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NEST-SUBALGEBRAS OF VON NEUMANN ALGEBRAS: COMMUTANTS MODULO COMPACTS AND DISTANCE ESTIMATES

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INTRODUCTION

In recent years several papers have appeared which focus on the structure of the commutant modulo compacts, or "essential commutant", of certain algebras of operators on Hilbert space. Perhaps the most important of these is the work of B. Johnson and S. Parrott [19] in which it was shown that the essential commutant of a von Neumann algebra which does not contain certain intractable type II_1 factors as direct summands decomposes as the sum of the algebraic commutant of the von Neumann algebra and the compact operators. Subsequently, answering a question of R. Douglas, K. Davidson [11] characterized the essential commutant of the analytic Toeplitz operators as the sum of the compact operators and those Toeplitz operators with symbol in $H^\infty + C$. More recently, E. Christensen and C. Peligrad [10] have shown that the essential commutant of an arbitrary nest algebra decomposes as scalar multiples of the identity plus the compact operators. Since the algebraic commutant of a nest algebra is trivial, this result is of the same basic type as [19].

In this paper we present a characterization of the essential commutant of a class of operator algebras which generalizes certain aspects of the work of [19] as well as that of [10]. In addition we obtain results concerning the Arveson distance estimate for an operator to certain operator algebras.

To a fixed von Neumann algebra $\mathscr{B}$ and a complete nest $\mathscr{N}$ of projections contained therein one associates the algebra $\mathscr{A}$ of all operators in $\mathscr{B}$ which leave invariant every projection in $\mathscr{N}$. So $\mathscr{A} = \mathscr{B} \cap \mathscr{A}_{\mathscr{N}}$ where $\mathscr{A}_{\mathscr{N}}$ denotes the nest algebra of $\mathscr{N}$ in $\mathscr{L}(H)$. $\mathscr{A}$ is then a reflexive operator algebra with invariant subspace (projection) lattice equal to the reflexive lattice generated by $\mathscr{N}$ together with the projections in the commutant of $\mathscr{B}$ in $\mathscr{L}(H)$. The algebra $\mathscr{A}$ is called the nest subalgebra of the von Neumann algebra $\mathscr{B}$ relative to the nest $\mathscr{N}$.

An investigation of the structure of nest subalgebras of von Neumann algebras was initiated by the present authors in [15] in which present terminology and basic elements of the theory were established. In the present paper we show that if $\mathscr{A}$ is a separably acting nest subalgebra of a von Neumann algebra $\mathscr{B}$ then the essential commutant of $\mathscr{A}$ admits the algebraic commutant plus compacts decomposition whenever the essential commutant of $\mathscr{B}$ admits a like decomposition. Also, the algebraic commutants of $\mathscr{A}$ and $\mathscr{B}$ are always equal. Moreover, even in cases where the decomposition for esscomm$\mathscr{B}$ may fail it remains true that esscomm$\mathscr{A}$ = = esscomm$\mathscr{B}$. The parallel results in [10] and [19] concerning derivations and automorphisms are not considered in this paper. The results in this paper were obtained independently and at about the same time as [10]. Our original proofs for the simple nest algebra case ($\mathscr{B} = \mathscr{L}$(H)) proceeded essentially the same as the results in [10].

We remark that there is some contact between the present work and the theory of algebras of analytic operators in von Neumann algebras associated with certain groups of $*$-automorphisms. This theory has its roots in the paper by W. Arveson [2] and was investigated and developed by R. Loebl and P. Muhly in [23]. In particular, nest subalgebras of von Neumann algebras are precisely those algebras of analytic operators which arise from ultraweakly continuous representations of **R** as groups of inner $*$-automorphisms on separably acting von Neumann algebras ([23], Theorem 4.2.3). Also, nest-subalgebras of *finite* von Neumann algebras are maximal subdiagonal algebras in the sense of Arveson [1]. In addition, nest subalgebras of factors with relative maximal abelian core are the most tractable special cases of the triangular subalgebras of factors defined and investigated by R. Kadison and I. M. Singer in [20].

In the process of our investigation we obtain some results of the type studied by Davidson [12] relating to Arveson's distance formula for nest algebras [4]. In particular we show that if $\mathscr{A}$ is a nest subalgebra of an arbitrary approximately finite dimensional von Neumann algebra there is a positive constant C such that

$$\operatorname{dist}(T, \mathscr{A}) \leqslant C \sup\{\|P^{\perp}TP\| : P \in \operatorname{Lat}\mathscr{A}\}$$

for every $T \in \mathscr{L}(H)$. Thus, via a result of Christensen such a distance estimate holds for an arbitrary von Neumann algebra $\mathscr{B}$ if and only if every derivation from its commutant $\mathscr{B}'$ into $\mathscr{L}(H)$ is inner (i.e., iff $H^1(\mathscr{B}', \mathscr{L}(H)) = 0$). This answers a question posed in [12] for the case in which $\mathscr{B}$ is a purely atomic m.a.s.a.. We were unable to show whether the corresponding result for nest subalgebras extends from the approximately finite dimensional case to the general setting.

After submission of this manuscript we received the thesis of Niels Anderson which contained, in particular, an alternate (and independent) proof that an abelian von Neumann algebra satisfies an Arveson distance formula with constant and an extension of this result to von Neumann algebras with property P. Moreover Eric Christensen has communicated to us that he has been aware for some time of the above mentioned results for von Neumann algebras and the connection between the Arveson distance estimate and the derivation problem for von Neumann algebras given in Remark 4.6 of this paper.

Finally, we remark that due to the "discrete" nature of compact operators our proofs do not require usage of the components of direct integral theory as was needed in our earlier paper [15]. Also, since certain of our key proofs depend on separability of the underlying Hilbert space we make the blanket assumption that all Hilbert spaces considered in this paper will be separable. However, certain of our results, especially those of § 2 and § 3, extend easily to the nonseparable case.

1. PRELIMINARIES

Throughout this paper all operators will be bounded, all subspaces closed, and all projections will be self-adjoint. We write $\mathscr{L}(H)$ for the collection of all bounded operators on a Hilbert space H, and we write $\mathscr{L}\mathscr{C}(H)$ for the ideal of compact operators in $\mathscr{L}(H)$.

If $A \in \mathscr{L}(H)$ the notation $[AH]$ will denote the closed range of A. Also, if T, R are operators and $\mathscr{S}$ a set of operators then $T\mathscr{S}R$ will denote the set $\{TSR : S \in \mathscr{S}\}$. We use the notation $\mathscr{S}'$ for the algebraic commutant of $\mathscr{S}$, and esscomm$\mathscr{S}$ will denote the set of all operators in $\mathscr{L}(H)$ that commute with every member of $\mathscr{S}$ modulo $\mathscr{L}\mathscr{C}(H)$.

Let $\mathscr{L}$ be a collection of subspaces containing $\{0\}$ and H which form a lattice under the operations $\vee$ and $\wedge$, where $M \vee N$ is the subspace generated by M and N while $M \wedge N$ is the intersection $M \cap N$. $\mathscr{L}$ is *commutative* if the projections on the subspaces commute pairwise, and is *complete* if the meet and join of every subset of $\mathscr{L}$ are also in $\mathscr{L}$. $\mathscr{L}$ is a *nest* (usually denoted by $\mathscr{N}$) if the lattice is linearly ordered by inclusion.

For convenience we shall disregard the distinction between a subspace of H and the orthogonal projection onto it. Thus a lattice will consist of either subspaces or projections depending on the context in which it is used.

As usual, we write Lat$\mathscr{S}$ for the lattice of all projections left invariant under every operator in a subset $\mathscr{S}$ of $\mathscr{L}(H)$, and dually Alg$\mathscr{L}$ denotes the algebra of all operators leaving each member of a set $\mathscr{L}$ of projections invariant. The term *subspace lattice* will denote a lattice of projections that is closed in the strong operator topology. An algebra $\mathscr{A}$ is *reflexive* if $\mathscr{A} = \text{AlgLat}\mathscr{A}$, and dually a lattice $\mathscr{L}$ is reflexive if $\mathscr{L} = \text{LatAlg}\mathscr{L}$. Subspace lattices need not be reflexive; however, *commutative* subspace lattices *are* reflexive [3, 12]. In particular, nests are reflexive.

If $\mathcal{N}$ is a nest of projections (subspaces) we use the notation $\mathcal{A}_{\mathcal{N}}$ to denote the nest algebra Alg$\mathcal{N}$. The *core* $\mathcal{C}_{\mathcal{N}}$ is the von Neumann algebra generated by the projections in $\mathcal{N}$. The *diagonal* of a nest algebra $\mathcal{A}_{\mathcal{N}}$ is the von Neumann algebra $\mathcal{D}_{\mathcal{N}} = \mathcal{A}_{\mathcal{N}} \cap \mathcal{A}_{\mathcal{N}}^*$. We have $\mathcal{D}_{\mathcal{N}} = \mathcal{C}'_{\mathcal{N}}$, and $\mathcal{C}_{\mathcal{N}}$ is the center of $\mathcal{D}_{\mathcal{N}}$. An *$\mathcal{N}$-interval* is a projection $E = M - N$ with $M, N \in \mathcal{N}$ and $M > N$. The projections M, N are called the *upper* and *lower* endpoints of E, respectively. The endpoints of a nonzero $\mathcal{N}$-interval are well-defined in the sense that if $M - N = M_1 - N_1 \neq 0$ with $M, N, M_1, N_1 \in \mathcal{N}$ then necessarily $M = M_1$ and $N = N_1$. We say that the $\mathcal{N}$-intervals E and F are *strictly ordered*, and write $E \ll F$, if the upper endpoint of E is contained in the lower endpoint of F. A von Neumann algebra is called *nonatomic* if it contains no minimal projections. A nest $\mathcal{N}$ of projections is said to be nonatomic if its core is nonatomic, or equivalently, if there exist no minimal $\mathcal{N}$-intervals. (This differs from standard lattice-theoretic terminology.)

Let $\mathcal{B}$ be a von Neumann algebra, let $\mathcal{N}$ be a complete nest of projections contained in $\mathcal{B}$, and let $\mathcal{A} = \mathcal{B} \cap \mathcal{A}_{\mathcal{N}}$ denote the nest subalgebra of $\mathcal{B}$ relative to $\mathcal{N}$. Then $\mathcal{A}$ is reflexive since it is the intersection of two reflexive algebras. If we let $\mathcal{M}$ denote the lattice of projections in $\mathcal{B}'$, then $\mathcal{A} = \mathrm{Alg}(\mathcal{M} \vee \mathcal{N})$, where $\mathcal{M} \vee \mathcal{N}$ denotes the subspace lattice generated by $\mathcal{M}$ and $\mathcal{N}$. The authors have shown that $\mathcal{M} \vee \mathcal{N}$ is reflexive (unpublished). We will for convenience adopt the notation *n.s.v.a.* to denote a nest subalgebra of a von Neumann algebra.

We present a few basic properties from [15] which are useful in the sequel. The notation is that of the previous paragraph.

LEMMA 1.1. *$\mathcal{A}$ will contain a m.a.s.a. iff $\mathcal{B}$ contains a m.a.s.a.*

LEMMA 1.2. *$PBP^{\perp} \in \mathcal{A}$ for every $B \in \mathcal{B}$ and $P \in \mathcal{N}$.*

LEMMA 1.3. *$\mathcal{A} + \mathcal{A}^*$ is ultraweakly dense in $\mathcal{B}$.*

Proof. This is a result of ([23], Theorem 3.15). See also ([1], p. 589).

LEMMA 1.4. *The center of $\mathcal{A}$ coincides with the center of $\mathcal{B}$.*

LEMMA 1.5. *$\mathcal{A}$ will equal $\mathcal{B}$ iff $\mathcal{A}$ is selfadjoint iff $\mathcal{N}$ lies in the center of $\mathcal{B}$.*

LEMMA 1.6. *If E is a central projection then $\mathcal{A}_E = \mathcal{A}|EH$ is the nest subalgebra of $\mathcal{B}_E = \mathcal{B}|EH$ relative to the nest $\mathcal{N}_E = \mathcal{N}|EH$.*

REMARK. Lemma 1.3 shows that a n.s.v.a. is always large in that it generates the von Neumann algebra it is defined in terms of. So a n.s.v.a. is associated with a uniquely determined von Neumann algebra. Different nests may give rise to the same n.s.v.a., however. Lemma 1.4 shows that there is no ambiguity in speaking of a "central projection" without specifying whether for $\mathcal{A}$ or for $\mathcal{B}$.

2. ALGEBRAIC COMMUTANTS AND LATTICES WITH COMPARABLE ELEMENTS

We must first show that the algebraic commutant of a n.s.v.a. coincides with that of its associated von Neumann algebra. The following lemma is useful.

LEMMA 2.1. *Let $\mathscr{B}$ be an operator algebra containing a projection P with the properties $\bigvee_{B\in\mathscr{B}} [PBP^{\perp}H] = PH$ and $\bigvee_{B\in\mathscr{B}} [P^{\perp}B^{*}PH] = P^{\perp}H$. Then the commutant of $\mathscr{B}$ equals the commutant of the subset $\{P, P\mathscr{B}P^{\perp}\}$.*

Proof. Suppose T commutes with P and with $P\mathscr{B}P^{\perp}$. For B_0, B arbitrary in $\mathscr{B}$ we have $[(PB_0P)T - T(PB_0P)]\,PBP^{\perp} = 0$ since $\mathscr{B}$ is an algebra containing P and T commutes with $P\mathscr{B}P^{\perp}$. Also, since T commutes with P the commutator $(PB_0P)T - T(PB_0P)$ has support in P. Hence, since $\bigvee_{B\in\mathscr{B}} [PBP^{\perp}H] = PH$ this commutator must be 0. So T commutes with $P\mathscr{B}P$. Similarly, we compute $PBP^{\perp}[(P^{\perp}B_0P^{\perp})T - T(P^{\perp}B_0P^{\perp})] = 0$ from which it follows that T commutes with $P^{\perp}\mathscr{B}P^{\perp}$. Finally, we compute using the above results that $PBP^{\perp}[(P^{\perp}B_0P)T - T(P^{\perp}B_0P)] = 0$ which implies T commutes with $P^{\perp}\mathscr{B}P$, and by hypothesis T commutes with $P\mathscr{B}P^{\perp}$, hence $T \in \mathscr{B}'$. ▨

Recall [16] that an element L in a lattice $\mathscr{L}$ is *comparable* for $\mathscr{L}$ if for every $L' \in \mathscr{L}$ either $L' \leqslant L$ or $L \leqslant L'$. Thus a lattice is a nest iff each of its elements is comparable.

LEMMA 2.2. *Let $\mathscr{L}$ be a subspace lattice containing a comparable element P. Then $P \in \mathrm{Alg}\mathscr{L}$ and $P\mathscr{L}(H)P^{\perp} \subset \mathrm{Alg}\mathscr{L}$.*

Proof. $P \in \mathrm{Alg}\mathscr{L}$ since it commutes with every member of $\mathscr{L}$. Now fix $L \in \mathscr{L}$ and suppose $A = PAP^{\perp}$ is an arbitrary member of $P\mathscr{L}(H)P^{\perp}$. If $L \leqslant P$ then $ALH = \{0\} \subseteq LH$. If $L \geqslant P$ then $ALH \subseteq PH \subseteq LH$. So $ALH \subseteq LH$ for every $L \in \mathscr{L}$. That is, $A \in \mathrm{Alg}\mathscr{L}$. ▨

COROLLARY 2.3. *If $\mathscr{L}$ is a subspace lattice containing a nontrivial comparable element then the algebraic commutant of* $\mathrm{Alg}\mathscr{L}$ *is trivial.*

Proof. If P is comparable for $\mathscr{L}$ with $P \neq 0, I$, just apply (2.1) with $\mathscr{B} = \mathscr{L}(H)$ noting that P satisfies the properties of the lemma and that the set $\{P, P\mathscr{B}P^{\perp}\}$ is contained in $\mathrm{Alg}\mathscr{L}$ by (2.2). ▨

The above corollary has an obvious extension. We do not assume $\mathscr{L}$ is entirely contained in $\mathscr{B}$.

COROLLARY 2.4. *If $\mathscr{L}$ is a subspace lattice containing a nontrivial comparable element P and if $\mathscr{B}$ is a von Neumann algebra containing P such that the central support of both P and $P^{\perp}$ in $\mathscr{B}$ is I then the algebraic commutant of the intersection* $(\mathrm{Alg}\mathscr{L}) \cap \mathscr{B}$ *equals the algebraic commutant of $\mathscr{B}$.*

Proof. By 2.2 we have $\{P, P\mathscr{B}P^{\perp}\} \subset (\mathrm{Alg}\mathscr{L}) \cap \mathscr{B}$. Now note that a projection P in a von Neumann algebra $\mathscr{B}$ satisfies the hypotheses of Lemma 2.1 if and only if both P and $P^{\perp}$ have central support I. ▨

THEOREM 2.5. *The algebraic commutant of an arbitrary n.s.v.a. equals the algebraic commutant of its von Neumann algebra.*

Proof. Let $\mathscr{A}$ be a nest subalgebra of a von Neumann algebra $\mathscr{B}$ relative to a nest $\mathscr{N} \subset \mathscr{B}$. So $\mathscr{A} = \mathscr{B} \cap (\mathrm{Alg}\mathscr{N})$. If $\mathscr{N} = \{0, I\}$ then $\mathscr{A} = \mathscr{B}$ so we are done. If for some $P \in \mathscr{N}$ both P and $P^{\perp}$ have central support I then Corollary 2.4 yields the desired result. If $\mathscr{B}$ is not a factor it can happen that no projection in a nontrivial nest $\mathscr{N} \subset \mathscr{B}$ satisfies this property. We first reduce to the case in which $\mathscr{A}$ is completely nonselfadjoint and show that for such an n.s.v.a. there exists a projection P in the subspace lattice generated by $\mathscr{N}$ together with the central projections $\mathscr{M}$ in $\mathscr{B}$ such that the central support of both P and $P^{\perp}$ is I and $P\mathscr{B}P^{\perp} \subset \mathscr{A}$.

Let E be the join of all central projections P with $\mathscr{A}P$ selfadjoint. Then $\mathscr{A}_E = \mathscr{A}|EH$ is selfadjoint and $\mathscr{A}_{E^{\perp}}$ is completely nonselfadjoint in the sense that it contains no central projection for which the compression algebra is selfadjoint. By Lemmas 1.5 and 1.6 $\mathscr{A}_E = \mathscr{B}_E$ and $\mathscr{A}_{E^{\perp}}$ is the n.s.v.a. of $\mathscr{B}_{E^{\perp}}$ relative to $\mathscr{N}_{E^{\perp}}$. The problem thus reduces to showing equality of the commutants of $\mathscr{A}_{E^{\perp}}$ and $\mathscr{B}_{E^{\perp}}$, and thus without loss of generality we can assume that our original algebra $\mathscr{A}$ is completely nonselfadjoint.

For each $N \in \mathscr{N}$ let E_N and F_N be the central supports of N and $N^{\perp}$ respectively, and let $G_N = E_N \wedge F_N$. If $G_N = 0$ then N is central, so $G_N \neq 0$ for some N. G_N is the central support of both NG_N and $N^{\perp}G_N$. Let $G = \bigvee_N G_N$. If $G \neq I$ then $\mathscr{N}_{G^{\perp}}$ consists of central projections of $\mathscr{B}_{G^{\perp}}$, so $\mathscr{A}_{G^{\perp}} = \mathscr{B}_{G^{\perp}}$ by 1.5 and 1.6 contradicting the hypothesis that $\mathscr{A}$ is completely nonselfadjoint. Thus $\bigvee_N G_N = I$. A simple Zorn's lemma argument now yields a family $\{P_\lambda\}$ of mutually orthogonal central projections with $\sum P_\lambda = I$ such that each P_λ is a subprojection of some G_N. For each λ choose $N_\lambda \in \mathscr{N}$ such that $P_\lambda \leqslant G_{N_\lambda}$, and set $P = \sum_\lambda N_\lambda P_\lambda$. Since the central support of both $N_\lambda P_\lambda$ and $N_\lambda^{\perp} P_\lambda$ is P_λ the central support of both P and $P^{\perp}$ is necessarily I.

We have $P \in \mathscr{N} \vee \mathscr{M}$ where $\mathscr{M}$ is the lattice of central projections in $\mathscr{B}$. Since each P_λ is central we have

$$N_\lambda P_\lambda \mathscr{B}(P_\lambda - N_\lambda P_\lambda) = N_\lambda \mathscr{B} N_\lambda^{\perp} P_\lambda \subset \mathscr{A}$$

and hence $P\mathscr{B}P^{\perp} \subset \mathscr{A}$. Now apply Lemma 2.1 noting that P satisfies the hypotheses of the lemma and $\{P, P\mathscr{B}P^{\perp}\} \subset \mathscr{A} \subseteq \mathscr{B}$. ▨

REMARK. The above proof is in fact valid for arbitrary Hilbert space. An alternate proof valid only in the separable case would utilize the fact that ([15], Theorem 4.4) $\mathscr{A}$ is equivalent to a direct integral of nest subalgebras of factor von Neumann algebras. Then the result would follow from Corollary 2.4 and the fact that the commutant of the direct integral of a measurable field of strongly closed algebras equals the direct integral of the commutants of the integrand algebras ([5], Lemma 4.6).

REMARK. Proposition 2.5 shows in particular that the commutant of an n.s.v.a. is selfadjoint. No more elementary proof of this fact is known to us. Indeed, if a simple proof that the commutant of an n.s.v.a. is selfadjoint were obtained then this together with Lemma 1.3 would immediately yield an alternate proof of Proposition 2.5.

3. ESSENTIAL COMMUTANTS OF CERTAIN OPERATOR ALGEBRAS

We begin with an "essential" version of Lemma 2.1. While the scope of its direct application is somewhat limited the ideas in its proof are suggestive of the approach taken in the sequel.

LEMMA 3.1. *Let $\mathscr{B}$ be an operator algebra containing a projection P with the properties:*

(i) *If $A \in \mathscr{L}(H)$ with PAP not compact there exists $B \in \mathscr{B}$ with $PAPBP^{\perp}$ not compact,*

(ii) *If $A \in \mathscr{L}(H)$ with $P^{\perp}AP^{\perp}$ not compact there exists $B \in \mathscr{B}$ with $PBP^{\perp}AP^{\perp}$ not compact.*

Then $\operatorname{esscomm}\mathscr{B} = \operatorname{esscomm}\{P, P\mathscr{B}P^{\perp}\}$.

Proof. Suppose $T \in \mathscr{L}(H)$ and T commutes modulo compacts with P and with all operators $PBP^{\perp}$, $B \in \mathscr{B}$. For $B_0, B \in \mathscr{B}$ the operator $[(PB_0P)T - T(PB_0P)]PBP^{\perp}$ is compact since $\mathscr{B}$ is an algebra containing P and T essentially commutes with $P\mathscr{B}P^{\perp}$. So since $TP - PT$ is compact the operator $(P(B_0T - TB_0)PBP^{\perp}$ must also be compact. Since $B \in \mathscr{B}$ is arbitrary we conclude that $P(B_0T - TB_0)P$ is compact and hence $(PB_0P)T - T(PB_0P)$ is compact. So T essentially commutes with $P\mathscr{B}P$. Similarly we compute that the operator $PBP^{\perp}[(P^{\perp}B_0P^{\perp})T - T(P^{\perp}B_0P^{\perp})]$ is compact from which it follows that T essentially commutes with $P^{\perp}\mathscr{B}P^{\perp}$. Finally we compute using the above results that $PBP^{\perp}[(P^{\perp}B_0P)T - T(P^{\perp}B_0P)]$ is compact. From this it follows that $PBP^{\perp}(B_0T - TB_0)P$ is compact. Let $S = P^{\perp}(B_0T - TB_0)P$. Then $SS^* = P^{\perp}SS^*P^{\perp}$, and $PBP^{\perp}SS^*P^{\perp}$ is compact for all $B \in \mathscr{B}$, so by hypothesis SS^* and hence S is compact. It follows that $(P^{\perp}B_0P)T - T(P^{\perp}B_0P)$ is compact. So T essentially commutes with $P^\perp \mathscr{B} P$. By hypothesis we already have that T essentially commutes with $P\mathscr{B}P^\perp$, so we conclude $T \in \operatorname{esscomm}\mathscr{B}$. ▨

COROLLARY 3.2. *Let $\mathscr{B}$ be an operator algebra containing an infinite rank and infinite corank projection P. If $\mathscr{B}$ contains a partial isometry S such that $SS^* - P$ and $S^*S - P^\perp$ are compact then* $\operatorname{esscomm}\mathscr{B} = \operatorname{esscomm}\{P, P\mathscr{B}P^\perp\}$.

COROLLARY 3.3. *If $\mathscr{L}$ is a subspace lattice with an infinite rank and infinite corank comparable element then* $\operatorname{esscomm}\operatorname{Alg}\mathscr{L} = \mathbf{C}I +$ *compacts.*

Recall that a finite *partition* of a projection P relative to a von Neumann algebra $\mathscr{B}$ is a finite set of mutually $\perp$ projections in $\mathscr{B}$ whose sum is P.

COROLLARY 3.4. *Let $\mathscr{A}$ be a nest subalgebra of a von Neumann algebra $\mathscr{B}$ relative to a nest $\mathscr{N} \subset \mathscr{B}$. If $\mathscr{N}$ contains a projection P with the property that P has a finite partition each of whose members is equivalent in $\mathscr{B}$ to a subprojection of $P^\perp$ and that $P^\perp$ has a finite partition each of whose members is equivalent to a subprojection of P, then* $\operatorname{esscomm}\mathscr{A} = \operatorname{esscomm}\mathscr{B}$.

REMARKS. Corollary 3.4 shows immediately that $\operatorname{esscomm}\mathscr{A} = \operatorname{esscomm}\mathscr{B}$ whenever $\mathscr{B}$ is a type I finite, type II finite or type III factor. Unfortunately, this is not enough to enable one to proceed directly from the factor case to cases in which the factor appears as a direct summand in the decomposition of a more general n.s.v.a.. Certain additional estimates are required relating norms of compact operators to norms of their corresponding derivations restricted to the summand n.s.v.a.s.. This is done in the proof of the general case. Also, Corollary 3.4 does not apply directly to the type I or type II infinite factor cases. The proofs for these cases are necessarily more technical.

The following example reveals limitations in the direct application of Lemma 3.1.

EXAMPLE 3.5. A projection with finite complement in a type $I_{\infty,\infty}$ factor $\mathscr{B}$ will fail to have the property in 3.1 relative to $\mathscr{B}$ even though $\mathscr{B}$ contains no compacts. Indeed, suppose $P^\perp$ is a minimal projection in $\mathscr{B}$ and let $\{E_i\}$ be a sequence of mutually $\perp$ projections in $\mathscr{B}$ with $\sum E_i = P$. For each i let T_i be a norm one compact operator in $\mathscr{L}(H)$ with $T_i = E_i T_i E_i$, and let $T = \sum T_i$. Then T is not compact but $PTPBP^\perp$ is compact for every $B \in \mathscr{B}$. Indeed, if we fix $B \in \mathscr{B}$ with $PBP^\perp \neq 0$ and let $Q = [PBP^\perp H]$ then Q is a minimal projection in $\mathscr{B}$ and consequently the sum $\sum E_i Q = PQ$ converges in *norm*. So $PTPBP^\perp = PTPQBP^\perp = \sum PTE_i QBP^\perp$ is compact since each summand is compact and the series is norm convergent. Clearly this argument extends to the case in which $P^\perp$ is a finite sum of minimal projections.

4. DERIVATION NORMS AND DISTANCE ESTIMATES

If T is an operator in $\mathscr{L}(H)$ we denote by δ_T the derivation $A \to AT - TA$ from $\mathscr{L}(H)$ into itself. If $\mathscr{A} \subseteq \mathscr{L}(H)$ is an algebra then $\|\delta_T\|_{\mathscr{A}}$ will denote the norm of the restriction of δ_T to $\mathscr{A}$.

An *expectation* from $\mathscr{L}(H)$ onto a von Neumann subalgebra $\mathscr{B}$ is a norm 1 idempotent positive linear map Φ from $\mathscr{L}(H)$ onto $\mathscr{B}$ with $\Phi(I) = I$ and with $\Phi(ATB) = A\Phi(T)B$, $A, B \in \mathscr{B}$, $T \in \mathscr{L}(H)$. There is a standard way of constructing an expectation from $\mathscr{L}(H)$ onto an approximately finite dimensional (AF) von Neumann algebra $\mathscr{B}$ via an invariant mean on an amenable group G of unitaries in the commutant of $\mathscr{B}$ which generates $\mathscr{B}'$ as a W^*-algebra. A. Connes has shown that the only separably acting von Neumann algebras which admit expectations (i.e., have property P) are the AF ones. We will require the formal machinery of the expectation construction in the case where $\mathscr{B}'$ is abelian.

Let G be a discrete abelian group. An invariant mean on G is a linear functional $M(\cdot)$ on the Banach space of all bounded complex-valued functions on G such that

(1) For real f, $\inf\{f(x) : x \in G\} \leqslant M(f) \leqslant \sup\{f(x) : x \in G\}$.

(2) For each $g \in G$, if $f_g(x) = f(gx)$, then $M(f_g) = M(f)$.

Every discrete abelian group has an invariant mean ([17], p. 231, Theorem 17.5) although means are not in general unique unless the group is finite.

Now let $\mathscr{E}$ be an abelian von Neumann algebra in $\mathscr{L}(H)$, let $\mathscr{U}$ be a group of unitary operators in $\mathscr{E}$ which generates $\mathscr{E}$ as a von Neumann algebra, and let M be an invariant mean on $\mathscr{U}$. Let $\mathscr{L}_*$ denote the ideal of trace class operators in $\mathscr{L}(H)$, and identify $\mathscr{L}(H)$ with $(\mathscr{L}_*)^*$ by writing $(T,f) = \mathrm{Tr}(Tf)$, $T \in \mathscr{L}(H)$, $f \in \mathscr{L}_*$. If $A \in \mathscr{L}(H)$, $f \in \mathscr{L}_*$, denote the mean of the bounded complex valued function $U \to (U^*AU, f)$ by $M_U(U^*AU, f)$, and define $\psi(A) \in \mathscr{L}(H)$ by $(\psi(A), f) = M_U(U^*AU, f), f \in \mathscr{L}_*$. Translation invariance of M implies that $U^*\psi(A)U = \psi(A)$ for every $U \in \mathscr{U}$, so $\psi(A) \in \mathscr{U}' = \mathscr{E}'$. Also, if $A \in \mathscr{E}'$ then A commutes with every U so $\psi(A) = A$. It is easily verified that $\psi : \mathscr{L}(H) \to \mathscr{E}'$ is a positive linear map (in fact a projection) of norm 1. It can be verified that ψ is completely positive. In addition, if $B \in \mathscr{L}(H)$ and $A \in \mathscr{E}'$ then $\psi(AB) = A\psi(B)$ and $\psi(BA) = \psi(B)A$, so ψ is an expectation of $\mathscr{L}(H)$ onto $\mathscr{E}'$.

We will refer to an expectation of $\mathscr{L}(H)$ onto the commutant of an abelian von Neumann algebra constructed in the above fashion as a *diagonal projection*. If the abelian von Neumann algebra $\mathscr{E}$ is not purely atomic the diagonal projection ψ will be neither faithful nor normal on $\mathscr{L}(H)$ and will not be ultraweakly continuous. However, by a standard separation theorem for each $T \in \mathscr{L}(H)$ the image $\psi(T)$ is in the ultraweakly closed convex hull of the operators U^*TU, $U \in \mathscr{U}$, and this is essentially the feature most frequently used.

LEMMA 4.1. *Let $\mathscr{E}$ be an abelian von Neumann algebra in $\mathscr{L}(H)$ and let $T \in \mathscr{L}(H)$ be arbitrary. If $\mathscr{U}$ is any unitary group which generates $\mathscr{E}$, and if M is any invariant mean on $\mathscr{U}$, then*

$$\frac{1}{2}\|\delta_T\|_{\mathscr{E}} \leqslant \operatorname{dist}(T, \mathscr{E}') \leqslant \|T - \psi(T)\| \leqslant$$

$$\leqslant \sup\{\|UT - TU\| : U \in \mathscr{U}\} \leqslant \|\delta_T\|_{\mathscr{E}},$$

where ψ is the diagonal projection onto $\mathscr{E}'$ associated with $\mathscr{U}$ and M.

Proof. For each trace class operator f we have

$$M_U(U^*(UT - TU), f) = M_U(T - U^*TU, f) = (T - \psi(T), f),$$

and

$$|M_U(U^*(UT - TU), f)| \leqslant \sup_U \|UT - TU\| \, \|f\|_1,$$

where $\|\cdot\|_1$ denotes the trace class norm. So for $\|f\|_1 = 1$ we have

$$|(T - \psi(T), f)| \leqslant \sup\{\|UT - TU\| : U \in \mathscr{U}\}$$

and hence

$$\|T - \psi(T)\| \leqslant \sup_U \|UT - TU\|.$$

The remaining inequalities are obvious. ▨

COROLLARY 4.2. *Let $\mathscr{E}$ and T be as in 4.1. If T is in the kernel of any diagonal projection onto $\mathscr{E}'$ then*

$$\|T\| \leqslant \|\delta_T\|_{\mathscr{E}} \leqslant 2\|T\|.$$

REMARK 4.3. In ([19], Theorem 2.1) Johnson and Parrott proved in fact (although more was shown) that if $\mathscr{E}$ is an abelian von Neumann algebra with $\mathscr{U}$ the full group of unitaries in $\mathscr{E}$, and if T is an operator in $\mathscr{L}(H)$ such that $AT-TA$ is compact for every $A \in \mathscr{E}$, then $T - \psi(T)$ is compact for every diagonal projection ψ associated with an invariant mean on $\mathscr{U}$. An inspection of their proof shows that for the proof of this particular item $\mathscr{U}$ can be replaced by any unitary group generating $\mathscr{E}$. This can prove useful.

The above remark together with Lemma 4.1 yields the following result used in later sections.

PROPOSITION 4.4. *Let $\mathscr{E}$ be an abelian von Neumann algebra, and let $T \in \mathscr{L}(H)$ be arbitrary. Then $T = T_0 + K$ where $T_0 \in \mathscr{E}'$ and $\|K\| \leqslant \|\delta_T\|_{\mathscr{E}} \leqslant 2\|K\|$. If in addition δ_T derives $\mathscr{E}$ into the compact operators then K can be taken to be compact.*

Proof. Let $\mathscr{U}$ be a unitary group generating $\mathscr{E}$, M an invariant mean on $\mathscr{U}$, and ψ the corresponding diagonal projection. Let $T_0 = \psi(T)$, $K = T - \psi(T)$, and apply 4.1 and 4.3 noting that $\delta_T \equiv \delta_K$ on $\mathscr{E}$. ▨

If $\mathscr{L}$ is a subspace lattice acting on H and if $T \in \mathscr{L}(H)$ let dist$(T, \mathrm{Alg}\mathscr{L})$ denote the distance from T to Alg$\mathscr{L}$. As Arveson noted in [4], it is always true that

$$\mathrm{dist}(T, \mathrm{Alg}\mathscr{L}) \geqslant \sup\{\|P^{\perp}TP\| : P \in \mathscr{L}\}$$

since for $A \in \mathrm{Alg}\mathscr{L}$ we have $P^{\perp}(T - A)P = P^{\perp}TP$. In [4] Arveson showed that equality holds provided $\mathscr{L}$ is a *nest* thus obtaining an important distance formula for nest algebras. (In [21] C. Lance obtained an independent proof of the Arveson result.) Although equality fails for many lattices, Davidson studied in [12] the question of whether the distance estimate holds with a *constant* for commutative subspace lattices. That is, given $\mathscr{L}$ is there a constant $\alpha > 0$ such that $\mathrm{dist}(T, \mathrm{Alg}\mathscr{L}) \leqslant \alpha \sup\{\|P^{\perp}TP\| : P \in \mathscr{L}\}$ for every $T \in \mathscr{L}(H)$? Classes were given for which such constants exist. The general question is unsettled and appears to be one of th e more interesting in the study of reflexive operator algebras.

The balance of this section shows that the Arveson distance estimate with onstant holds for arbitrary nest subalgebras of A. F. von Neumann algebras. The question of whether either the triangle inequality or the distance estimate holds for more general n.s.v.a. has not been resolved.

Our original proof of the next result used an invariant mean technique, however we use a simpler proof provided by the referee. As we noted in the introduction both (4.5) and (4.6) were known to Christensen for some time though unpublished by him.

PROPOSITION 4.5. *Let $\mathscr{B}$ be an arbitrary von Neumann algebra and let $\mathscr{P}$ denote the lattice of projections in $\mathscr{B}$. For every $T \in \mathscr{L}(H)$ we have*

$$\|\delta_T\|_{\mathscr{B}} \leqslant 4 \sup\{\|P^{\perp}TP\| : P \in \mathscr{P}\} \leqslant 4\|\delta_T\|_{\mathscr{B}}.$$

Proof. Let $T \in \mathscr{L}(H)$ and $\|P^{\perp}TP\| \leqslant \alpha$ for all $P \in \mathscr{P}$. Let S be a symmetry in $\mathscr{B}$; thus $S = P - P^{\perp}$ for some $P \in \mathscr{P}$. Then

$$\|ST - TS\| = 2\|PT - TP\| = 2\|PTP^{\perp} - P^{\perp}TP\| \leqslant 2\alpha.$$

A consequence of the spectral theorem is that the weakly closed convex hull of the symmetries is the set of self adjoint contractions. Thus $\|AT - TA\| \leqslant 2\alpha$ if A is a self adjoint contraction in $\mathscr{B}$ and thus $\|AT - TA\| \leqslant 4\alpha$ for every contraction in $\mathscr{B}$.

REMARK 4.6. The following are equivalent properties for a von Neumann algebra $\mathscr{B}$ acting on a separable Hilbert space H.

(i) There exists a constant $K > 0$ such that

$$\operatorname{dist}(T, \mathscr{B}) \leqslant K \sup\{\|P^\perp TP\| : P \in \operatorname{Lat}\mathscr{B}\}$$

for every $T \in \mathscr{L}(H)$,

(ii) There exists a constant $K > 0$ such that

$$\operatorname{dist}(T, \mathscr{B}) \leqslant K\|\delta_T\|_{\mathscr{B}'},$$

for every $T \in \mathscr{L}(H)$,

(iii) For every derivation δ of $\mathscr{B}'$ into $\mathscr{L}(H)$ there exists $T \in \mathscr{L}(H)$ such that $\delta = \delta_T|\mathscr{B}'$.

Proof. The equivalence of (ii) and (iii) is due to Christensen [8, 9]. The equivalence of (i) and (ii) is immediate from Proposition 4.5 since Lat $\mathscr{B}$ is the projection lattice of $\mathscr{B}'$. ▨

LEMMA 4.7. *Let $\mathscr{B}$ be an approximately finite dimensional von Neumann algebra and let Φ be an expectation of $\mathscr{L}(H)$ onto $\mathscr{B}$. Then* $\operatorname{dist}(T, \mathscr{B}) \geqslant \geqslant \frac{1}{2}\|T - \Phi(T)\|$ *for all $T \in \mathscr{L}(H)$.*

Proof. Let $S = T - \Phi(T)$, so $\Phi(S) = 0$ and $\operatorname{dist}(S, \mathscr{B}) = \operatorname{dist}(T, \mathscr{B})$. Suppose there exists $S_0 \in \mathscr{B}$ with $\|S - S_0\| < \frac{1}{2}\|S\|$. Then

$$\frac{1}{2}\|S\| > \|S - S_0\| \geqslant \|\Phi(S) - \Phi(S_0)\| = \|S_0\|.$$

But $S = (S - S_0) + S_0$ so

$$\|S\| \leqslant \|S - S_0\| + \|S_0\| < \frac{1}{2}\|S\| + \frac{1}{2}\|S\| = \|S\|,$$

a contradiction.

LEMMA 4.8. *Let $\mathscr{B}$ be an approximately finite dimensional von Neumann algebra and let $\mathscr{L}$ be a lattice of projections contained in $\mathscr{B}$. Then for each $T \in \mathscr{B}$ we have*

$$\operatorname{dist}(T, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}) = \operatorname{dist}(T, \operatorname{Alg}\mathscr{L}).$$

Proof. Let Φ be an expectation from $\mathscr{L}(H)$ onto $\mathscr{B}$. Suppose $A \in \operatorname{Alg}\mathscr{L}$. Then $P^\perp AP = 0$, $P \in \mathscr{L}$, so since $P \in \mathscr{B}$ we have $P^\perp\Phi(A)P = \Phi(P^\perp AP) = 0$. So $\Phi(A) \in \mathscr{B} \cap \operatorname{Alg}\mathscr{L}$. (We have in fact shown that Φ is an expectation of $\operatorname{Alg}\mathscr{L}$ onto $\mathscr{B} \cap \operatorname{Alg}\mathscr{L}$.) We have thus that

$$\|T - A\| \geqslant \|\Phi(T - A)\| = \|T - \Phi(A)\|.$$

▨

LEMMA 4.9. *Let $\mathscr{B}$ be an approximately finite dimensional von Neumann algebra and let $\mathscr{L}$ be a projection lattice contained in $\mathscr{B}$. Let Φ be an expectation of $\mathscr{L}(H)$ onto $\mathscr{B}$. For every $T \in \mathscr{L}(H)$ we have* $\operatorname{dist}(\Phi(T), \operatorname{Alg}\mathscr{L}) \leqslant \operatorname{dist}(T, \operatorname{Alg}\mathscr{L})$.

Proof. If $A \in \operatorname{Alg}\mathscr{L}$ then so is $\Phi(A)$ from the proof of 4.10, and so

$$\|\Phi(T) - \Phi(A)\| = \|\Phi(T - A)\| \leqslant \|T - A\|.$$ ▨

PROPOSITION 4.10. (Triangle inequality). *Let $\mathscr{B}$ be an approximately finite dimensional von Neumann algebra and let $\mathscr{L}$ be a lattice of projections contained in $\mathscr{B}$. For every $T \in \mathscr{L}(H)$ we have*

$$\operatorname{dist}(T, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}) \leqslant 2 \operatorname{dist}(T, \mathscr{B}) + \operatorname{dist}(T, \operatorname{Alg}\mathscr{L}).$$

Proof. Let Φ be an expectation of $\mathscr{L}(H)$ onto $\mathscr{B}$. Then if $T \in \mathscr{L}(H)$ let $T_1 = \Phi(T)$, $S = T - \Phi(T)$. Then

$$\operatorname{dist}(T, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}) \leqslant \operatorname{dist}(T_1, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}) + \operatorname{dist}(S, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}).$$

By Lemma 4.8 we have

$$\operatorname{dist}(T_1, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}) = \operatorname{dist}(T_1, \operatorname{Alg}\mathscr{L})$$

and by Lemma 4.9 we have

$$\operatorname{dist}(T_1, \operatorname{Alg}\mathscr{L}) \leqslant \operatorname{dist}(T, \operatorname{Alg}\mathscr{L}).$$

Also, by Lemma 4.7 we have

$$\operatorname{dist}(S, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}) \leqslant \|S\| \leqslant 2 \operatorname{dist}(S, \mathscr{B}) = 2 \operatorname{dist}(T, \mathscr{B}).$$

Hence

$$\operatorname{dist}(T, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}) \leqslant 2 \operatorname{dist}(T, \mathscr{B}) + \operatorname{dist}(T, \operatorname{Alg}\mathscr{L}).$$ ▨

REMARK 4.11. Proposition 4.10 yields as an immediate consequence the Arveson distance estimate for nest subalgebras of approximately finite dimensional von Neumann algebras. Also, if 4.10 were known independently then 4.8 would be an immediate consequence. Moreover, 4.10 show that for $T \in \operatorname{Alg}\mathscr{L}$ we have $\operatorname{dist}(T, \mathscr{B} \cap \operatorname{Alg}\mathscr{L}) \leqslant 2 \operatorname{dist}(T, \mathscr{B})$ for $\mathscr{B}$ an AF algebra. Since these inequalities can be stated independently of the existence of an expectation it becomes a natural question as to whether any or all of these hold for arbitrary von Neumann algebras. More generally one might define operator algebras $\mathscr{A}_1$, $\mathscr{A}_2$ to be in *triangular position* if there exist finite constants K_1, K_2 such that for arbitrary $T \in \mathscr{L}(H)$ one has

$$\operatorname{dist}(T, \mathscr{A} \cap \mathscr{A}_2) \leqslant K_1 \operatorname{dist}(T, \mathscr{A}_1) + K_2 \operatorname{dist}(T, \mathscr{A}_2).$$

Proposition 4.10 shows that a reflexive operator algebra is in triangular position with any approximately finite dimensional von Neumann algebra which contains its invariant subspace lattice. Triangle-type inequalities are likely to hold only for very special cases, but these may well turn out to be cases of interest. Some limitations are revealed by the easily shown fact that a nest algebra $\mathscr{A}_{\mathscr{N}}$ and its adjoint algebra $\mathscr{A}_{\mathscr{N}}^{*}$ are *never* in triangular position if the nest is infinite.

The following theorem can be stated in terms of either algebras or subspace lattices. We prefer the latter here since it suggests further structure questions for lattices.

THEOREM 4.12. *If* $\mathscr{L}$ *is a subspace lattice for which a constant* $K_0 > 0$ *exists such that*

$$\operatorname{dist}(T, \mathrm{Alg}\mathscr{L}) \leqslant K_0 \sup\{\|P^{\perp}TP\| : P \in \mathscr{L}\}$$

for every $T \in \mathscr{L}(H)$, *and if* $\mathscr{M}$ *is an orthogonally complemented subspace lattice which generates an approximately finite dimensional von Neumann algebra and which commutes pairwise with* $\mathscr{L}$, *then there exists* $K_1 > 0$ *such that*

$$\operatorname{dist}(T, \mathrm{Alg}(\mathscr{L} \vee \mathscr{M})) \leqslant K_1 \sup\{\|P^{\perp}TP\| : P \in \mathscr{L} \vee \mathscr{M}\}$$

for all $T \in \mathscr{L}(H)$.

Proof. Let $\mathscr{B} = \mathscr{M}'$, then $\mathscr{B}$ is also an approximately finite dimensional von Neumann algebra. We have $\mathscr{M} = \mathrm{Lat}\mathscr{B}$, $\mathscr{L} \subset \mathscr{B}$, and $\mathrm{Alg}(\mathscr{L} \vee \mathscr{M}) = \mathscr{B} \cap \mathrm{Alg}\mathscr{L}$. Now apply Proposition 4.10 and 4.6 noting that every approximately finite dimensional von Neumann algebra satisfies (ii) and (iii) of that theorem.

REMARK 4.13. It is presently an open question whether the join of a reflexive lattice $\mathscr{L}$ with an orthogonally complemented lattice $\mathscr{M}$ (the projection lattice of a von Neumann algebra) in its commutant is necessarily reflexive. The answer is unknown even in the case where $\mathscr{M}$ is commutative.

COROLLARY 4.14. *If* $\mathscr{A}$ *is a nest subalgebra of an approximately finite dimensional von Neumann algebra then there exists* $K > 0$ *such that for every* $T \in \mathscr{L}(H)$ *we have*

$$\operatorname{dist}(T, \mathscr{A}) \leqslant K \sup\{\|P^{\perp}TP\| : P \in \mathrm{Lat}\mathscr{A}\}.$$

5. NEST SUBALGEBRAS OF FACTORS

We require a decomposition result for the essential commutant of a n.s.v.a. of a factor together with a simultaneous norm estimate for the decomposition for all except the type II_1 case. In the I_∞, II_∞ and III (i.e. infinite) cases we show that the nest under consideration can be enlarged if necessary so that there exists an infinite sequence $\{P_n\}$ in $\mathcal{N}$, either strictly increasing or strictly decreasing, with the property that the $\mathcal{N}$-intervals $P_{n+1} - P_n$ (or $P_n - P_{n+1}$ if the sequence is decreasing) are all mutually equivalent in $\mathcal{B}$. This property is then used to deduce the required result for the infinite factor case. We prefer to consider this case first reserving the technically much simpler finite type I case until later.

Our main aim of this section is to prove the following theorem.

THEOREM 5.1. *Let $\mathcal{A}$ be a nest subalgebra of an infinite factor $\mathcal{B}$. If $T \in \text{esscomm}\mathcal{A}$ then $T = \tilde{T} + K$ where $\tilde{T} \in \mathcal{B}'$ and K is compact with $\|K\| \leqslant C\|\delta_T\|$, where C is a positive finite constant simultaneoulsy valid for all infinite factors and all nests contained therein.*

REMARK. The constant C can perhaps be taken to be 1, although we have not shown this.

Before proceeding with the proof we require some preliminary results.

REMARK 5.2. First note that if $\mathcal{A}$ is a n.s.v.a. of a von Neumann algebra $\mathcal{B}$ relative to a nest $\mathcal{N} \subset \mathcal{B}$ then increasing the nest will decrease the n.s.v.a. relative to $\mathcal{B}$ so that esscomm$\mathcal{A}$ may increase and $\|\delta_T\|_{\mathcal{A}}$ may decrease. In this case the proof of 5.1 for the new n.s.v.a. will imply the corresponding result for the original n.s.v.a.. A simple Zorn's lemma argument shows that an arbitrary nest in $\mathcal{B}$ is contained in some maximal nest in $\mathcal{B}$ ($\mathcal{N}$ is maximal in $\mathcal{B}$ if it is not properly contained in a larger nest in $\mathcal{B}$), so we may assume without loss of generality than $\mathcal{N}$ is maximal in $\mathcal{B}$. It is clear that a nest $\mathcal{N}$ in a von Neumann algebra is maximal in $\mathcal{B}$ if and only if for each $N \in \mathcal{N}$ the immediate predecessor $N_- = \vee\{L \in \mathcal{N} : L < N\}$ and the immediate successor $N_+ = \wedge\{L \in \mathcal{N} : L > N\}$ differ from N by at most a minimal projection in $\mathcal{B}$. In particular, if $\mathcal{B}$ is type II or III a nest in $\mathcal{B}$ is maximal iff its core is nonatomic. In general, a nest with nonatomic core is maximal in $\mathcal{L}(H)$ so also in every von Neumann algebra containing it.

If $\mathcal{B}$ is a type I or type II factor let $\text{Tr}(\cdot)$ denote a trace function on $\mathcal{B}$.

LEMMA 5.3. *Let $\mathcal{N}$ be a maximal nest in a type* II *factor $\mathcal{B}$ and let $\mathcal{N}_f = \{P \in \mathcal{N} : \text{Tr}(P) < \infty\}$. Then $\text{Tr}(\mathcal{N}_f)$ is a closed connected subset of reals.*

Proof. Let P and Q be in $\mathcal{N}_f$. Then either $P \leqslant Q$ or $Q \leqslant P$. Thus $\text{Tr}(P) \leqslant \text{Tr}(Q)$ implies $P \leqslant Q$ since otherwise P would be equivalent to a proper subprojection of itself. Similarly $\text{Tr}(P) = \text{Tr}(Q)$ implies $P = Q$ for $P, Q \in \mathcal{N}_f$.

Assume $\text{Tr}(P_0) = b \neq 0$ for P_0 in $\mathcal{N}_f$ and let $0 < a < b$. Let $R = \sup\{P \in \mathcal{N}_f : \text{Tr}(P) \leqslant a\}$ and $Q = \inf\{P \in \mathcal{N}_f : \text{Tr}(P) \geqslant a\}$. Clearly R and Q are in $\mathcal{N}_f$ with $R \subset Q$ and $\text{Tr}(R) \leqslant a \leqslant \text{Tr}(Q)$. If $\text{Tr}(R) < \text{Tr}(Q)$, then $R \neq Q$ and there is a member S of $\mathcal{N}$ strictly between R and Q. This follows since $\mathcal{B}$ has no minimal projections and $\mathcal{N}$ was taken to be maximal in $\mathcal{B}$. Since S is also in $\mathcal{N}_f$ this would contradict the definition of R or Q. So $\mathrm{Tr}(\mathcal{N}_f)$ is connected. Let $N_0 = \sup\{N : N \in \mathcal{N}_f\}$. If N_0 is finite then $\mathrm{Tr}(\mathcal{N}_f) = [0, \mathrm{Tr}(N_0)]$ and if N_0 is infinite then $\mathrm{Tr}(\mathcal{N}_f) = [0, \infty)$. ▨

LEMMA 5.4. *Let $\mathcal{N}$ be a nest in an infinite factor $\mathcal{B}$ and let E and F be mutually orthogonal $\mathcal{N}$-intervals with E infinite. Assume that an operator $T \in \mathcal{L}(H)$ essentially commutes with the nest subalgebra $\mathcal{A} = \mathcal{B} \cap \mathcal{A}_{\mathcal{N}}$ and with $E\mathcal{B}E$, and commutes with the core $\mathcal{C}_{\mathcal{N}}$. Then T essentially commutes with $(E+F)\mathcal{B}(E+F)$.*

Proof. Replacing $\mathcal{N}$ with $\mathcal{N}^{\perp} = \{I - N : N \in \mathcal{N}\}$, $\mathcal{A}$ with $\mathcal{A}^*$, and T with T^* if necessary, we may assume that $E \leqslant F$ so that $E\mathcal{B}F \subset \mathcal{A}$. Let S be a partial isometry in $\mathcal{B}$ with support F and range a subprojection E_0 of E (not necessarily an interval). Since S is in $\mathcal{A}$ we have that $ST - TS$ is compact and hence $STS^* - TE_0$ is compact. So since T essentially commutes with $E_0\mathcal{B}E_0$ and with F it follows that T essentially commutes with $F\mathcal{B}F$. Now a straightforward calculation using the fact that $S^*TS - TF$ is compact yields that T essentially commutes with $F\mathcal{B}E$. ▨

LEMMA 5.5. *In addition to the hypotheses in 5.4 suppose $T|EH = T_0 + K$ where $T_0 \in (E\mathcal{B}|EH)'$ and K is compact. Then T_0 extends to an operator $T_1 \in ((E+F)\mathcal{B}|(E+F)H)'$ and K to K_1 on $(E+F)H$ so that $T|(E+F)H = T_1 + K_1$, K_1 is compact, $K_1F = FK_1$, and $\|K_1\| \leqslant \|K\| + \|\delta_T\|_{\mathcal{A}}$.*

Proof. Let S be as in 5.4. Calculations similar to the above show that S^*T_0S commutes with $F\mathcal{B}|F$ and that $T_1 = T_0 \oplus S^*T_0S$ commutes with all of $(E+F)\mathcal{B}|((E+F)H)$. Let $K_1 = T_1 - T|(E+F)H$. We have

$$\|K_1\| = \sup\{\|K\|, \|T_1|FH - T|FH\|\},$$

and

$$\|T_1|FH - T|FH\| = \|S^*T_0S - TF\| \leqslant \|S^*T_0S - S^*TS\| + \|S^*TS - TF\| \leqslant$$

$$\leqslant \|S^*(T_0 - T)S\| + \|TS - ST\| \leqslant \|K\| + \|\delta_T\|_{\mathcal{A}}.$$

Compactness of K_1 follows from compactness of both K and $S^*TS - TF$. ▨

LEMMA 5.6. *Let $\mathcal{B}$ be an infinite factor, and let $\mathcal{N}$ be a maximal nest of projections in $\mathcal{B}$. Then $\mathcal{N}$ contains either (perhaps both) an infinite increasing sequence $P_1 < P_2 < \ldots$ with the $\mathcal{N}$-intervals $P_{n+1} - P_n$ mutually equivalent in $\mathcal{B}$ or an infinite decreasing sequence $P_1 > P_2 > \ldots$ with the $\mathcal{N}$-intervals $P_n - P_{n+1}$ mutually equivalent in $\mathcal{B}$.*

Proof. If $\mathcal{B}$ is type I_∞ then $\mathcal{B}$ is $*$-isomorphic to $\mathcal{L}(H)$ with $\dim H = \infty$ so the result is obvious since $\mathcal{N}$ is assumed maximal in $\mathcal{B}$. If $\mathcal{B}$ is type III the result follows trivially from the fact that all nonzero projections in $\mathscr{B}$ are equivalent. So we need only be concerned with type II_∞.

Firstly, if $\mathscr{N}$ contains a nonzero $\mathscr{B}$-finite projection and if the join of the finite projections in $\mathscr{N}$ is $\mathscr{B}$-infinite then $\mathrm{Tr}(\mathscr{N}_f) = [0, \infty)$ by Lemma 5.3. So noting that $N - \mathrm{Tr}(N)$ is an increasing function on $\mathscr{N}_f$ there exists an increasing sequence $P_1 < P_2 < \ldots$ in $\mathscr{N}$ with $\mathrm{Tr}(P_n) = n$ for each n. The $\mathscr{N}$-intervals $P_{n+1} - P_n$, $n \geqslant 1$, have trace 1 so are equivalent projections in $\mathscr{B}$, as desired.

Secondly, if the dual nest $\mathscr{N}^\perp = \{I - P : P \in \mathscr{N}\}$ satisfies the properties in the above paragraph we obtain by similar reasoning a sequence $P_1 > P_2 > \ldots$ in $\mathscr{N}$ with the desired properties.

Next, if E is an arbitrary nonzero $\mathscr{B}$-infinite $\mathscr{N}$-interval consider the nest $\mathscr{N}_E = \{P|EH : P \in \mathscr{N}\}$ contained in the type II_∞ factor $\mathscr{B}_E = \{EB|EH : B \in \mathscr{B}\}$. $\mathscr{N}_E$ is maximal since its core is obviously nonatomic, and the trace on $\mathscr{B}_E$ is the restriction of that on $\mathscr{B}$. These statements together with the argument above show that if $\vee\{PE : P \in \mathscr{N}, PE \text{ finite}\}$ or $\vee\{P^\perp E : P \in \mathscr{N}, P^\perp E \text{ finite}\}$ is $\mathscr{B}$-infinite then there exists a sequence $P_1 < P_2 < \ldots$ or $P_1 > P_2 > \ldots$ in $\mathscr{N}$ between the lower and upper endpoints of E having the desired properties.

The above paragraph shows that if there exists a $\mathscr{B}$-infinite $\mathscr{N}$-interval that cannot be written as the sum of two $\mathscr{B}$-infinite $\mathscr{N}$-intervals then we are done. For if E is such an interval then either PE or $P^\perp E$ is finite for all $P \in \mathscr{N}$ and one of $\vee\{P^\perp E : P \in \mathscr{N}, P^\perp E \text{ finite}\}$ or $\vee\{PE : P \in \mathscr{N}, P^\perp E \text{ finite}\}$ must be infinite.

Finally, if every infinite $\mathscr{N}$-interval can be written as the sum of two infinite $\mathscr{N}$-intervals then since I is infinite an obvious construction yields a sequence $P_1 < P_2 < \ldots$ in $\mathscr{N}$ such that the projections $P_{n+1} - P_n$ are $\mathscr{B}$-infinite and hence mutually equivalent in $\mathscr{B}$. ▨

REMARK. In case $\mathscr{B}$ is type I_∞ the statement of Lemma 5.6 can be amplified so that either $0 = P_1 < P_2 < \ldots$ with $\bigvee_n P_n$ having finite complement or $I = P_1 > P_2 > \ldots$ with $\bigwedge_n P_n$ finite. This is not always possible in the type II_∞ case. A technically more complicated amplification can be given for the II_∞ case, although we will not show it here as it does not lead to simplicity of our arguments.

We now return to the proof of Theorem 5.1. One may view this as paralleling to a certain extent the proof of [10] for a nest algebra in $\mathscr{L}(H)$, although it is necessarily technically more complicated. We make explicit usage of the Johnson-Parrot result ([19], Theorem 2.1) for commutative von Neumann algebras, although the corresponding noncommutative results in that paper are not used in this proof and are in fact picked up along the way. Knowledge of the noncommutative results would not simplify our proof to any extent.

Proof (*Theorem 5.1*). By Remark 5.2 we may assume that, $\mathscr{N}$ is a maximal nest in an infinite factor $\mathscr{B}$ and that $\mathscr{A} = \mathscr{B} \cap \mathscr{A}_\mathscr{N}$.

Let us first consider the case in which $\mathcal{N}$ contains an infinite sequence $0 = P_0 < < P_1 < P_2 < \ldots$ with $P_n \to I$ strongly such that the $\mathcal{N}$-intervals $P_{n+1} - P_n$ are equivalent in $\mathcal{B}$. Denote by E_n the interval $P_n - P_{n-1}$ and by V_n a partial isometry in $\mathcal{B}$ from E_1 onto E_n. Since $\mathcal{C}_{\mathcal{N}} \subseteq \mathcal{A}$, where $\mathcal{C}_{\mathcal{N}}$ denotes the core of $\mathcal{N}$, we have $T \in \text{esscomm}\mathcal{C}_{\mathcal{N}}$. Hence by Proposition 4.4 T decomposes $T = T_0 + K_0$ where $T \in \mathcal{C}'_{\mathcal{N}} = \mathcal{D}_{\mathcal{N}}$ and K_0 is compact with $\|K_0\| \leqslant \|\delta_T|\mathcal{C}_{\mathcal{N}}\| \leqslant \|\delta_T|\mathcal{A}\|$. Then $T_0 \in \text{esscomm}\mathcal{A}$ and T_0 is reduced by the projections $\{E_i\}$.

Now let $T_i = V_i^* E_i T_0 E_i V_i$. We first claim that $\{T_i\}$ forms a Cauchy sequence in $\mathcal{L}(E_1 H)$. If not, then there exists a sequence n_i of indexes so that $\|T_{n_i} - T_{n_{i-1}}\| \geqslant \geqslant \varepsilon > 0$; i.e.,

$$\|V_{n_i}^* T_0 V_{n_i} - V_{n_{i-1}}^* T_0 V_{n_{i-1}}\| \geqslant \varepsilon.$$

Let S be the partial isometry in $\mathcal{B}$ mapping each E_{n_i} onto $E_{n_{i-1}}$ defined by $SE_{n_i} = = V_{n_{i-1}} V_{n_i}^*$, $i = 1,2, \ldots$. We shall show that $T_0 S - ST_0$ is not compact, yet S as constructed belongs to $\mathcal{A}$ since each summand does by Lemma 1.2.

Let $B = T_0 S - ST_0$. Then

$$E_{n_{i-1}} B E_{n_i} = E_{n_{i-1}} T_0 S E_{n_i} - E_{n_{i-1}} S T_0 E_{n_i} =$$

$$= E_{n_{i-1}} T_0 E_{n_{i-1}} S E_{n_i} - E_{n_{i-1}} S E_{n_i} T_0 E_{n_i} =$$

$$= E_{n_{i-1}} T_0 E_{n_{i-1}} V_{n_{i-1}} V_{n_i}^* - E_{n_{i-1}} V_{n_{i-1}} V_{n_i}^* T_0 E_{n_i}.$$

Now taking the norm of $E_{n_{i-1}} B E_{n_i}$ after multiplying on the left by $V_{n_{i-1}}^*$ and on the right by V_{n_i} we get

$$\|E_{n_{i-1}} B E_{n_i}\| = \|V_{n_{i-1}}^* T_0 V_{n_{i-1}} - V_{n_i}^* T_0 V_{n_i}\| \geqslant \varepsilon.$$

Since this is true for each i it follows that B cannot be compact which contradicts our hypothesis. Thus there is an operator T_∞ on $E_1 H$ so that $T_n \to T_\infty$ in norm.

Next, for $m, n \geqslant 1$ we compute

$$V_m^*(T_0 V_m V_n^* - V_m V_n^* T_0) V_n = T_m - T_n.$$

If $n > m$ then $V_m V_n^* \in \mathcal{A}$ hence the left hand side is compact by hypothesis. Thus $T_m - T_n$ is compact for all $m, n \geqslant 1$. Also, for fixed $n \geqslant 1$ we have $T_n - T_\infty = = \lim_m (T_n - T_m)$ so $T_n - T_\infty$ is compact, $n \geqslant 1$.

Let $\tilde{T} = \sum V_n T_\infty V_n^*$ on H. Since $\tilde{T}$ and T_0 are reduced by the projections E_n we have

$$T_0 - \tilde{T} = \sum E_n (T_0 - \tilde{T}) E_n = \sum (V_n T_n V_n^* - V_n T_\infty V_n^*) = \sum V_n (T_n - T_\infty) V_n^*.$$

Since $T_n - T_\infty \to 0$ in norm and is compact for each n we may conclude that $T_0 - \tilde{T} = K_1$ is compact. We have

$$\|K_1\| = \sup\|T_n - T_\infty\| \leqslant \sup\|T_n - T_m\|.$$

From the preceding paragraph we have

$$T_n - T_m = V_m^*(T_0 V_m V_n^* - V_m V_n^* T_0) V_n$$

so we conclude that $\|K_1\| \leqslant \|\delta_{T_0}\|_{\mathscr{A}}$. Finally,

$$\|\delta_{T_0}\|_{\mathscr{A}} \leqslant \|\delta_T\|_{\mathscr{A}} + \|\delta_{K_0}\|_{\mathscr{A}} \leqslant \|\delta_T\|_{\mathscr{A}} + 2\|K_0\| \leqslant 3\|\delta_T\|_{\mathscr{A}},$$

and thus

$$\|K_0 + K_1\| \leqslant \|K_0\| + \|K_1\| \leqslant \|\delta_T\|_{\mathscr{A}} + 3\|\delta_T\|_{\mathscr{A}} = 4\|\delta_T\|_{\mathscr{A}}.$$

Now we can write $T = \tilde{T} + K$ where K is compact with $\|K\| \leqslant 4\|\delta_T\|_{\mathscr{A}}$ and $\tilde{T} = \sum V_n T_\infty V_n^*$. Notice that $\tilde{T}$ essentially commutes with $\mathscr{A}$. We will show that in fact $\tilde{T} \in \mathscr{B}'$.

Since $\sum E_i = I$, to show $\tilde{T} \in \mathscr{B}'$ it will suffice to show that $\tilde{T}$ commutes with each operator $E_i B E_j$ with $B \in \mathscr{B}$, $i, j \geqslant 1$. From the computation

$$V_i^*(TE_iBE_j - E_iBE_jT)V_j = V_i^*(V_iT_\infty V_i^* BE_j - E_iBV_jT_\infty V_j^*)V_j = T_\infty V_i^* BV_j - V_i^* BV_j T_\infty$$

it follows that the assertion that $\tilde{T}$ commutes with each E_iBE_j is equivalent to the assertion that T_∞ commutes with each operator $V_i^* B V_j$, $B \in \mathscr{B}$, $i, j \geqslant 1$. We prove the latter.

Assume there exists $B_0 \in \mathscr{B}$ and $n, m \geqslant 1$ such that

$$B_{nm} = T_\infty V_m^* B_0 V_n - V_m^* B_0 V_n T_\infty \neq 0.$$

Since $B_{nm}\colon E_1 H \to E_1 H$ and is nonzero it follows that $\sum_{i=2}^{\infty} V_{i-1} B_{nm} V_i^*$ is not compact in $\mathscr{L}(H)$. But

$$\sum_{i=2}^{\infty} V_{i-1} B_{nm} V_i^* = \sum (V_{i-1} T_\infty V_m^* B_0 V_n V_i^* - V_{i-1} V_m^* B_0 V_n T_\infty V_i^*) =$$

$$= \sum_{i=2}^{\infty} V_{i-1} T_\infty V_{i-1}^* (V_{i-1} V_m^* B_0 V_n V_i^*) -$$

$$- \sum_{i=2}^{\infty} (V_{i-1}(V_m^* B_0 V_n V_i^*) V_i T_\infty V_i^* =$$

$$= \tilde{T}\left(\sum_{i=2}^{\infty} V_{i-1} V_m^* B_0 V_n V_i^*\right) - \left(\sum_{i=2}^{\infty} V_{i-1} V_m^* B_0 V_n V_i^*\right)\tilde{T}.$$

However, $A = \sum V_{i-1}V_m^* B_0 V_n V_i^*$ is in $\mathscr{B}$ and is also in the nest algebra determined by $\mathscr{N}$ so $A \in \mathscr{A}$. This contradicts our hypothesis. If follows that T_∞ commutes with $V_m^* \mathscr{B} V_n$ for all n and m, and the proof for the first case is complete.

Secondly, if the dual nest $\mathscr{N}^\perp = \{I - P : P \in \mathscr{N}\}$ satisfies the properties of the first case note that $\mathscr{B} \cap \mathscr{A}_{\mathscr{N}^\perp} = \mathscr{A}^*$ and interchange T with T^* noting that $\mathscr{B}$ is self-adjoint to obtain the desired result.

For the general case apply Lemma 5.6 yielding either $P_1 < P_2 < \dots$ or $P_1 > P_2 > \dots$ with the $\mathscr{N}$-intervals $P_{n+1} - P_n$ (resp. $P_n - P_{n+1}$) mutually equivalent in $\mathscr{B}$. If increasing let $P_\infty = \vee P_n$ and set $E = P_\infty - P_1$. If decreasing let $P_0 = \wedge P_n$ and set $E = P_1 - P_0$. Let $\mathscr{N}_E = \{P|EH : P \in \mathscr{N}\}$ and $\mathscr{B}_E = \{EB|EH : B \in \mathscr{B}\}$, so $\mathscr{B}_E$ is an infinite factor and $\mathscr{N}_E$ is a maximal nest in $\mathscr{B}_E$. Since E is an $\mathscr{N}$-interval it follows easily that $\mathscr{B}_E \cap \mathrm{Alg}(\mathscr{N}_E) = \{EA|EH : A \in \mathscr{A}\} = \mathscr{A}_E$. Now note that $\mathscr{N}_E$ lies in either the first or second cases considered above relative to the sequence $\{P_n|EH\}$. So first apply 4.4 obtaining $T = T_0 + K_0$ with $T_0 \in \mathscr{C}'_{\mathscr{N}}$ and K_0 compact with $\|K_0\| \leqslant \|\delta_T\|_{\mathscr{A}}$, and then apply the results for the first and second cases above to T_0 to obtain a decomposition $T_0 E = T_1 + K_1$ where $T_1 = ET_1E$, $K_1 = EK_1E$, T_1 commutes with $E\mathscr{B}E$, and K_1 is compact with

$$\|K_1\| \leqslant 4\|\delta_{T_0}\|_{E\mathscr{A}E} \leqslant 4\|\delta_{T_0}\|_{\mathscr{A}} .$$

In particular this shows that T_0 essentially commutes with $E\mathscr{B}E$. Now write $E = M - N$ where M, N are the upper and lower endpoints of E in $\mathscr{N}$, and apply Lemma 5.5 to T_0 with $F = N$ to conclude that $T_0 M = T_2 + K_2$ where $T_2 = MT_2M$, $K_2 = MK_2M$, T_2 commutes with $M\mathscr{B}M$, and K_2 is compact with

$$\|K_2\| \leqslant \|K_1\| + \|\delta_{T_0}\|_{\mathscr{A}} \leqslant 5\|\delta_{T_0}\|_{\mathscr{A}} .$$

Again apply Lemma 5.5 to T_0 with $F = M^\perp$ to conclude that $T_0 = \tilde{T} + K_3$ where $\tilde{T} \in \mathscr{B}'$ and K_3 is compact with

$$\|K_3\| \leqslant \|K_2\| + \|\delta_{T_0}\|_{\mathscr{A}} \leqslant 6\|\delta_{T_0}\|_{\mathscr{A}} .$$

Since $T_0 = T - K_0$ we have

$$\|\delta_{T_0}\|_{\mathscr{A}} \subseteq \|\delta_T\|_{\mathscr{A}} + \|\delta_{K_0}\|_{\mathscr{A}} \leqslant \|\delta_T\|_{\mathscr{A}} + 2\|K_0\| \leqslant 3\|\delta_T\|_{\mathscr{A}} ,$$

so $\|K_3\| \leqslant 18\|\delta_T\|_{\mathscr{A}}$. Now set $K = K_0 + K_3$. We thus have $T = \tilde{T} + K$ where $\tilde{T} \in \mathscr{B}'$ and K is compact with

$$\|K\| \leqslant \|K_0\| + \|K_3\| \leqslant 19\|\delta_T\|_{\mathscr{A}} .$$

▨

PROPOSITION 5.7 *Let $\mathscr{A}$ be a nest subalgebra of a finite factor $\mathscr{B}$ of type* I. *If $T \in \operatorname{esscomm}\mathscr{A}$ then $T = \tilde{T} + K$ where $\tilde{T} \in \mathscr{B}'$ and K is compact with $\|K\| \leqslant C\|\delta_T\|_{\mathscr{A}}$, where C is a positive finite constant simultaneously valid for all finite type* I *factors and all nests contained therein.*

Proof. Assume $\mathscr{B}$ is a factor of type $\mathrm{I}_{n\alpha}$ on $H_n \otimes H_\alpha$. From Example 4.5 in [15] we may assume that $\mathscr{A} = \mathscr{A}_{\mathscr{N}_0} \otimes I_\alpha$ where $\mathscr{N}_0$ is a nest in $\mathscr{L}(H_n)$. Enlarge $\mathscr{N}$ to be maximal in $\mathscr{B}$ if necessary thus possibly decreasing $\mathscr{A}$ and $\mathscr{A}_0$ noting that a proof for this new nest will imply the required result for the original nest. $\mathscr{N}_0$ will then be maximal in $\mathscr{L}(H_n)$, and so since H_n is *finite* dimensional the core of $\mathscr{N}_0$ will be a m.a.s.a. in $\mathscr{L}(H_n)$. Apply Proposition 4.4 writing $T = T_0 + K_0$ with $T_0 \in \mathscr{C}'_{\mathscr{N}}$ and K_0 compact with

$$\|K_0\| \leqslant \|\delta_T\|_{\mathscr{C}_{\mathscr{N}}} \leqslant \|\delta_T\|_{\mathscr{A}} .$$

The structure of T_0 will be of the form $T_1 \oplus \ldots \oplus T_n$ where the T_i are operators on H_α, or equivalently, T_0 can be matricially written as $\operatorname{diag}(T_1, \ldots, T_n)$. As in the proof of the first case in Theorem 5.1 we easily check that $T_i - T_j$ is necessarily compact for $1 \leqslant i, j \leqslant n$ since T_0 essentially commutes with $\mathscr{A}$. Let $\tilde{T} = \operatorname{diag}(T_1, \ldots, T_1) = I_n \otimes T_1$. Then $\tilde{T} \in \mathscr{B}'$, and as in 5.1 we can easily construct an operator in $\mathscr{A}$ to show that $\|T_0 - \tilde{T}\| \leqslant \|\delta_{T_0}\|_{\mathscr{A}}$. We have that $T_0 - \tilde{T} = K_1$ is compact since each $T_i - T_1$ is compact. Finally, setting $K = K_0 + K_1$ we compute as in 5.1 that $\|K\| \leqslant 4\|\delta_T\|_{\mathscr{A}}$. ▨

REMARK. The techniques in the proof of 5.1 and 5.7 do not apply for obvious reasons in case $\mathscr{B}$ is a type II_1 factor. While we know that $\operatorname{esscomm}\mathscr{A} = \operatorname{esscomm}\mathscr{B}$ for this case, the question as to whether the decomposition or indeed the norm estimate of 5.1 holds for the general II_1 case remains open. A positive answer would of course imply a positive resolution of the Johnson-Parrott question for type II_1 factors.

6. THE ESSENTIAL COMMUTANT OF AN N.S.V.A.

The following simple observation and its immediate consequence was used by the authors of [19] to reduce immediately to the direct sum of factors case.

LEMMA 6.1. *Let $\mathscr{E}$ be a nonatomic abelian von Neumann algebra. Then* $\cap \mathscr{L}\mathscr{C}(H) = \{0\}$.

Proof. If T were a nonzero compact operator in the commutant of $\mathscr{E}$ then $\mathscr{E}'$ would contain nonzero finite rank spectral projections for T^*T and hence $\mathscr{E}'$ would have minimal projections. But $\mathscr{E} = \operatorname{cent}(\mathscr{E}')$ and the central support of a minimal projection for $\mathscr{E}'$ would be minimal in $\mathscr{E}$, a contradiction.

LEMMA 6.2. *Let $\mathscr{A}$ be an arbitrary operator algebra. If the center of $\mathscr{A}$ contains a nonatomic von Neumann algebra containing I, then* $\operatorname{esscomm}\mathscr{A} = \mathscr{A}' + \mathscr{L}\mathscr{C}(H)$.

Proof. Let $\mathscr{E}$ be a nonatomic von Neumann algebra in the center of $\mathscr{A}$. If $T \in \operatorname{esscomm}\mathscr{A}$ then $T \in \operatorname{esscomm}\mathscr{E}$ so by [19], Theorem 2.1 $T = T_0 + K$ where $T_0 \in \mathscr{E}'$ and $K \in \mathscr{L}\mathscr{C}(H)$. We have $\mathscr{A} \subseteq \mathscr{E}'$, so if for some $A \in \mathscr{A}$ we have $AT_0 - T_0A \neq 0$ then this commutator would be a nonzero compact operator in $\mathscr{E}'$, a contradiction. Thus $T_0 \in \mathscr{A}'$. ▨

REMARK. In the hypothesis of 6.2 we require the nonatomic subalgebra of $\operatorname{cent}\mathscr{A}$ to contain the identity for $\mathscr{L}(H)$. So in case $\operatorname{cent}\mathscr{A}$ is a von Neumann algebra the requirement becomes in fact that $\operatorname{cent}\mathscr{A}$ is nonatomic, *not* simply that $\mathscr{A}$ contains a projection Q such that $\operatorname{cent}\mathscr{A} \mid QH$ is nonatomic. In practice we let P be the sum of all minimal projections in $\operatorname{cent}\mathscr{A}$ and note that if $T \in \operatorname{esscomm}\mathscr{A}$ then $T \mid PH$ (resp. $T \mid (I-P)H$) is in $\operatorname{esscomm}\mathscr{A} \mid PH$ (resp. $\mathscr{A} \mid (I-P)H$) enabling one to reduce the problem to that of a direct sum of algebras each summand of which generates a factor as a von Neumann algebra. This is the procedure utilized in [19], Theorem 3.4. We note that a direct sum of algebras each of which has essential commutant equal to its algebraic commutant + compacts need not itself have essential commutant which decomposes thus. Considerable pathology can occur in general. Thus the problem of "lifting" from the factor case to the general von Neumann algebra situation is nontrivial, and at least in the type I_∞ and II_∞ cases it is necessary to know the norm estimates of Theorem 5.1 for such "lifting". Alternate methods are available for showing that $\operatorname{esscomm}\mathscr{A} = \operatorname{esscomm}\mathscr{B}$ whenever $\mathscr{B}$ does not contain summands of types I_∞ or II_∞, and indeed we use such a method to prove this latter result for $\mathscr{B}$ a direct sum of type II_1 factors. This could be adapted to direct sums of finite type I and type III factors, although no significant degree of simplification would be obtained since the difficulty in proof essentially lies in types I_∞ and II_∞. Moreover, our proofs for all except type II_1 absorb in a natural way the corresponding Johnson-Parrot results for von Neumann algebras, and we feel that this is desirable.

THEOREM 6.3. *Let $\mathscr{A}$ be a n.s.v.a. with nest $\mathscr{N}$ and von Neumann algebra $\mathscr{B}$. Assume $\mathscr{B}$ has no type II_1 factor as a direct summand. If T is an operator in $\mathscr{L}(H)$ which commutes with $\mathscr{A}$ modulo the compact operators then $T = \tilde{T} + K$ where $\tilde{T} \in \mathscr{B}'$ and K is compact.*

Proof. Let $\mathscr{E} = \operatorname{cent}\mathscr{A} = \operatorname{cent}\mathscr{B}$. By [19], Theorem 2.1 $T = T_0 + K_0$ where $T_0 \in \mathscr{E}'$ and $K_0 \in \mathscr{L}\mathscr{C}(H)$. Let $\{E_\lambda\}$ be the set of minimal projections of $\mathscr{E}$ and let $E = \vee E_\lambda$. The argument in 6.2 shows that $T_0 \mid (I-E)H$ is in the commutant of $\mathscr{A} \mid (I-E)H$, and by 2.5 this is the commutant of $\mathscr{B} \mid (I-E)H$. Each algebra $\mathscr{A} \mid E_\lambda H$ is the nest subalgebra of the factor $\mathscr{B} \mid E_\lambda H$ relative to the nest $\mathscr{N}_{E_\lambda} =$

$= \mathcal{N} \mid E_\lambda H$, and we have $T_0 \mid E_\lambda H \in \mathrm{esscomm}(\mathcal{A} \mid E_\lambda H)$. Let $T_\lambda = T_0 \mid E_\lambda H$ and $\mathcal{A}_\lambda = \mathcal{A} \mid E_\lambda H$. Since each factor $\mathcal{B} \mid E_\lambda H$ is of type I, II_∞ or III, by Theorems 5.1 and 5.7 for each λ we can write $T_\lambda = \tilde{T}_\lambda + K_\lambda$ where $\tilde{T}_\lambda$, $K_\lambda \in \mathcal{L}(E_\lambda H)$, $\tilde{T}_\lambda \in (\mathcal{B} \mid E_\lambda H)'$, and K_λ is compact with $\|K_\lambda\| \leqslant C\|\delta_{T_\lambda}\|_{\mathcal{A}_\lambda}$ where $C>0$ is independent of λ. Let $\hat{K}$ be the direct sum of the operators K_λ together with the zero operator on $E^\perp H$. Since $\mathcal{A}$ contains the direct sum of the $\mathcal{A}_\lambda$ and since T_0 derives $\mathcal{A}$ into $\mathcal{L}\mathcal{C}(H)$ it follows that for each $\varepsilon > 0$ we have $\|\delta_{T_\lambda}\|_{\mathcal{A}_\lambda} < \varepsilon$ for all but a finite number of λ, and hence $\hat{K}$ is compact. Now let $\tilde{T} = T_0 - \hat{K}$ and $K = K_0 + \hat{K}$. The proof is complete.

Combining the above results we can prove the equality of the essential commutants for $\mathcal{A}$ and $\mathcal{B}$ in every case.

THEOREM 6.4. *Let $\mathcal{A}$ be a nest subalgebra of an arbitrary von Neumann algebra $\mathcal{B}$. Then* $\mathrm{esscomm}\mathcal{A} = \mathrm{esscomm}\mathcal{B}$.

Proof. We have $\mathcal{A} = \mathcal{A}_{\mathcal{N}} \cap \mathcal{B}$ where $\mathcal{N}$ is a nest in $\mathcal{B}$. Let $\mathcal{E} = \mathrm{cent}\mathcal{A} = \mathrm{cent}\mathcal{B}$. If $\mathcal{B}$ has no type II_1 factor as a direct summand then Theorem 6.3 completes the proof. Otherwise let $\{E_\lambda\}$ be the set of minimal projections, of $\mathcal{E}$ such that $\mathcal{B} \mid E_\lambda H$ is type II_1 and let $E = \vee E_\lambda$. By 6.3 we have $\mathrm{esscomm}\mathcal{A} \mid (I-E)H = \mathrm{esscomm}\mathcal{B} \mid (I-E)H$, so we must show $\mathrm{esscomm}\mathcal{A} \mid EH = \mathrm{esscomm}\mathcal{B} \mid EH$. Let $\mathcal{A}_\lambda = \mathcal{A} \mid E_\lambda H$, $\mathcal{B}_\lambda = \mathcal{B} \mid E_\lambda H$, $\mathcal{N}_\lambda = \mathcal{N} \mid E_\lambda H$, and let I_λ denote the identity in $\mathcal{L}(E_\lambda H)$. For each λ let $\mathrm{Tr}_\lambda(\cdot)$ denote the trace on $\mathcal{B}_\lambda$ with $\mathrm{Tr}_\lambda(I_\lambda) = 1$. Let $\hat{\mathcal{N}}_\lambda$ be a maximal nest in $\mathcal{B}_\lambda$ containing $\mathcal{N}_\lambda$. By Lemma 5.3 there exists $P_\lambda \in \hat{\mathcal{N}}_\lambda$ with $\mathrm{Tr}_\lambda(P_\lambda) = 1/2$. Then P_λ is equivalent to $P_\lambda^\perp \in \mathcal{B}_\lambda$, and also $P_\lambda \mathcal{B}_\lambda P_\lambda^\perp \subset \mathrm{Alg}(\hat{\mathcal{N}}_\lambda) \subseteq \mathcal{A}_\lambda$. Let P be the direct sum of the $\{P_\lambda\}$. Then $P \in \mathcal{A} \mid EH$, $P(\mathcal{B} \mid EH)P^\perp \subset \mathcal{A} \mid EH$, and P is equivalent to $P^\perp$ via a partial isometry in $\mathcal{B} \mid EH$. Now Corollary 3.2 implies that $\mathrm{esscomm}\mathcal{A} \mid EH = \mathrm{esscomm}\mathcal{B} \mid EH$. ▨

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