$ is Cauchy. Since $\tilde{T}$ is $\left(-\frac{1}{p}\right)$-homogeneous (with respect to φ'),

$$\tilde{T}x'_n\xi_\varphi = \sigma^{\varphi'}_{-\frac{i}{p}}(x'_n)\tilde{T}\xi_\varphi = \sigma^{\varphi'}_{-\frac{i}{p}}(x'_n)|\xi|$$

belongs to the initial space of u so that $\{\tilde{T}x'_n\xi_\varphi\}_n$ is also a Cauchy sequence. Thus the closedness of $\tilde{T}$ implies that $\lim_n \tilde{T}x'_n\xi_\varphi = 0$ so that $\lim_n u\tilde{T}x'_n\xi_\varphi = 0$.

Let T be the closure of the restriction of $u\tilde{T}$ to $\mathscr{M}'_0\xi_\varphi$. By construction, $\xi_\varphi \in \mathscr{D}(T)$ and $T\xi_\varphi = \xi$. Furthermore, Lemma 2.1 in [10], guarantees that T is affiliated with $L^p(\mathscr{M};\varphi')$. Q.E.D.

In the above corollary, we assumed that $2 < p \leqslant \infty$ so that the exponent $\frac{1}{p} - \frac{1}{2}$ of Δ_φ appearing in (i) is strictly negative. On the other hand, we know that $\Delta_{\psi\varphi}^{\frac{1}{2}}\Delta_\varphi^{\frac{1}{p}-\frac{1}{2}}$ is closable and $(\Delta_{\psi\varphi}^{\frac{1}{2}}\Delta_\varphi^{\frac{1}{p}-\frac{1}{2}})^{-}$ belongs to $L^p(\mathscr{M};\varphi')$ whenever $\frac{1}{p} - \frac{1}{2} \geqslant 0$, that is, $p \leqslant 2$. (See [8].) Thus, combining Corollary 2.2 and the second half of Theorem 1.2, we have:

THEOREM 2.3. *Let* φ *be as in the theorem;*

(i) $(2 < p \leqslant \infty)$. *If* $\mathscr{M}$ *is finite, for any* $\xi \in \mathscr{H}$ *there exists an operator* T *affiliated with* $L^p(\mathscr{M};\varphi')$ *such that* $\xi = T\xi_\varphi$.

(ii) ($1 \leqslant p \leqslant 2$, $\mathcal{M}$ *is arbitrary*). *For any* $\xi \in \mathcal{H}$ *there exists a (unique)* T *in* $L^p(\mathcal{M}; \varphi')$ *satisfying* $\xi = T\xi_\varphi$.

We remark that, for the special value of $p = \infty$, the statement (i) of Theorem 2.3 is exactly the celebrated T-theorem of Murray-von Neumann.

DEFINITION 2.4. For a pair $(\mathcal{M}, \varphi)$ consisting of a von Neumann algebra $\mathcal{M}$ and a faithful functional in $\mathcal{M}_*^+$, we say that *the T-theorem for* $L^p(\mathcal{M}; \varphi')$ *holds*, if one (hence all) of the four conditions in Corollary 2.2 is satisfied.

Although we fixed a standard form at the beginning of § 1, everything thereafter does not depend on the choice of our standard form so that the above definition is legitimate. Thus, the above Theorem 2.3, (i), can be rephrased as follows:

THEOREM 2.3′. (T-theorem for L^p-spaces, $2 < p \leqslant \infty$). *If* $\mathcal{M}$ *is finite, the T-theorem for* $L^p(\mathcal{M}; \varphi')$ *holds for each faithful functional* φ *in* $\mathcal{M}_*^+$ *and* $2 < p \leqslant \infty$.

As stated in the introduction, the main result of the paper is the converse of the above "T-theorem". For the special value of $p = \infty$, the converse was proved by Dye [5]. (See also [12].) We thus assume $2 < p < \infty$ *in the rest of the paper*, unless the contrary is stated. In the rest, we will concentrate on the proofs of the following two theorems, which are the main results of the paper:

THEOREM 2.5. *If, for* **any** *faithful* $\varphi \in \mathcal{M}_*^+$ *(and a single* p, $2 < p \leqslant \infty$*), the T-theorem for* $L^p(\mathcal{M}; \varphi')$ *holds, then* $\mathcal{M}$ *is finite.*

THEOREM 2.6. *Assume that* $\mathcal{M}$ *is a factor of type either* I_∞ *or* III_λ, $0 < \lambda < 1$, *and that* $p \in \,]2, \infty]$. *Then the T-theorem for* $L^p(\mathcal{M}; \varphi)$ *holds for* **no** φ.

3. TECHNICAL LEMMAS

To prove the two main results, we prepare some technical lemmas. We prove the stability of our "T-theorem" under certain perturbations and a normal projection of norm 1.

LEMMA 3.1. *For an automorphism* α *of* $\mathcal{M}$, *the T-theorem for* $L^p(\mathcal{M}; \varphi')$ *holds if and only if the same is true for* $L^p(\mathcal{M}; (\varphi \circ \alpha)')$.

Proof. Let $\alpha = \mathrm{Ad}v^*$ be the canonical implementation, [1], [6]. As a consequence of the uniqueness of the polar decomposition, it is shown that $\Delta_{\psi\varphi} = v^* \Delta_{\psi\circ\alpha, \varphi\circ\alpha} v$, $\varphi, \psi \in \mathcal{M}_*^+$. One thus obtains $\Delta_{\psi\varphi}^{\frac{1}{2}} \Delta_\varphi^{\frac{1}{p} - \frac{1}{2}} = v^* \Delta_{\psi\circ\alpha, \varphi\circ\alpha}^{\frac{1}{2}} \Delta_{\varphi\circ\alpha}^{\frac{1}{p} - \frac{1}{2}} v$ so that the result follows from Corollary 2.2 (i). Q.E.D.

LEMMA 3.2. *Assume that faithful* φ, ψ *in* $\mathcal{M}_*^+$ *satisfy* $\varphi \leqslant l_1 \psi$ and $\psi \leqslant l_2 \varphi$ *with some* $l_1, l_2 \geqslant 0$. *For any* $-\dfrac{1}{2} \leqslant \alpha \leqslant \dfrac{1}{2}$, $\mathcal{D}(\Delta_\varphi^\alpha)$ is exactly $\mathcal{D}(\Delta_\psi^\alpha)$.

Before going to its proof, we recall that $z \to (\mathrm{D}\varphi : \mathrm{D}\psi)_z \in \mathscr{M}$ is bounded and (σ-weakly) continuous on $-\frac{1}{2} \leqslant \mathrm{Im}z \leqslant 0$ and analytic in the interior if and only if $\varphi \leqslant l\psi$ for some $l \geqslant 0$. (See U. Haagerup, Operator valued weights in von Neumann algebras. I, *J. Functional Analysis*, **32**(1979), 175—206; Lemma 3.3.) Thus, when $\varphi \leqslant l_1\psi$ and $\psi \leqslant l_2\varphi$, both of $(\mathrm{D}\varphi : \mathrm{D}\psi)_z$ and $(\mathrm{D}\psi : \mathrm{D}\varphi)_z$ $\left(-\frac{1}{2} \leqslant \mathrm{Im}z \leqslant 0\right)$ make sense as elements in $\mathscr{M}$ and satisfy

$$(\mathrm{D}\varphi : \mathrm{D}\psi)_z\, (\mathrm{D}\psi : \mathrm{D}\varphi)_z = (\mathrm{D}\psi : \mathrm{D}\varphi)_z\, (\mathrm{D}\varphi : \mathrm{D}\psi)_z = 1$$

because of the uniqueness of analytic continuation. (For $z = t \in \mathbf{R}$, this equality is well-known.) In other words, $(\mathrm{D}\varphi : \mathrm{D}\psi)_z \in \mathscr{M}$ is invertible and its inverse is $(\mathrm{D}\psi : \mathrm{D}\varphi)_z$. (Furthermore, due to $(\mathrm{D}\varphi : \mathrm{D}\psi)_t = ((\mathrm{D}\varphi : \mathrm{D}\psi)_t^{-1})^* = (\mathrm{D}\psi : \mathrm{D}\varphi)_t^*$, the above mentioned facts remain valid for $|\mathrm{Im}z| \leqslant \frac{1}{2}$.)

Proof of Lemma 3.2. At first we remark

(i) $J\Delta_\varphi^\alpha J = \Delta_\varphi^{-\alpha}$ (and $J\Delta_\psi^\alpha J = \Delta_\psi^{-\alpha}$),

(ii) $J\Delta_{\psi\varphi}^\alpha J = \Delta_{\psi\varphi}^{-\alpha}$,

(iii) $\mathscr{M}\xi_\varphi$ is a common core for $\Delta_\varphi^{\frac{1}{2}}$ and $\Delta_{\psi\varphi}^{\frac{1}{2}}$.

In fact, (i) is well-known, [13], and (iii) is obvious from the construction. Also, (ii) follows from (i) and the well-known 2×2-matrix argument. Due to (i), we may and do assume that $0 \leqslant \alpha \leqslant \frac{1}{2}$ in what follows. We now consider the following two functions (for each $x \in \mathscr{M}$):

$$z \to \Delta_{\psi\varphi}^{iz} x\xi_\varphi$$

$$z \to (\mathrm{D}\psi : \mathrm{D}\varphi)_z\, \Delta_\varphi^{iz} x\xi_\varphi .$$

Because of $x\xi_\varphi \in \mathscr{D}(\Delta_{\psi\varphi}^{\frac{1}{2}}) \cap \mathscr{D}(\Delta_\varphi^{\frac{1}{2}})$ and the remark before the proof, these two functions are bounded and continuous on $-\frac{1}{2} \leqslant \mathrm{Im}z \leqslant 0$ and analytic in the interior. For $z = t \in \mathbf{R}$, one obviously has

$$(\mathrm{D}\psi : \mathrm{D}\varphi)_t\, \Delta_\varphi^{it} x\xi_\varphi = \Delta_{\psi\varphi}^{it}\Delta_\varphi^{-it}\Delta_\varphi^{it} x\xi_\psi = \Delta_{\psi\varphi}^{it} x\xi_\varphi .$$

Therefore, the uniqueness of analytic continuation implies:

$$\Delta_{\psi\varphi}^\alpha x\xi_\varphi = (\mathrm{D}\psi : \mathrm{D}\varphi)_{-i\alpha}\Delta_\varphi^\alpha x\xi_\varphi .$$

It is well-known that $\mathscr{M}\xi_\varphi$ is a common core for $\Delta^\alpha_{\psi\varphi}$ and Δ^α_φ ((iii) and Lemma 4, [1]). Furthermore, $(D\psi : D\varphi)_{-i\alpha}$ is invertible (so that $\mathscr{M}\xi_\varphi$ is a core for $(D\psi : D\varphi)_{-i\alpha}\Delta^\alpha_\varphi$). Therefore, the above equality actually means

$$\Delta^\alpha_{\psi\varphi} = (D\psi : D\varphi)_{-i\alpha}\Delta^\alpha_\varphi = \tag{1}$$

$$= \Delta^\alpha_\varphi (D\psi : D\varphi)^*_{-i\alpha}$$

(due to the self-adjointness of $\Delta^\alpha_{\psi\varphi}$, Δ^α_φ and the boundedness of $(D\psi : D\varphi)_{-i\alpha}$). Taking the inverse of this non-singular positive self-adjoint operator, we have

$$\Delta^{-\alpha}_{\psi\varphi} = ((D\psi : D\varphi)^*_{-i\alpha})^{-1}\Delta^{-\alpha}_\varphi = (D\varphi : D\psi)^*_{-i\alpha}\Delta^{-\alpha}_\varphi .$$

Then (i) and (ii) yield:

$$\Delta^\alpha_{\varphi\psi} = J(D\varphi : D\psi)^*_{-i\alpha} J J \Delta^{-\alpha}_\varphi J =$$

$$= J(D\varphi : D\psi)^*_{-i\alpha} J \Delta^\alpha_\varphi .$$

By changing the roles of φ and ψ, we now have

$$\Delta^\alpha_{\psi\varphi} = J(D\psi : D\varphi)^*_{-i\alpha} J \Delta^\alpha_\psi . \tag{2}$$

Finally, (1) and (2) yield:

$$\Delta^\alpha_\varphi = (D\psi : D\varphi)^{-1}_{-i\alpha}\Delta^\alpha_{\psi\varphi} =$$

$$= (D\psi : D\varphi)^{-1}_{-i\alpha} J (D\psi : D\varphi)^*_{-i\alpha} J \Delta^\alpha_\psi ,$$

so that the invertibility of $(D\psi : D\varphi)^{-1}_{-i\alpha} J(D\psi : D\varphi)^*_{-i\alpha} J$ gives the result. Q.E.D.

By Corollary 2.2 (iv) we have :

COROLLARY 3.3. *For φ and ψ in the above lemma, the T-theorem for $L^p(\mathscr{M}; \varphi')$ holds if and only if the same is true for $L^p(\mathscr{M}; \psi')$.*

LEMMA 3.4. *Let ε be a normal projection of norm 1 from $\mathscr{M}$ onto a von Neumann subalgebra $\mathscr{N}$, and φ be a faithful functional in $\mathscr{N}^+_*$. If the T-theorem for $L^p(\mathscr{M}; (\varphi\circ\varepsilon)')$ holds, then the same is true for $L^p(\mathscr{N}; \varphi')$.*

Proof. We may and do assume that $\varphi\circ\varepsilon = \omega_{\xi_0}$ with a cyclic and separating vector ξ_0 and that ξ_0 is also a cyclic and separating vector in the subspace $\mathscr{K} = [\mathscr{N}\xi_0]$ for the restricted von Neumann algebra $\mathscr{N} \mid \mathscr{K}$. In this set up, the modular operator associated with $(\mathscr{M}, \varphi\circ\varepsilon)$ is diagonalized as follows:

$$\Delta_{\varphi\circ\varepsilon} = \begin{bmatrix} \Delta' & 0 \\ 0 & \Delta_\varphi \end{bmatrix}$$

where Δ_φ is the modular operator on $\mathscr{K}$ determined by $(\mathscr{N}, \varphi)$. (See [14].) In particular, we have

$$\mathscr{D}(\Delta^{\alpha}_{\varphi\circ\varepsilon}) \cap \mathscr{K} = \mathscr{D}(\Delta^{\alpha}_{\varphi}),$$

so that the lemma follows from Corollary 2.2 (iv). Q.E.D.

4. A SPECIAL CASE (A FACTOR OF TYPE I_∞)

In this section, we prove (a stronger result than) our main result for a factor of type I_∞.

Let $\mathscr{K}$ be an infinite dimensional separable Hilbert space and $\mathscr{M} = B(\mathscr{K})$. We realize a standard form for $\mathscr{M}$ by the quadruple $(\mathscr{M}, \mathscr{H}, J = {}^*, \mathscr{H}_+)$. Here $\mathscr{H}$ denotes the Hilbert space of all Hilbert-Schmidt class operators on $\mathscr{K}$ and $\mathscr{M} = B(\mathscr{K})$ acts on $\mathscr{H}$ as left multiplications. We denote the usual trace on $\mathscr{M}$ by Tr. For a faithful φ (resp. an arbitrary ψ) in $\mathscr{M}_*^+$, there exists a unique non-singular positive trace class operator h_φ on $\mathscr{K}$ (resp. a unique positive trace class operator h_ψ on $\mathscr{K}$) satisfying $\varphi = \mathrm{Tr}(h_\varphi \cdot)$ (resp. $\psi = \mathrm{Tr}(h_\psi \cdot)$). In other words, $h_\varphi^{\frac{1}{2}}$ (resp. $h_\psi^{\frac{1}{2}}$) is a unique implementing vector in $\mathscr{H}_+$ for φ (resp. ψ).

The next result is essentially due to Dixmier:

LEMMA 4.1. *For faithful $\varphi, \psi \in \mathscr{M}_*^+$, there always exists a unitary operator u on $\mathscr{K}$ such that the (densely-defined) operator $h_\chi^{\frac{1}{2}} h_\varphi^{\frac{1}{p}-\frac{1}{2}}$ on $\mathscr{K}$ is not closable, where $\chi = u\psi u^*$ and $h_\chi = uh_\psi u^*$.*

Proof. Since h_φ is a non-singular compact operator on $\mathscr{K}$ and $\frac{1}{p} - \frac{1}{2} < 0$, $h_\varphi^{\frac{1}{p}-\frac{1}{2}}$ is not bounded. Similarly, $h_\psi^{-\frac{1}{2}}$ is not either.

Then, by Lemma 8.3, [4], there always exists a unitary u on $\mathscr{K}$ satisfying

$$\mathscr{D}(h_\varphi^{\frac{1}{p}-\frac{1}{2}}) \cap u\mathscr{D}(h_\varphi^{-\frac{1}{2}}) = \mathscr{D}(h_\varphi^{\frac{1}{p}-\frac{1}{2}}) \cap \mathscr{D}(h_\chi^{-\frac{1}{2}}) = \{0\}.$$

Therefore, one computes

$$\mathscr{D}(h_\varphi^{\frac{1}{p}-\frac{1}{2}} h_\chi^{\frac{1}{2}}) = h_\chi^{-\frac{1}{2}}\{\mathscr{D}(h_\varphi^{\frac{1}{p}-\frac{1}{2}}) \cap \mathscr{D}(h_\chi^{-\frac{1}{2}})\} = \{0\}.$$

As $h_\chi^{\frac{1}{2}}$ being a bounded operator on $\mathscr{K}$, the adjoint of $h_\chi^{\frac{1}{2}} h_\varphi^{\frac{1}{p}-\frac{1}{2}}$ is exactly $h_\varphi^{\frac{1}{p}-\frac{1}{2}} h_\chi^{\frac{1}{2}}$, whose domain is zero alone, so that $h_\chi^{\frac{1}{2}} h_\varphi^{\frac{1}{p}-\frac{1}{2}}$ is not closable. Q.E.D.

For vectors ξ, ζ in $\mathscr{K}$, the rank-one operator $\xi \otimes \zeta$ on $\mathscr{K}$ is defined by $(\xi \otimes \zeta)(\eta) = (\eta \mid \zeta)\ \xi, \eta \in \mathscr{K}$, as usual. Clearly, $\xi \otimes \zeta$ belongs to $\mathscr{H}$, on which $\mathscr{M}$ acts. The following results can be checked by straightforward calculation:

LEMMA 4.2. *Let φ_1 (resp. φ_2) be a faithful (resp. an arbitrary) element in $\mathscr{M}_*^+$ and ξ, ζ be vectors in $\mathscr{K}$.*

(i) *If $\zeta \in \mathscr{D}(h_{\varphi_1}^{-\frac{1}{2}})$, then $\xi \otimes \zeta \in \mathscr{D}(\Delta_{\varphi_2\varphi_1}^{\alpha})$, $0 \leqslant \alpha \leqslant \frac{1}{2}$, and*

$$\Delta_{\varphi_2\varphi_1}^{\alpha}(\xi \otimes \zeta) = (h_{\varphi_2}^{\alpha}\xi) \otimes (h_{\varphi_1}^{-\alpha}\zeta).$$

(ii) *If $\xi \in \mathscr{D}(h_{\varphi_2}^{-\frac{1}{2}})$, then $\xi \otimes \zeta \in \mathscr{D}(\Delta_{\varphi_2\varphi_1}^{-\alpha})$, $0 \leqslant \alpha \leqslant \frac{1}{2}$, and*

$$\Delta_{\varphi_2\varphi_1}^{-\alpha}(\xi \otimes \zeta) = (h_{\varphi_2}^{-\alpha}\xi) \otimes (h_{\varphi_1}^{\alpha}\zeta).$$

LEMMA 4.3. *For φ and χ in Lemma 4.1, the (densely-defined) operator $\Delta_{\chi\varphi}^{\frac{1}{2}}\Delta_{\varphi}^{\frac{1}{p}-\frac{1}{2}}$ on $\mathscr{H}$ is* **not** *closable.*

Proof. By the previous lemma, $(\Delta_{\chi\varphi}^{\frac{1}{2}}\Delta_{\varphi}^{\frac{1}{p}-\frac{1}{2}})(\xi \otimes \zeta) = h_{\chi}^{\frac{1}{2}}h_{\varphi}^{\frac{1}{p}-\frac{1}{2}}\xi \otimes h_{\varphi}^{-\frac{1}{p}}\zeta$, whenever $\xi \in \mathscr{D}(h_{\chi}^{\frac{1}{2}}h_{\varphi}^{\frac{1}{p}-\frac{1}{2}})$ and $\zeta \in \mathscr{D}(h_{\varphi}^{-\frac{1}{p}})$. At first, we fix a non-zero ζ_0 in $\mathscr{D}(h_{\varphi}^{-\frac{1}{p}})$. Then, it follows from Lemma 4.1 that there exists a sequence $\{\xi_n\}$ in $\mathscr{D}(h_{\chi}^{\frac{1}{2}}h_{\varphi}^{\frac{1}{p}-\frac{1}{2}})$ such that $\{\xi_n\}$ converges to 0 and $\{h_{\chi}^{\frac{1}{2}}h_{\varphi}^{\frac{1}{p}-\frac{1}{2}}\xi_n\}$ converges to a non-zero ξ (in $\mathscr{K}$).

Checking a sequence $\{\xi_n \otimes \zeta_0\}_n$ in $\mathscr{D}(\Delta_{\chi\varphi}^{\frac{1}{2}}\Delta_{\varphi}^{\frac{1}{p}-\frac{1}{2}})$, we easily conclude that $\Delta_{\chi\varphi}^{\frac{1}{2}}\Delta_{\varphi}^{\frac{1}{p}-\frac{1}{2}}$ is not closable. Q.E.D.

Summing up the arguments in this section, one obtains:

PROPOSITION 4.4. *Let $\mathscr{M}$ be a factor of type I_∞, and φ, ψ be faithful functionals in $\mathscr{M}_*^+$. There always exists a unitary u in $\mathscr{M}$ such that $\Delta_{\chi\varphi}^{\frac{1}{2}}\Delta_{\varphi}^{\frac{1}{p}-\frac{1}{2}}$ is* **not** *closable. Here χ is given by $\chi = u\psi u^*$. In particular, the T-theorem for $L^p(\mathscr{M}; \varphi')$ never holds.*

5. PROOFS OF MAIN RESULTS

We still keep our assumption: $2 < p < \infty$.

Proof of Theorem 2.5. To show the theorem by contradiction, we assume that the T-theorem for $L^p(\mathscr{M}; \varphi')$ holds for any faithful $\varphi \in \mathscr{M}_*^+$ and that $\mathscr{M}$ is not finite.

Cutting $\mathscr{M}$ by the largest finite central projection, we may and do assume that $\mathscr{M}$ is actually properly infinite. Thus $\mathscr{M}$ can be written as the tensor product of a von Neumann subalgebra and a factor $\mathscr{N}$ of type I_∞. Then we choose a normal projection ε of norm 1 from $\mathscr{M}$ onto $\mathscr{N}$ and a faithful functional φ in $\mathscr{N}_*^+$. By the assumption, the T-theorem for $L^p(\mathscr{M}; (\varphi\circ\varepsilon)')$ holds so that the same is true for $L^p(\mathscr{N}; \varphi')$ by Lemma 3.4, which contradicts Proposition 4.4. Q.E.D.

Proof of Theorem 2.6. We may assume that $\mathscr{M}$ is a factor of type III_λ due to Proposition 4.4. To show the theorem by contradiction, we also assume that the T-theorem for $L^p(\mathscr{M}; \varphi')$ holds for **some** faithful $\varphi\in\mathscr{M}_*^+$. Due to Theorem 2.5, it suffices to show that the T-theorem for $L^p(\mathscr{M}; \psi')$ holds for a generic faithful $\psi\in\mathscr{M}_*^+$. However, since $\mathscr{M}$ is a factor of type III_λ, $0<\lambda<1$, there always exists an inner automorphism α and positive numbers l_1, $l_2\geqslant 0$ satisfying $\varphi\leqslant l_1\psi\circ\alpha$, $\psi\circ\alpha\leqslant l_2\varphi$. (See [3], Ch. II, Corollary 4.2.) Thus, Lemma 3.1 and Corollary 3.3 yield that the T-theorem for $L^p(\mathscr{M}; \varphi')$ would hold as well. Q.E.D.

Acknowledgement. We would like to thank the referee for useful suggestions. Among them, the referee informed us that a lemma closely related to Dixmier's lemma used in the proof of Lemma 4.1 was obtained by von Neumann (Zur theorie der unbeschrankten Matrizen, *J. für Mathematik*, **161**(1929), 208–234; Satz 18).

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Received March 19, 1981 ; revised August 19, 1981.

Added in proof. Recently, the author showed that the mapping in Theorem 1.2 is actually a homeomorphism with respect to the norm topologies.