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COMPACT PERTURBATIONS OF DEFINITIZABLE OPERATORS. II

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The aim of the present note is to continue the study of perturbations of definitizable J-selfadjoint operators in a Kreĭn space from [10].

In this paper we assume that the unperturbed operator A is a J-positive J-selfadjoint operator similar to a selfadjoint operator (with respect to some positive definite scalar product).

We first consider, in Theorem 1, perturbations which produce J-positive J-selfadjoint operators again. These perturbations are assumed to have, in a certain sense, a small size with respect to A (see (2.10)). Theorem 1 extends a result of K. Veselić ([15]) for A having a bounded inverse. On the one hand in our context Theorem 1 is a tool for the investigation of perturbations which, in some sense, are compact. On the other hand it is of interest on its own right. It includes conditions for the preservation of regularity of the critical points and, hence, criteria for the perturbed operator B to be similar to a selfadjoint operator. Theorem 2, which is a consequence of Theorem 1, specifies a class of J-positive perturbations which preserve the similarity to a selfadjoint operator.

In Theorem 3 we consider a class of perturbations which produce definitizable J-selfadjoint operators. These perturbations are defined by some compactness properties with respect to the unperturbed operator A. Though, for a general operator A, not every finite-dimensional operator belongs to the perturbations considered here, this class is of interest for another reason: There is a simple criterion for the regularity of the critical point 0 of the perturbed operator. For the perturbations considered here the difference of the resolvents of A and B (B the perturbed J-selfadjoint operator, $\rho(B)\neq\emptyset$) is compact in a common point of their resolvent sets. We remark that this property, by a result of [10], is not sufficient for B to be definitizable.

Roughly speaking, Theorem 3 is giving a class of perturbations such that the perturbed operators have a spectral function with no more than a finite number of singular points.

We remark that perturbations of the type considered in this paper, which can be defined by forms as well, are investigated in the scattering theory of differential operators.

Besides the results of [10] we heavily rely on results of K. Veselić ([15]).

The results of this note can be partly carried over to unperturbed operators which are not similar to a selfadjoint one and also to operators in spaces with indefinite scalar product which are not Kreĭn spaces (see [8]).

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1. NOTATIONS AND PRELIMINARIES

1.1. Let $\mathscr{H}$ be a Kreĭn space with the indefinite scalar product $[x, y]$, $x, y \in \mathscr{H}$. We use the notations and definitions of [10]. In addition, we need the notion of regularity for critical points (see e.g. [12]): A critical point $t \in \overline{\mathbf{R}}$ *) of the definitizable J-selfadjoint operator H with $\rho(H) \neq \emptyset$ is called *regular* if there exists an open neighbourhood $\Delta_0 \subset \overline{\mathbf{R}}$ of t, $\bar{\Delta}_0 \cap c(H) = \{t\}$, such that the projectors $E_H(\Delta)$, $\bar{\Delta} \subset \Delta_0 \setminus \{t\}$, Δ connected, are uniformly bounded. Here $c(H)$ and E_H denote the set of critical points and the spectral function of H, respectively. The set of regular critical points of H is denoted by $c_r(H)$. The elements of $c_s(H) = c(H) \setminus c_r(H)$ are called *singular* critical points.

1.2. In the following let A be a J-positive J-selfadjoint operator:

$$[Ax, x] > 0, \quad 0 \neq x \in \mathscr{D}(A).$$

We assume $\rho(A) \neq \emptyset$. As a consequence the spectrum of A is real. Furthermore, we assume

$$c_s(A) = \emptyset.$$

Therefore, the spectral function E of A is bounded and

$$(x, y)_0 := \lim_{n \to \infty} [\{E((n^{-1}, n)) - E((-n, -n^{-1}))\}x, y], \quad x, y \in \mathscr{H},$$

defines a positive definite scalar product in $\mathscr{H}$ such that A is selfadjoint with respect to $(\cdot, \cdot)_0$ and $[Ax, y] = (|A|x, y)_0$, $x \in \mathscr{D}(A)$, $y \in \mathscr{H}$.

In the following we shall need some further scalar products and norms: On the linear set $D(A) \cap D(A^{-1})$ we define

$$(1.1) \qquad \begin{gathered} (x, y)_j := (|A|^{\frac{j}{2}} x, |A|^{\frac{j}{2}} y)_0, \quad \|x\|_j^2 := (x, x)_j, \\ j = -2, -1, 0, 1, 2; \quad x, y \in \mathscr{D}(A) \cap \mathscr{D}(A^{-1}). \end{gathered}$$

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*) $\overline{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ is regarded as the one point compactification of $\mathbf{R}$.

It is easy to see that these norms are pairwise compatible. Here two norms defined on a linear set are said to be *compatible* if the following holds: If a sequence is a Cauchy sequence with respect to both norms and converges to zero with respect to one norm, then it converges to zero with respect to the other norm.

Further, we define the quadratic norms

$$\|x\|_{0\wedge j} := (\|x\|_0^2 + \|x\|_j^2)^{\frac{1}{2}},$$

$$\|x\|_{0\vee j} := \inf\{\|x_1\|_0^2 + \|x_2\|_j^2 : x = x_1 + x_2;\ x_1, x_2 \in \mathscr{D}(A) \cap \mathscr{D}(A^{-1})\}^{\frac{1}{2}},$$

$$j = -2, -1, 0, 1, 2, \quad x \in \mathscr{D}(A) \cap \mathscr{D}(A^{-1}).$$

It is easy to see that

$$\|x\|_{0\wedge j} = \|(I + |A|^j)^{\frac{1}{2}} x\|_0, \quad \|x\|_{0\vee j} = \|(I + |A|^{-j})^{-\frac{1}{2}} x\|_0, \tag{1.2}$$

$$j = -2, -1, 0, 1, 2, \quad x \in \mathscr{D}(A) \cap \mathscr{D}(A^{-1}).$$

In what follows indices from the set $\{j, 0\wedge j, 0\vee j : j = -2, -1, 0, 1, 2\}$ are denoted by the letters r and s. Evidently, the norms $\|\cdot\|_r$ are pairwise compatible.

The completion of $\mathscr{D}(A) \cap \mathscr{D}(A^{-1})$ with respect to $\|\cdot\|_r$ is denoted by $\mathscr{H}_r$. We have

$$\mathscr{H}_0 = \mathscr{H}, \ \mathscr{H}_{0\wedge j} = \mathscr{D}(|A|^{\frac{j}{2}}), \quad j = -2, -1, 1, 2.$$

The supremum norm of $\mathscr{L}(\mathscr{H}_r, \mathscr{H}_s)$ will be denoted by $\|\cdot\|_{s,r}$.

Obviously, the norms

$$\|x\|'_{0\wedge j} := \|(I + |A|^{2j})^{\frac{1}{4}} x\|_0, \quad \|x\|'_{0\vee j} := \|(I + |A|^{-2j})^{-\frac{1}{4}} x\|_0, \tag{1.3}$$

$$j = -1, 1, \quad x \in \mathscr{D}(A) \cap \mathscr{D}(A^{-1}),$$

are equivalent to $\|\cdot\|_{0\wedge j}$ and $\|\cdot\|_{0\vee j}$, respectively. We shall need the norms (1.3) on $\mathscr{H}_{0\wedge j}$ and $\mathscr{H}_{0\vee j}$, $j = -1, 1$, respectively.

If $\|\cdot\|_r$ is finer than $\|\cdot\|_s$ then in a natural way $\mathscr{H}_r$ is densely embedded in $\mathscr{H}_s$. The embedding operator is denoted by $E_{s,r}$.

In the sequel the operator $A_{-1,1}$ defined as follows will frequently be used: The operator A maps $\mathscr{H}_{0\wedge 2}(\|\cdot\|_1)$ isometrically onto $\mathscr{H}_{0\wedge -2}(\|\cdot\|_{-1})$. The extension by continuity of this linear mapping to an isometric linear operator of $\mathscr{H}_1$ onto $\mathscr{H}_{-1}$ is denoted by $A_{-1,1}$.

1.3. Setting

$$\bar{j} := -j, \ \overline{0\wedge j} := 0 \vee -j, \ \overline{0\vee j} := 0 \wedge -j, \quad j = -2, -1, 0, 1, 2,$$

from (1.1) and (1.2) one easily derives

$$\|x\|_r = \sup\{|[x, y]| : \|y\|_{\bar{r}} \leq 1,\ y \in \mathscr{D}(A) \cap \mathscr{D}(A^{-1})\}, \quad x \in \mathscr{D}(A) \cap \mathscr{D}(A^{-1}).$$

Therefore, the hermitian sesquilinear form $[\cdot,\cdot]$ on $\{\mathscr{D}(A) \cap \mathscr{D}(A^{-1})\} \times \{D(A) \cap D(A^{-1})\}$ can be extended to sesquilinear forms on $\mathscr{H}_r \times \mathscr{H}_{\bar{r}}$ (denoted by the same symbol) such that

$$[x, y] = \overline{[y, x]}, \quad x \in \mathscr{H}_r,\ y \in \mathscr{H}_r,$$

holds.

By (1.1) and the definition of $A_{-1,1}$ we find

$$(1.4) \qquad (x, y)_1 = [A_{-1,1}x, y] = [x, A_{-1,1}y], \quad x, y \in \mathscr{H}_1.$$

Therefore, for every $y \in \mathscr{H}_1$ there exists a unique $z_y \in \mathscr{H}_{-1}$ such that $(x, y)_1 = [x, z_y]$, $x \in \mathscr{H}_1$, and the linear mapping $y \mapsto z_y\ (= A_{-1,1})$ is an isometry of $\mathscr{H}_r$ onto $\mathscr{H}_{-1}$. In a similar way one verifies that similar statements hold for $[\cdot,\cdot]$ on $\mathscr{H}_r \times \mathscr{H}_{\bar{r}}$, r arbitrary.

A linear operator V defined on $\mathscr{H}_r$ with values in $\mathscr{H}_{\bar{r}}$ is called $[\cdot,\cdot]$-*symmetric* if

$$[Vx, y] = [x, Vy], \quad x, y \in \mathscr{H}_r.$$

Obviously, $V \in \mathscr{L}(\mathscr{H}_r, \mathscr{H}_{\bar{r}})$. A linear operator V defined on $\mathscr{H}_r$ with range in $\mathscr{H}$ is called $[\cdot,\cdot]$-non-negative if

$$[Vx, x] \geq 0, \quad x \in \mathscr{H}_r.$$

We remark that we have defined in 1.2 and 1.3 what is called partial inner product space in [3]. In this article, we have chosen notations similar to [3].

1.4. We shall need the following criterion of K. Veselić ([15]) for regularity of the critical point ∞. In order this article to be self-contained we give a proof of it. For this and similar criteria for regularity see also [9].

LEMMA 1. *Let $\mathscr{H}$ be a Kreĭn space and let H be a definitizable operator in $\mathscr{H}$ with $\rho(H) \neq \emptyset$. Then $\infty \notin c_s(H)$ if and only if for some real $\xi_0, \eta_0, \eta_0 > \max\{|\mathrm{Im} z| : z \in \sigma(H)\}$, the operators*

$$(1.5) \qquad \int_{\xi_0 + i\eta_0}^{\xi_0 + i\eta} (H - \zeta I)^{-1} d\zeta + \int_{\xi_0 - i\eta}^{\xi_0 - i\eta_0} (H - \zeta I)^{-1} d\zeta = i \int_{\eta_0}^{\eta} ((H - (\xi_0 + i\mu)I)^{-1} + (H - (\xi_0 - i\mu)I)^{-1}) d\mu$$

are uniformly bounded.

Proof. We may suppose $\xi_0 = 0$. Let E be the spectral function of H. Taking into account the well-known behaviour at ∞ of the resolvent of a bounded operator it remains to prove the statement for the restriction of H to $E(\Delta)\mathscr{H}$, where $\Delta \subset \overline{\mathbf{R}}$ is some connected open neighbourhood of ∞. Therefore we may confine ourselves to the case of a selfadjoint (with respect to a definite scalar product) operator H, $(-1, 1) \subset \rho(H)$, and to the case of a J-positive operator H, $(-1, 1) \subset \rho(H)$. Hence we may assume $\eta_0 = 0$.

Let

$$f_\eta(t) := \int_0^\eta ((t - i\mu)^{-1} + (t + i\mu)^{-1})\, d\mu, \qquad \eta \in (0, \infty).$$

Obviously, $f_\eta(t) > 0$ for $t > 0$ and $f_\eta(t) < 0$ for $t < 0$. For $|t| \geqslant 1$ we find

$$|f_\eta(t)| = 2\int_0^\eta \frac{|t|}{|t|^2 + \mu^2}\, d\mu = 2 \arctan \frac{\eta}{|t|} \leqslant \pi . \tag{1.6}$$

Hence, in the case of a selfadjoint operator H, $(-1, 1) \subset \rho(H)$, the operators $f_\eta(H)$ are uniformly bounded.

It remains to show that for a J-positive operator H, $(-1, 1) \subset \rho(H)$, the uniform boundedness of the operators $f_\eta(H)$ implies $\infty \notin c_s(H)$.

From (1.6) we obtain that for every M, $1 \leqslant M < \infty$, and $M \leqslant \eta < \infty$ we have

$$\frac{\pi}{2} \leqslant |f_\eta(t)|, \quad 1 \leqslant |t| \leqslant M. \tag{1.7}$$

Denoting by χ_δ the characteristic function of some real interval δ, relation (1.7) implies

$$\chi_{[1, M]}(t) \leqslant \frac{2}{\pi} f_\eta(t), \qquad t \in (0, \infty),$$

and

$$-\chi_{[-M, -1]}(t) \geqslant \frac{2}{\pi} f_\eta(t), \qquad t \in (-\infty, 0),$$

for $\eta \geqslant M$. Therefore, making use of simple properties of the functional calculus for J-positive operators, we find

$$\begin{aligned} &[E([1, M])x, x] \leqslant \frac{2}{\pi}[f_\eta(H)x, x], \\ &|[E([-M, -1])x, x]| \leqslant \frac{2}{\pi}[f_\eta(H)x, x], \qquad x \in \mathscr{H}, \end{aligned} \tag{1.8}$$

for $\eta \geqslant M$. Now assume that the operators $f_\eta(H)$ are uniformly bounded. Then by (1.8) the spectral function of H is bounded, i.e., $\infty \notin c_s(H)$.

2. PERTURBATIONS OF J-POSITIVE J-SELFADJOINT OPERATORS PRESERVING DEFINITIZABILITY

2.1. Let A be an operator as in Section 1. In the following theorem perturbations are considered which preserve J-positiveness. In the special case of a uniformly J-positive operator A (i.e., $[Ax, x] \geq \alpha \|x\|^2$, $x \in \mathscr{D}(A)$, for some $\alpha > 0$, see [4]) this theorem (in a form similar to Corollary 1) was proved by K. Veselić [15]. In a part of the proof we follow [15].

THEOREM 1. *Let V be a $[\cdot,\cdot]$-symmetric linear operator defined on $\mathscr{H}_1$ with range in $\mathscr{H}_{-1}$ such that*

$$\|V\|_{-1,1} < 1. \tag{2.1}$$

Then the following holds.

(i) *There exists a J-positive J-selfadjoint operator B in $\mathscr{H}$ with $\rho(B) \neq \emptyset$ such that $\mathscr{D}(B)$ is dense in $\mathscr{H}_{0\wedge 1}$, $\mathscr{R}(B)$ is contained in $\mathscr{H}_{0\wedge -1}$ and dense in $\mathscr{H}_{-1}$,*

$$Bx = (A_{-1,1} + V)x, \quad x \in \mathscr{D}(B). \tag{2.2}$$

Moreover,

$$\infty \notin c_s(B). \tag{2.3}$$

(ii) *Let T be a closed operator in $\mathscr{H}$ with $\rho(T) \neq \emptyset$ such that $\mathscr{D}(T)$ is contained in $\mathscr{H}_{0\wedge 1}$ and dense in $\mathscr{H}_1$, $\mathscr{R}(T)$ is contained in $\mathscr{H}_{0\wedge -1}$,*

$$Tx = (A_{-1,1} + V)x, \quad x \in \mathscr{D}(T).$$

Then T coincides with B.

(iii) *If, in addition, we assume that*

$$\|(A_{-1,1} + V)^{-1} - A_{-1,1}^{-1}\|_{1,-1} = \|(I + A_{-1,1}^{-1}V)^{-1} - I\|_{1,1} < 1, \tag{2.4}$$

then we have

$$0 \notin c_s(B).$$

REMARK 1. Under the stronger assumption $\|V\|_{-1,1} < \frac{1}{2}$, instead of (2.1), (2.4) is fulfilled.

REMARK 2. If, in addition to the assumptions of Theorem 1, $V \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_{-1})$ is a $[\cdot,\cdot]$-non-negative operator, then by (1.4) $A_{-1,1}^{-1}V$ is a non-negative bounded operator in the Hilbert space $\mathscr{H}_1$. In this case, evidently, (2.4) is fulfilled.

Proof of Theorem 1. (i) For the construction of the operator B we make use of a method of [3].

We define

$$A^{(0)} := E_{0\vee -1, -1} A_{-1,1} E_{1, 0\wedge 1}, \qquad V^{(0)} := E_{0\vee -1, -1} V E_{1, 0\wedge 1}.$$

From (2.1) it is easy to see that

$$\| V^{(0)}x\|'_{0\vee -1} \leqslant \| V\|_{-1,1}\|x\|'_{0\wedge 1}, \quad x \in \mathscr{H}_{0\wedge 1}. \tag{2.5}$$

Now we claim that the operator B in $\mathscr{H}$ defined by

$$\mathscr{D}(B) := \{x \in \mathscr{H}_{0\wedge 1} : (A^{(0)} + \mathrm{i}E_{0\vee -1, 0\wedge 1} + V^{(0)})x \in \mathscr{H}\}$$

$$Bx := (A^{(0)} + V^{(0)})x, \quad x \in \mathscr{D}(B).$$

fulfils assertion (i).

It is easy to verify that the operators $A^{(0)} \pm \mathrm{i}E_{0\vee -1, 0\wedge 1}$ map $\mathscr{H}_{0\wedge 1}(\|\cdot\|'_{0\wedge 1})$ isometrically onto $\mathscr{H}_{0\vee -1}(\|\cdot\|'_{0\vee -1})$. Hence, in view of (2.5), the operators $A^{(0)} \pm \mathrm{i}E_{0\vee -1, 0\wedge 1} + V^{(0)}$ map $\mathscr{H}_{0\wedge 1}$ one-to-one onto $\mathscr{H}_{0\vee -1}$. Therefore, the restrictions to $\mathscr{H}$ of the inverses of these operators regarded as operators in $\mathscr{H}$ are bounded and their ranges are dense in $\mathscr{H}_{0\wedge 1}$. By definition of B these restrictions are the inverses of $B \pm \mathrm{i}I$, respectively. Hence $\mathscr{D}(B)$ is dense in $\mathscr{H}_{0\wedge 1}$ and $\mp\mathrm{i} \in \rho(B)$. Relation (2.2) follows from the definition of B.

Relation (2.1) implies that $A_{-1,1} + V$ is a one-to-one linear operator of $\mathscr{H}_1$ onto $\mathscr{H}_{-1}$. Hence, on account of (2.2), $\mathscr{R}(B)$ is dense in $\mathscr{H}_{-1}$.

According to (1.4) and the remark following it we find

$$[Bx, x] = [A_{-1,1}x + Vx, x] = (x, x)_1 + [Vx, x] \geqslant$$

$$\geqslant (1 - \| V\|_{-1,1})(x, x)_1, \quad x \in \mathscr{D}(B).$$

To prove (i) it remains to show (2.3). Here we shall follow the lines of K. Veselić's proof.

Let μ be a real number with $|\mu| \geqslant 1$. Then we have

$$\begin{aligned} &\sup\{\|(A - \mathrm{i}\mu I)^{-1}x\|'_{0\wedge 1} : x \in \mathscr{H}, \ \|x\|'_{0\vee -1} \leqslant 1\} = \\ &= \sup\{\|(|A|^2 + I)^{\frac{1}{2}}(A - \mathrm{i}\mu I)^{-1}y\|_0 : y \in \mathscr{H}, \ \|y\|_0 \leqslant 1\} \leqslant 1. \end{aligned} \tag{2.6}$$

Denoting the extension by continuity of

$$E_{0\wedge 1, 0\wedge 2}(A - \mathrm{i}\mu I)^{-1} : \mathscr{H}(\|\cdot\|_{0\vee -1}) \to \mathscr{H}_{0\wedge 1}$$

by $S(\mathrm{i}\mu) \in \mathscr{L}(\mathscr{H}_{0\vee -1}, \mathscr{H}_{0\wedge 1})$ we easily find

$$\begin{aligned} &h(\mathrm{i}\mu) := [(B - \mathrm{i}\mu I)^{-1}x, x] - [(A - \mathrm{i}\mu I)^{-1}x, x] = \\ &= \sum_{n=1}^{\infty} [S(\mathrm{i}\mu)\{-V^{(0)}S(\mathrm{i}\mu)\}^n E_{0\vee -1, 0}x, E_{0\vee -1, 0}x], \quad x \in \mathscr{H}. \end{aligned} \tag{2.7}$$

According to (2.5) and (2.6) this series converges absolutely and uniformly for $|\mu| \geq 1$. By Lemma 1 it is sufficient to prove that the integral

$$\int\limits_{[-\eta,-1]\cup[1,\eta]} h(\mathrm{i}\mu)\,\mathrm{d}\mu$$

is uniformly bounded for $\eta > 1$.

In view of (2.5), by an easy computation we obtain

$$\Big|\int\limits_{[-\eta,-1]\cup[1,\eta]} h(\mathrm{i}\mu)\,\mathrm{d}\mu\Big| \leq$$

$$\leq \sum_{n=1}^{\infty} \int\limits_{[-\eta,-1]\cup[1,\eta]} |[V^{(0)}\{S(\mathrm{i}\mu)V^{(0)}\}^{n-1}S(\mathrm{i}\mu)E_{0\vee -1,0}x,\ S(-\mathrm{i}\mu)E_{0\vee -1,0}x]|\,\mathrm{d}\mu \leq$$

$$\leq \sum_{n=1}^{\infty} \|V\|^{n}_{-1,1}\Big(\int\limits_{[-\eta,-1]\cup[1,\eta]} \|(A-\mathrm{i}\mu I)^{-1}x\|_1^2\,\mathrm{d}\mu\Big)^{\frac{1}{2}}\Big(\int\limits_{[-\eta,-1]\cup[1,\eta]} \|(A+\mathrm{i}\mu I)^{-1}x\|_1^2\,\mathrm{d}\mu\Big)^{\frac{1}{2}}.$$

Further, we find

$$\int\limits_{[-\eta,-1]\cup[1,\eta]} \|(A \pm \mathrm{i}\mu I)^{-1}x\|_1^2\,\mathrm{d}\mu = \int\limits_{[-\eta,-1]\cup[1,\eta]} (|A|(A-\mathrm{i}\mu I)^{-1}(A+\mathrm{i}\mu I)^{-1}x, x)_0\,\mathrm{d}\mu =$$

$$(2.8)\qquad = \frac{1}{2}\int\limits_{[-\eta,-1]\cup[1,\eta]} (\{(|A|-\mathrm{i}\mu I)^{-1} + (|A|+\mathrm{i}\mu I)^{-1}\}x, x)_0\,\mathrm{d}\mu.$$

Hence, by Lemma 1, the expression on the right side of (2.8) is uniformly bounded for $\eta > 1$. Thus (i) is proved.

(ii) The extension by continuity of

$$T\colon \mathscr{D}(T)(\|\cdot\|_1) \to \mathscr{H}_{0\wedge -1}(\|\cdot\|_{-1})$$

coincides with $A_{-1,1} + V$. By the definition of B it is easy to verify that B is an extension of T. Since B is a J-positive J-selfadjoint operator with $\rho(B) \neq \emptyset$ and, therefore, has real spectrum, we obtain $\rho(B) \cap \rho(T) \neq \emptyset$. This implies $B = T$.

(iii) Setting

$$U := (A_{-1,1} + V)^{-1} - A^{-1}_{-1,1},$$

interchanging the roles of A and A^{-1} and making use of (i) we find a J-positive J-selfadjoint operator B_1 in $\mathscr{H}$ with real spectrum such that $\mathscr{D}(B_1)$ is dense in $\mathscr{H}_{0\wedge -1}$, $\mathscr{R}(B_1)$ is contained in $\mathscr{H}_{0\wedge 1}$ and dense in $\mathscr{H}_1$,

$$B_1 x = (A^{-1}_{-1,1} + U)x, \quad x \in \mathscr{D}(B_1).$$

We have

$$(2.9)\qquad \infty \notin c_s(B_1).$$

We claim that $B_1 = B^{-1}$. Indeed, by the definition of B the set $\mathscr{R}(B) = \mathscr{D}(B^{-1})$ is contained in $\mathscr{H}_{0\wedge -1}$ and is dense in $\mathscr{H}_{-1}$ and $\mathscr{H}$. Furthermore, we have

$$B^{-1}x = (A_{-1,1} + V)^{-1}x = (A_{-1,1}^{-1} + U)x, \quad x \in \mathscr{D}(B^{-1}).$$

Then by (ii) $B_1 = B^{-1}$.

From (2.9) we obtain $\infty \notin c_s(B^{-1})$ and, hence, $0 \notin c_s(B)$ (see [7]).

It is easy to see that for $[\cdot,\cdot]$-non-negative V (see Remark 2) the perturbed operator B and the perturbation V, instead of A and V, respectively, also satisfy the assumptions of Theorem 1 with the same spaces $\mathscr{H}_1, \mathscr{H}_{-1}$ (with norms equivalent to the former ones). Thus we obtain an operator which, loosely speaking, differs from A by $2V$. Repeating this procedure we derive:

THEOREM 2. *Let $\hat{V}$ be a $[\cdot,\cdot]$-non-negative linear operator on $\mathscr{H}_1$ with range in $\mathscr{H}_{-1}$. Then the following holds.*

(i) *There exists a J-positive J-selfadjoint operator $\hat{B}$ in $\mathscr{H}$ with $\rho(\hat{B}) \neq \emptyset$ such that $\mathscr{D}(\hat{B})$ is dense in $\mathscr{H}_{0\wedge 1}$, $\mathscr{R}(\hat{B})$ is contained in $\mathscr{H}_{0\wedge -1}$ and dense in $\mathscr{H}_{-1}$,*

$$\hat{B}x = (A_{-1,1} + \hat{V})x, \quad x \in \mathscr{D}(\hat{B}).$$

Moreover,

$$c_s(\hat{B}) = \emptyset,$$

i.e., $\hat{B}$ is similar to a selfadjoint operator.

(ii) *Let $\hat{T}$ be a closed operator in $\mathscr{H}$ with $\rho(\hat{T}) \neq \emptyset$ such that $\mathscr{D}(\hat{T})$ is contained in $\mathscr{H}_{0\wedge 1}$ and dense in $\mathscr{H}_1$, $\mathscr{R}(\hat{T})$ is contained in $\mathscr{H}_{0\wedge -1}$,*

$$\hat{T}x = (A_{-1,1} + \hat{V})x, \quad x \in \mathscr{D}(\hat{T}).$$

Then $\hat{T}$ coincides with $\hat{B}$.

REMARK 3. By results from [2] and [6] it is known that there exist J-nonnegative perturbations (even one-dimensional, [2]) of J-positive operators which do not preserve the regularity of the critical points 0 and ∞.

The assertions (i) and (iii) of the following corollary are easy consequences of the construction of B in the proof of Theorem 1 and of Theorem 1, (iii). The last assertion is an immediate consequence of Theorem 2.

COROLLARY 1. *Let G be a J-symmetric operator in $\mathscr{H}$ such that $\mathscr{D}(G) \subset \mathscr{D}(A) = \mathscr{H}_{0\wedge 2}$, $\mathscr{R}(G) \subset \mathscr{R}(|A|^{\frac{1}{2}}) = \mathscr{H}_{0\wedge -1}$ and $\mathscr{D}(G)$ is dense in $\mathscr{H}_{0\wedge 1}$. Assume that*

$$\|Gx\|_{-1} \leqslant \beta \|x\|_1, \quad x \in \mathscr{D}(G),$$

or, equivalently,

$$\||A|^{-\frac{1}{2}}G|A|^{-\frac{1}{2}}x\|_0 \leqslant \beta\|x\|_0, \quad x \in |A|^{\frac{1}{2}}\mathscr{D}(G), \tag{2.10}$$

for some positive $\beta < 1$. *We note that* $|A|^{\frac{1}{2}}\mathscr{D}(G)$ *is dense in* $\mathscr{H}$. *Then the following holds.*

(i) *There exists a J-positive J-selfadjoint extension B of* $A + G$ *with* $\rho(B) \neq \emptyset$ *such that* $\mathscr{D}(B)$ *is dense in* $\mathscr{H}_{0\wedge 1}$, $\mathscr{R}(B)$ *is contained in* $\mathscr{H}_{0\wedge -1}$ *and dense in* $\mathscr{H}_{-1}$, *and* $\infty \notin c_s(B)$.

(ii) *Every J-symmetric extension T of* $A + G$ *with* $\mathscr{D}(T) \subset \mathscr{H}_{0\wedge 1}$, $\mathscr{R}(T) \subset \mathscr{H}_{0\wedge -1}$ *and* $\rho(T) \neq \emptyset$ *coincides with B.*

(iii) *Denote the closure of the operator* $\operatorname{sign}(A)|A|^{-\frac{1}{2}}G|A|^{-\frac{1}{2}}$ *(see (2.10)) by K. Then K is a bounded selfadjoint operator in the Hilbert space* $\mathscr{H}$ *and*

$$\|(I + K)^{-1} - I\|_{0,0} < 1 \tag{2.11}$$

implies $0 \notin c_s(B)$.

Further, if G is J-non-negative and (2.10) *holds with an arbitrary positive number* β *(not necessarily* < 1), *then* (i) *and* (ii) *are valid and we have* $c_s(B) = \emptyset$, *i.e. B is similar to a selfadjoint operator.*

Proof. To prove (ii) we shall see first that T considered as operator from $\mathscr{H}_1$ to $\mathscr{H}_{-1}$ is closable. Indeed, assume $x_n \in \mathscr{D}(T)$, $n = 1, 2, \ldots, y \in \mathscr{H}_{-1}$, $\|x_n\|_1 \to 0$, $\|Tx_n - y\|_{-1} \to 0$. Then for every $w \in \mathscr{D}(T) \subset \mathscr{H}_{0\wedge 1}$

$$[y, w] = \lim_{n\to\infty}[Tx_n, w] = \lim_{n\to\infty}[x_n, Tw] = 0.$$

Since $\mathscr{D}(T)$ is dense in $\mathscr{H}_{0\wedge 1}$ we find $y = 0$.

Setting V for the closure of G considered as operator from $\mathscr{H}_1$ to $\mathscr{H}_{-1}$, we derive that the closure of T considered as operator from $\mathscr{H}_1$ to $\mathscr{H}_{-1}$ is equal to $A_{-1,1} + V$. Then, applying Theorem 1 (ii), we obtain (ii).

In the following corollary we introduce more special assumptions on G which imply that $A + G = B$ is J-selfadjoint.

COROLLARY 2. *Let G be a J-symmetric operator in* $\mathscr{H}$ *with*

$$\mathscr{D}(G) \supset \mathscr{D}(A), \quad \mathscr{R}(G) \subset \mathscr{R}(A). \tag{2.12}$$

Assume that

$$\|Gx\|_0 \leqslant \beta\|Ax\|_0 \ \text{ or } \ \|A^{-1}Gx\|_0 \leqslant \beta\|x\|_0, \quad x \in \mathscr{D}(A), \tag{2.13}$$

for some positive $\beta < 1$.

Then $A+G$ *is a J-positive J-selfadjoint operator with* $\rho(A+G)\neq\emptyset$ *and* $\infty\notin c_s(A+G)$. *Assertion* (iii) *from Corollary* 1 *holds with* $B=A+G$.

Further, if G is J-non-negative and (2.13) *holds with an arbitrary positive number* β *(not necessarily* <1*), then* $A+G$ *is a J-positive J-selfadjoint operator with* $\rho(A+G)\neq\emptyset$ *and we have* $c_s(A+G)=\emptyset$, *i.e.* $A+G$ *is similar to a selfadjoint operator.*

Proof. By (2.12) and the J-symmetry of A and G we find that both inequalities of (2.13) are equivalent. By (2.12) and (2.13) G is A-bounded with A-bound smaller than 1. Hence by well-known perturbation results $A+G$ is J-selfadjoint and we have $\rho(A+G)\neq\emptyset$.

It is easy to see that the restriction of G to $\mathscr{D}(A)$ fulfils the assumptions of Corollary 1. On account of the Heinz inequality (or a suitable interpolation method, see Corollary 4 below) relation (2.13) implies (2.10). Hence the present corollary follows from the preceding one.

2.2. Now making use of a result of [10] we see that a perturbation of A which is a sum of an operator of the kind considered in Theorem 1 (roughly speaking: of small size) and a finite-dimensional J-selfadjoint operator produces a definitizable operator.

In the following theorem we shall consider a more special class of perturbations. This restriction allows a simple criterion for regularity of the critical point 0 (see (iii) of the following theorem).

THEOREM 3. *Let* W *be a* $[\cdot,\cdot]$*-symmetric compact linear operator defined on* $\mathscr{H}_1$ *with range in* $\mathscr{H}_{-1}$. *Then the following holds.*

(i) *There exists a definitizable J-selfadjoint operator* H *in* $\mathscr{H}$ *with* $\rho(H)\neq\emptyset$ *such that* $\mathscr{D}(H)$ *is dense in* $\mathscr{H}_{0\wedge 1}$, $\mathscr{R}(H)$ *is contained in* $\mathscr{H}_{0\wedge -1}$ *and*

$$Hx=(A_{-1,1}+W)x,\quad x\in\mathscr{D}(H). \tag{2.14}$$

We have

$$\infty\notin c_s(H)$$

and the following holds.

(a) *If* E *is the spectral function of* H *and* Δ_1, Δ_2 *are bounded intervals with* $\bar\Delta_1\subset(0,\infty)$, $\bar\Delta_2\subset(-\infty,0)$ *whose endpoints are not critical we have* $\varkappa_-(E(\Delta_1)\mathscr{H})<\infty$ *and* $\varkappa_+(E(\Delta_2)\mathscr{H})<\infty$ *(see* [10]*). In particular, all finite critical points* $\neq 0$ *of* H *have finite rank of indefiniteness.*

(b) *The non-real eigenvalues of* H *have finite algebraic multiplicities.*

(ii) *Every closed operator* S *in* $\mathscr{H}$ *with* $\rho(S)\neq\emptyset$ *such that* $\mathscr{D}(S)$ *is contained in* $\mathscr{H}_{0\wedge 1}$ *and dense in* $\mathscr{H}_1$, $\mathscr{R}(S)$ *is contained in* $\mathscr{H}_{0\wedge -1}$, *and*

$$Sx=(A_{-1,1}+W)x,\quad x\in\mathscr{D}(S),$$

coincides with H.

(iii) *Assume that $A_{-1,1}+W$ maps $\mathcal{H}_1$ one-to-one onto $\mathcal{H}_{-1}$ or, equivalently, that the operator $I+A_{-1,1}^{-1}W\in\mathcal{L}(\mathcal{H}_1)$ is invertible. We note that $A_{-1,1}^{-1}W$ is a compact selfadjoint operator in $\mathcal{H}_1$.*

Then we have

$$0\notin c_s(H).$$

Proof. (i) By assumption $A_{-1,1}^{-1}W$ is a compact operator in $\mathcal{H}_1$. Since, in view of (1.4),

$$(A_{-1,1}^{-1}Wx,x)_1=[Wx,x]=[x,Wx]=(x,A_{-1,1}^{-1}Wx)_1$$

the operator $A_{-1,1}^{-1}W$ is selfadjoint in $\mathcal{H}_1$. Hence we can choose $z_j\in\mathcal{D}(A)\subset\mathcal{H}_1$ and $\varepsilon_j=\pm 1$, $j=1,\dots,k$, such that the finite-dimensional bounded selfadjoint operator

$$\sum_{j=1}^{k}\varepsilon_j z_j(\cdot,z_j)_1$$

satisfies

$$\left\|A_{-1,1}^{-1}W-\sum_{j=1}^{k}\varepsilon_j z_j(\cdot,z_j)_1\right\|_{1,1}=\left\|W-\sum_{j=1}^{k}\varepsilon_j A_{-1,1}z_j(\cdot,z_j)_1\right\|_{-1,1}<1.$$

Denote by B the operator constructed in Theorem 1 and starting from

$$V:=W-\sum_{j=1}^{k}\varepsilon_j A_{-1,1}z_j(\cdot,z_j)_1\,.$$

Setting

$$H_1:=\sum_{j=1}^{k}\varepsilon_j Az_j[\cdot,Az_j],\qquad H:=B+H_1$$

we claim that H fulfils assertion (i).

Since we have $\infty\notin c_s(B)$, for every $\eta_0>0$ we can find a constant M such that

$$\|(B-\lambda I)^{-1}\|\leqslant M|\mathrm{Im}\lambda|^{-1},\qquad |\mathrm{Im}\lambda|\geqslant\eta_0\,.$$

Then, observing that H_1 is a finite-dimensional bounded J-selfadjoint operator, we find $\rho(H)\neq\emptyset$ and, making use of [10], we find that H is a definitizable J-selfadjoint operator satisfying the properties (a) and (b). By Theorem 1, $\mathcal{D}(B)=\mathcal{D}(H)$ is dense in $\mathcal{H}_{0\wedge 1}$. We have $\mathcal{R}(B)\subset\mathcal{H}_{0\wedge -1}$, $\mathcal{R}(H_1)\subset\mathcal{H}_{0\wedge -1}$ and, therefore, $\mathcal{R}(H)\subset\mathcal{H}_{0\wedge -1}$. The relation (2.2) implies (2.14). Expressing $(H-\lambda I)^{-1}=(B+H_1-\lambda I)^{-1}$ in terms of $(B-\lambda I)^{-1}$ and H_1 for sufficiently large $|\mathrm{Im}\lambda|$, and making use of Lemma 1 we obtain $\infty\notin c_s(H)$.

(ii) If we set $T=S-H_1$ the uniqueness statement follows from Theorem 1 (ii).

(iii) Under our additional assumption

$$(A_{-1,1}+W)^{-1}-A_{-1,1}^{-1}$$

is a $[\cdot,\cdot]$-symmetric and compact linear operator from $\mathscr{H}_{-1}$ in $\mathscr{H}_1$. Thus the assumptions of this theorem are satisfied with A^{-1} instead of A.

On the other hand our additional assumption implies that $\mathscr{R}(H)$ is dense in $\mathscr{H}_{-1}$. Therefore, H must be invertible. Indeed, if there exists a $x \in \mathscr{D}(H) \subset \mathscr{H}_{0\wedge 1}$ with $Hx = 0$, then for every $y \in \mathscr{D}(H)$ we find $[Hy, x] = 0$. This contradicts the density of $\mathscr{R}(H)$ in $\mathscr{H}_{-1}$. Now in the same way as in the proof of Theorem 1 (iii), we find $0 \notin c_s(H)$.

REMARK 4. Under the assumptions of Theorem 3 the difference of the resolvents of H and A in a common point of their resolvent sets is compact. This follows from relation (2.7).

COROLLARY 3. *Let G be a J-symmetric operator in $\mathscr{H}$ such that $\mathscr{D}(G) \subset \mathscr{D}(A) = \mathscr{H}_{0\wedge 2}$, $\mathscr{R}(G) \subset \mathscr{D}(|A|^{-\frac{1}{2}}) = \mathscr{H}_{0\wedge -1}$ and $\mathscr{D}(G)$ is dense in $\mathscr{H}_{0\wedge 1}$. Assume that G considered as a mapping of $\mathscr{D}(G)(\|\cdot\|_1)$ in $\mathscr{D}(|A|^{-\frac{1}{2}})(\|\cdot\|_{-1})$ is compact or, equivalently, $|A|^{-\frac{1}{2}}G|A|^{-\frac{1}{2}}$ (densely defined in $\mathscr{H}$, see Corollary 1) is compact. Then the following holds.*

(i) *There exists a definitizable J-selfadjoint extension H of $A+G$ with $\rho(H)\neq\emptyset$ satisfying all conditions mentioned in Theorem* 3 (i) *(except relation* (2.14)*).*

(ii) *Every J-symmetric extension S of $A+G$ with $\mathscr{D}(S)\subset\mathscr{H}_{0\wedge 1}$, $\mathscr{R}(S)\subset\mathscr{H}_{0\wedge -1}$ and $\rho(S)\neq\emptyset$ coincides with H.*

(iii) *Denote the closure of* $\operatorname{sign}(A)|A|^{-\frac{1}{2}}G|A|^{-\frac{1}{2}}$ *in $\mathscr{H}$ by K. K is a compact selfadjoint operator in $\mathscr{H}$. If $I+K$ is invertible, then we have* $0 \notin c_s(H)$.

(iv) *If, in addition, we assume that for some complex number μ,* $\operatorname{Im}\mu\neq 0$, *the closure of $A+G-\mu I$ is a Fredholm operator, then every J-selfadjoint extension H' of $A+G$ with $\rho(H')\neq\emptyset$ is definitizable and has the properties* (a) *and* (b) *from Theorem* 3.

Proof. Denoting the extension by continuity of

$$G\colon \mathscr{D}(G)(\|\cdot\|_1) \to \mathscr{H}_{0\wedge -1}(\|\cdot\|_{-1})$$

to a bounded operator of $\mathscr{H}_1$ in $\mathscr{H}_{-1}$ by W, the assertions (i) and (iii) are simple consequences of Theorem 3. (ii) is proved in the same way as Corollary 1 (ii).

To prove (iv) we make use of the finite-dimensional J-selfadjoint bounded operator H_1 from the proof of Theorem 3. Denote the closure of $A+G$ by $[A+G]$. On account of a well-known perturbation result, the assumption of (iv) implies that $[A+G]-H_1-\mu I$ is a Fredholm operator. Observing that $[A+G]-H_1$ is J-positive and making use of [13; Lemma 1.2] we conclude that $[A+G]-H_1-zI$ and, hence, $[A+G]-zI$ are Fredholm operators for every nonreal z.

Let H' be an araitrary J-selfadjoint extension of $A+G$ with $\rho(H')\neq\emptyset$. Since the nonreal part of the spectrum of H consists of no more than a finite number of points we have $\rho(H) \cap \rho(H') \neq \emptyset$. For an arbitrary nonreal point z_0 of $\rho(H) \cap \rho(H')$ the operators $(H - z_0 I)^{-1}$ and $(H' - z_0 I)^{-1}$ coincide on $\mathscr{R}([A + G] - z^0 I)$. Hence the difference

$$(H' - z_0 I)^{-1} - (H - z_0 I)^{-1}$$

is finite-dimensional. Then by [10], it follows that H' is definitizable and satisfies the conditions (a) and (b) from Theorem 3.

In the following corollary, as in Corollary 2, we introduce more special assumptions on G which imply that $A + G = H$ is J-selfadjoint.

COROLLARY 4. *Let G, $\mathscr{D}(G) \supset \mathscr{D}(A)$, be an A-compact J-symmetric operator. Let $\mathscr{R}(G) \subset \mathscr{D}(A^{-1})$. Assume that*

$$G: \mathscr{D}(A)(\|\cdot\|_2) \to \mathscr{H}(\|\cdot\|_0) \tag{2.15}$$

is compact or, equivalently,

$$G: \mathscr{D}(G)(\|\cdot\|_0) \to \mathscr{D}(A^{-1})(\|\cdot\|_{-2}) \tag{2.16}$$

is compact.

Then $A + G$ is a definitizable J-selfadjoint operator with $\rho(A + G) \neq \emptyset$ satisfying (a) *and* (b) *from Theorem* 3 *and $\infty \notin c_s(A + G)$. Moreover, assertion* (iii) *from Corollary* 3 *holds with $H = A + G$.*

Proof. By a well-known perturbation result we find $\rho(A + G) \neq \emptyset$ and that $A + G$ is J-selfadjoint.

It remains to prove that the restriction of G to $\mathscr{D}(A)$ fulfils the assumptions of Corollary 3. Thus we have to show that the compactness of the operators (2.15) and (2.16) implies the compactness of

$$G: \mathscr{D}(A)(\|\cdot\|_1) \to \mathscr{D}(A^{-1})(\|\cdot\|_{-1}). \tag{2.17}$$

Denote the closures of the operators (2.15) and (2.16) with respect to the indicated norms by $G_{0,2}$ and $G_{-2,0}$, respectively. Since the J-adjoint of the closure of GA^{-1} coincides with the closure of $A^{-1}G$, the compactness of one of the operators $G_{0,2}$ and $G_{-2,0}$ implies the compactness of the other.

Now we consider the interpolation pairs (see [14]) $\{\mathscr{H}_2, \mathscr{H}\}$ and $\{\mathscr{H}, \mathscr{H}_{-2}\}$, where $\mathscr{H}_2, \mathscr{H}$ are looked upon as contained in $\mathscr{H}_{0\vee 2}$ and $\mathscr{H}, \mathscr{H}_{-2}$ in $\mathscr{H}_{0\vee -2}$. The linear mapping

$$\tilde{G}: \mathscr{H}_{0\vee 2} = \mathscr{H} + \mathscr{H}_2 \to \mathscr{H}_{0\vee -2} = \mathscr{H} + \mathscr{H}_{-2}$$

defined by

$$\tilde{G}: x_0 + x_2 \mapsto G_{-2,0}x_0 + G_{0,2}x_2, \quad x_0 \in \mathscr{H}, \quad x_2 \in \mathscr{H}_2,$$

is a morphism of $\{\mathscr{H}_2, \mathscr{H}\}$ to $\{\mathscr{H}, \mathscr{H}_{-2}\}$ (see [14]).

Now, making use of the L-method, which is equivalent to the K-method ([14]), we easily find

$$\|x\|^{(L)\,2}_{(\mathscr{H}_2,\,\mathscr{H};\,2,\,2,\,\frac{1}{2},\,1)} = \int_0^\infty t^{-\frac{1}{2}} \|(I + t^{-1}|A|^{-2})^{-\frac{1}{2}} x\|_0^2 \frac{\mathrm{d}t}{t} =$$

$$= \pi(|A|x, x), \quad x \in \mathscr{H}_{0\wedge 2},$$

and

$$\|x\|^{(L)\,2}_{(\mathscr{H},\,\mathscr{H}_{-2};\,2,\,2,\,\frac{1}{2},\,1)} = \int_0^\infty t^{-\frac{1}{2}} \|(I + t^{-1}|A|^{2})^{-\frac{1}{2}} x\|_0^2 \frac{\mathrm{d}t}{t} =$$

$$= \pi(|A|^{-1}x, x), \quad x \in \mathscr{H}_{0\wedge -2}.$$

According to a theorem of K. Hayakawa ([5]) compactness of $G_{0,2}$ and $G_{-2,0}$ implies compactness of the operator (2.17).

REMARK 5. Evidently, in the case of a uniformly J-positive operator A (or, equivalently, A positive, $0 \in \rho(A)$) the assumptions of Corollary 4 reduce to A-compactness and J-symmetry of G. In this case the compactness of the operator (2.17) is a consequence of a theorem of M. G. Kreĭn [11] (see [8]).

In the case of a bounded operator A the assumptions of Corollary 4 reduce to the following: G is a J-selfadjoint bounded operator which is a compact mapping of $\mathscr{H}$ into $\mathscr{D}(A^{-1})$ furnished with the graph norm.

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