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SOME INVARIANTS FOR SEMI-FREDHOLM SYSTEMS OF ESSENTIALLY COMMUTING OPERATORS

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The aim of this paper is to give a sequence of integers associated in a natural way to a matrix (T_{ij}) of essentially commuting operators on a Banach space, which is semi-Fredholm and satisfies some condition (e.g. there is a right or left essential inverse matrix (S_{ji}), such that all the commutators $[S_{ji}, S_{kl}], [S_{ji}, T_{lk}]$ are compact). These numbers are invariant for compact or small norm perturbations of the T_{ij}'s and we will study their properties.

The invariants will be computed as indices of some Fredholm complexes associated with the matrix $\mathscr{T} = (T_{ij})$. A Fredholm complex is an extension of the notion of complex of Banach spaces with finite dimensional cohomology and the index is a generalization of the Euler characteristic of such a complex. To do the construction of complexes we isolate the properties of $\mathscr{T}$ in a universal framework:

Consider the polynomial ring $A = \mathbf{C}[X_{ij}]$ with mn generators, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, and consider also the map given by the matrix $X = (X_{ij})$:

$$X : A^n \to A^m.$$

It turns out that the kernel of X has a finite, free, minimal resolution. This resolution specializes in our concrete situation to a Fredholm complex. The same procedure applied to some operators obtained from $\mathscr{T}$ by twisting with symmetric powers will give an entire sequence of Fredholm complexes associated with $\mathscr{T}$ whose indices will be denoted by $\operatorname{ind}_p\mathscr{T}$, $p \in \mathbf{Z}$.

The first section contains the algebraic preliminaries, namely the construction of the complexes and of certain homotopies. These complexes are obtained by mixing Koszul complexes corresponding to each line in the $m \times n$-matrix with the symmetric algebra with m generators, in a manner which reminds Spencer complexes [16]. Let us remark that one of these complexes appears in [4], related to some problems in algebraic dimension theory.

In § 2 we use a notion of essential Fredholm complex of Banach spaces which is a particular case of the notion of Fredholm complex of vector bundles of G. Segal [14]. For a natural definition of the numerical index of such a complex, we obtain stability results. This invariant and its properties agree in particular cases with the indices of [2], [5] or [17].

The third section contains the definition of the indices associated to a semi--Fredholm system of essentially commuting operators, the stability theorem for the invariants and some of their properties. The theory for systems with one line is better understood [2], [5], [17] and we add some results.

In § 4 the problem of independence of $\mathrm{ind}_p\mathscr{T}$ for a right invertible matrix in an essentially commutative algebra is considered, using K-theory. An universal construction over the Stiefel manifold $V_{m,n-m}$ of m-frames in $\mathbf{C}^n$ is made and the main result of this section is that the elements corresponding in $\mathrm{K}^1(V_{m,n-m})$ to the complexes used to compute the indices, generate the ring $\mathrm{K}^*(V_{m,n-m})$. Moreover, the relations between these generators are found, which turns out to be useful for proving other properties of the indices. For example, all the sequence $(\mathrm{ind}_p\mathscr{T})_{p\in\mathbf{Z}}$ is determined by every m-consecutive part of it, or by $\mathrm{ind}_0\mathscr{T}$, $\mathrm{ind}_0(\mathscr{T}$ minus a row), $\mathrm{ind}_0(\mathscr{T}$ minus two rows),... . This part is relatively independent from the rest of the paper and may be of independent interest.

The last part contains applications to elliptic (on the right or on the left) systems of pseudodifferential operators on a compact manifold. Also systems with right or left invertible symbol of Toeplitz operators are illuminating examples for our constructions.

1. ALGEBRAIC PRELIMINARIES

Let A be a commutative, unital, C-algebra and $a = (a_1, \ldots, a_n)$ a n-tuple of elements of A. The Koszul complex $K_{\bullet}(a)$ associated to a can be defined as follows:

$$K_p(a) = \Lambda^p[Y, A], \quad p \in \mathbf{Z}$$

(1.1)
$$\delta_p : K_p(a) \to K_{p-1}(a)$$

$$\delta_p(aY_{i_1} \wedge \ldots \wedge Y_{i_p}) = \sum_{j=1}^{p} (-1)^j a_{i_j} a Y_{i_1} \wedge \ldots \wedge \hat{Y}_{i_j} \wedge \ldots \wedge Y_{i_p}$$

where we make the convention $1 \leqslant i_1 < \ldots < i_p \leqslant n$.

If there are elements $b_1, \ldots, b_n \in A$ such that $a_1b_1 + \ldots + a_nb_n = 1$, then the maps

$$\varepsilon^p : K_p(a) \to K_{p+1}(a), \quad p \in \mathbf{Z}$$

(1.2)
$$\varepsilon^p(aY_{i_1} \wedge \ldots \wedge Y_{i_p}) = \sum_{k=1}^{n} b_k a Y_k \wedge Y_{i_1} \wedge \ldots \wedge Y_{i_p}$$

give a trivial homotopy for $K_{\bullet}(a)$.

We have denoted by $\Lambda^{\cdot}[Y, A]$ the exterior algebra with n generators $Y_1, \ldots, Y_n$ and with coefficients in A.

There is a dual notion, that of cochains Koszul complex:

$$K^p(a) = \Lambda^p[Y, A], \quad \delta^p: K^p(a) \to K^{p+1}(a),$$

$$\delta^p(aY_{i_1} \wedge \ldots \wedge Y_{i_p}) = \sum_{k=1}^{n} a_k a Y_k \wedge Y_{i_1} \wedge \ldots \wedge Y_{i_p}.$$

If the ideal generated by $a_1, \ldots, a_n$ coincides with A, then there exists a trivial homotopy for $K^{\cdot}(a)$, similar with (1.1).

Let now K_i be m complexes, $1 \leqslant i \leqslant m$, of A-modules, which differ only by their boundary operators $\partial_{i_\bullet} =: \partial_i$. We shall suppose

$$\partial_i \partial_j + \partial_j \partial_i = 0, \quad 1 \leqslant i, j \leqslant m \tag{1.3}$$

the compositions being made in all possible combinations.

Let $S^{\cdot}$ be the symmetric algebra with m independent generators $X_1, \ldots, X_m$, with coefficients in A. We shall identify S^p with the set of homogenous polynomials of degree p.

We shall define the complex $K_\bullet = K_\bullet(K_1, \ldots, K_m)$:

$$\ldots \xrightarrow{D} K_1 \otimes_A S^1 \xrightarrow{D} K_0 \otimes_A S^0 \xrightarrow{\partial_1 \ldots \partial_m} K_{-m} \otimes_A S^0 \xrightarrow{D'} K_{-m-1} \otimes_A S^1 \xrightarrow{D'} \ldots \tag{1.4}$$

where the tensor products are on A and the operators D, D' work as follows:

$$D(xX_{i_1}^{\alpha_{i_1}} \ldots X_{i_p}^{\alpha_{i_p}}) = \sum_{j=1}^{p} \partial_{i_j} x X_{i_1}^{\alpha_{i_1}} \ldots X_{i_j}^{\alpha_{i_j}-1} \ldots X_{i_p}^{\alpha_{i_p}}, \tag{1.5}$$

$x \in K_h$, $1 \leqslant i_1 < \ldots < i_p \leqslant m$, $\alpha_{i_1} + \ldots + \alpha_{i_p} = h$ and all $\alpha_{i_j} > 0$;

$$D'(xX_{i_1}^{\alpha_{i_1}} \ldots X_{i_p}^{\alpha_{i_p}}) = \sum_{k=1}^{m} \partial_k x X_k X_{i_1}^{\alpha_{i_1}} \ldots X_{i_p}^{\alpha_{i_p}}, \tag{1.5'}$$

where $x \in K_{-h-m}$, $1 \leqslant i_1 < \ldots < i_p \leqslant m$, $\alpha_{i_1} + \ldots + \alpha_{i_p} = h$, $\alpha_{i_j} > 0$.

It is easy to prove that $K_\bullet$ is a complex.

PROPOSITION 1.1. *Assume, with the above notations, that there are the homotopy operators e_i on the complexes K_i, respectively, with the following properties:*

$$\begin{aligned} e_i \partial_i + \partial_i e_i &= 1, \quad 1 \leqslant i \leqslant m, \\ e_i \partial_j + \partial_j e_i &= 0, \quad 1 \leqslant i, j \leqslant m, \ i \neq j, \\ e_i e_j + e_j e_i &= 0, \quad 1 \leqslant i, j \leqslant m. \end{aligned} \tag{1.6}$$

Then the complex $K_\bullet$ is homotopy-trivial.

Proof. Let us define the maps, also of degree $+1$:

$$\tilde{e}_1 = e_1 e_2 \partial_2 \dots e_m \partial_m, \quad \tilde{e}_2 = e_2 e_3 \partial_3 \dots e_m \partial_m, \dots, \quad \tilde{e}_m = e_m$$

$$\tilde{e}'_1 = e_1 \partial_2 e_2 \dots \partial_m e_m, \quad \tilde{e}'_2 = e_2 \partial_3 e_3 \dots \partial_m e_m, \dots, \quad \tilde{e}'_m = e_m .$$

With this one we shall define the homotopy operators for $K_\bullet$:

$$\begin{array}{ccccccccc}
\dots \to S^1 \otimes K_1 & \xrightarrow{D} & S^0 \otimes K_0 & \xrightarrow{\partial_1 \dots \partial_m} & S^0 \otimes K_{-m} & \xrightarrow{D'} & S^1 \otimes K_{-m-1} \dots \\
 & \swarrow^{E} & & \swarrow^{e_m \dots e_1} & & \swarrow^{E'} & \\
\dots \to S^1 \otimes K_1 & \xrightarrow{D} & S^0 \otimes K_0 & \xrightarrow{\partial_1 \dots \partial_m} & S^0 \otimes K_{-m} & \xrightarrow{D'} & S^1 \otimes K_{-m-1} \dots
\end{array}$$

(1.7) $$E(xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}}) = \sum_{k \geqslant i_p} \tilde{e}_k x X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} X_k ,$$

the conventions for the indices being those in (1.5),

(1.7)′ $$E'(xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}}) = \tilde{e}'_{i_p} x X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}},$$

the indices being like in (1.5)′, except the case $x \in K_{-m}$.

First we verify the homotopy relations on the terms $S^0 \otimes K_0$ and $S^0 \otimes K_{-m}$:

$$(e_m \dots e_1 \partial_1 \dots \partial_m + DE)x = e_1 \partial_1 e_2 \partial_2 \dots e_m \partial_m x + D\left(\sum_{k=1}^{m} e_k x X_k\right) =$$

$$= e_1 \partial_1 e_2 \partial_2 \dots e_m \partial_m x + \sum_k \partial_k e_k x = e_1 \partial_1 e_2 \partial_2 \dots e_m \partial_m x + \partial_1 e_1 e_2 \partial_2 \dots e_m \partial_m x +$$

$$+ \partial_2 e_2 e_3 \partial_3 \dots e_m \partial_m x + \dots + \partial_m e_m x = (e_1 \partial_1 + \partial_1 e_1) e_2 \partial_2 \dots e_m \partial_m x + \dots +$$

$$+ \partial_m e_m x = (e_2 \partial_2 + \partial_2 e_2) e_3 \partial_3 \dots e_m \partial_m x + \dots + \partial_m e_m x = (e_m \partial_m + \partial_m e_m) x = x,$$

$$(\partial_1 \dots \partial_m e_m \dots e_1 + E'D')x = \partial_1 e_1 \partial_2 e_2 \dots \partial_m e_m x +$$

$$+ E'\left(\sum_{k=1}^{m} \partial_k x X_k\right) = \partial_1 e_1 \partial_2 e_2 \dots \partial_m e_m x + \dots + e_m \partial_m x =$$

$$= (\partial_1 e_1 + e_1 \partial_1) \partial_2 e_2 \dots \partial_m e_m x + \dots + e_m \partial_m x =$$

$$= (\partial_2 e_2 + e_2 \partial_2) \partial_3 e_3 \dots \partial_m e_m x + \dots + e_m \partial_m x = \dots = (\partial_m e_m + e_m \partial_m) x = x.$$

For the terms on the left of $S^0 \otimes K_0$ one verifies the homotopy relations as follows:

$$ED(xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}}) = E\left(\sum_{j=1}^{p} \partial_{i_j} xX_{i_1}^{\alpha_{i_1}} \dots X_{i_j}^{\alpha_{i_j}-1} \dots X_{i_p}^{\alpha_{i_p}}\right) =$$

$$(1.8) \quad = \sum_{j=1}^{p-1} \sum_{k \geq i_p} \tilde{e}_k \partial_{i_j} xX_{i_1}^{\alpha_{i_1}} \dots X_{i_j}^{\alpha_{i_j}-1} \dots X_{i_p}^{\alpha_{i_p}} X_k + \tilde{e}_{i_p} \partial_{i_p} xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} +$$

$$+ \sum_{k > i_p} \tilde{e}_k \partial_{i_p} xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} X_k .$$

The term $\tilde{e}_{i_{p-1}} \partial_{i_p} xX_{i_1}^{\alpha_{i_1}} \dots X_{i_{p-1}}^{\alpha_{i_{p-1}}+1}$, if it appears, does not change the sum, because $\tilde{e}_{i_{p-1}} \partial_{i_p} = 0$.

$$(1.8)' \qquad DE(xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}}) = D\left(\sum_{k \geq i_p} \tilde{e}_k xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} X_k\right) =$$

$$= \sum_{k \geq i_p} \sum_{j=1}^{p-1} \partial_{i_j} \tilde{e}_k xX_{i_1}^{\alpha_{i_1}} \dots X_{i_j}^{\alpha_{i_j}-1} \dots X_{i_p}^{\alpha_{i_p}} X_k + \sum_{k > i_p} \partial_{i_p} \tilde{e}_k xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}-1} X_k +$$

$$+ \sum_{k \geq i_p} \partial_k \tilde{e}_k xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} .$$

From (1.3) and (1.6),

$$\partial_j \tilde{e}_k + \tilde{e}_k \partial_j = \partial_j e_k e_{k+1} \partial_{k+1} \dots e_m \partial_m + e_k \partial_j e_{k+1} \partial_{k+1} \dots e_m \partial_m = 0$$

for $1 \leq j < k \leq m$, and

$$\tilde{e}_j \partial_j + \sum_{k \geq j} \partial_k \tilde{e}_k = 1, \quad 1 \leq j \leq m.$$

Indeed

$$\tilde{e}_j \partial_j + \sum_{k \geq j} \partial_k e_k = e_j \partial_j e_{j+1} \partial_{j+1} \dots e_m \partial_m +$$

$$+ \partial_j e_j e_{j+1} \partial_{j+1} \dots e_m \partial_m + \partial_{j+1} e_{j+1} e_{j+2} \partial_{j+2} \dots e_m \partial_m + \dots + \partial_m e_m =$$

$$= (e_j \partial_j + \partial_j e_j) e_{j+1} \partial_{j+1} \dots e_m \partial_m + \dots + \partial_m e_m = \dots = e_m \partial_m + \partial_m e_m = I.$$

Finally by adding (1.8) to (1.8)' one obtains $ED + DE = I$.

On the right of $S^0 \otimes K_{-m}$ one computes in the same way:

$$E'D'(xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}}) = E'\left(\sum_{k \leq i_p} \partial_k x X_k X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} + \right.$$

$$(1.9) \qquad \left. + \sum_{k > i_p} \partial_k x X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} X_k\right) = \sum_{k \leq i_p} \tilde{e}'_{i_p} \partial_k x X_k X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}-1} +$$

$$+ \sum_{k > i_p} \tilde{e}'_k \partial_k x X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} ;$$

$$D'E'(xX_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}}) = D'(\tilde{e}'_{i_p} x X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}-1}) =$$

$$(1.9)' \qquad = \sum_{k < i_p} \partial_k \tilde{e}'_{i_p} x X_k X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}-1} + \partial_{i_p} \tilde{e}'_{i_p} x X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} +$$

$$+ \sum_{k > i_p} \partial_k \tilde{e}'_{i_p} x X_{i_1}^{\alpha_{i_1}} \dots X_{i_p}^{\alpha_{i_p}} X_k .$$

The last term of (1.9)′ is zero because $\partial_k \tilde{e}'_{i_p} = 0$.

Remarking that $\partial_k \tilde{e}'_j + \tilde{e}'_j \partial_k = 0$ for $k < j$ and that $\partial_j \tilde{e}'_j + \sum_{k \geq j} \tilde{e}'_k \partial_k = 1$, $1 \leq j \leq m$, the equality $E'D' + D'E' = I$ holds.

Let $\mathscr{A}$ be a $m \times n$ matrix (a_{ij}) with the elements in a commutative algebra A. We shall denote by $a_i = (a_{i1}, \dots, a_{in})$ the rows of $\mathscr{A}$ and by $a^j = (a_{1j}, \dots, a_{mj})'$ the columns of $\mathscr{A}$. We shall apply the above proposition to the Koszul complexes $K_i = K_\bullet(a_i)$ or to $K^j = K^\bullet(a^j)$. If we denote by ∂_i and ∂^j the corresponding boundary operators, the relations (1.3) hold.

We shall denote by $C_\bullet(p)$ the translations of a complex $C_\bullet$: $C_q(p) = C_{p+q}$, $\partial_q(p) = \partial_{p+q}$, $p, q \in \mathbf{Z}$.

COROLLARY 1.2. *If the matrix A is left or right invertible, then the complexes $K_\bullet(K^1(p), \dots, K^n(p)) \otimes_A M$, respectively $K_\bullet(K_1(p), \dots, K_m(p)) \otimes_A M$ are exact, for each $p \in \mathbf{Z}$ and each A-module M.*

Proof. Let $\mathscr{B} = (b_{ji})$ be a left inverse for $\mathscr{A}$. Then $m \geq n$ and if one denotes by b_j the rows of $\mathscr{B}$:

$$b_j a^j = \sum_i b_{ji} a_{ij} = 1, \quad 1 \leq j \leq n,$$

$$(1.10) \qquad b_j a^k = \sum_i b_{ji} a_{ik} = 0, \quad 1 \leq j, k \leq n, \ j \neq k.$$

Then one defines with the elements of $\mathscr{B}$, in analogy with (1.1), the operators of degree -1, $e_k : K^j \to K^j$,

$$e_k(xY_{i_1} \wedge \dots \wedge Y_{i_p}) = \sum_{h=1}^{p} (-1)^h b_{ki_h} xY_{i_1} \wedge \dots \wedge \hat{Y}_{i_h} \wedge \dots \wedge Y_{i_p},$$

and the relations (1.6) follow from (1.10).

For a right invertible matrix the procedure is dual.

By Proposition (1.1) the complexes $K_\bullet(K^1(p), \dots, K^n(p))$, respectively $K_\bullet(K_1(p), \dots, K_m(p))$ are homotopy-trivial, which is enough for the proof of the corollary.

2. ESSENTIAL FREDHOLM COMPLEXES

There are many directions to extend the theory of Fredholm operators to complexes of operators, for example the Hilbert space context [2] where there exists a good notion of Fredholm complex with nice stability properties, or the Banach space context and unbounded operators on them [17], where there are still open problems. We shall relate these ways by a notion of essential Fredholm complex of Banach spaces and associate an index with good stability properties.

We denote by X, Y, ...; δ, ε, ... Banach spaces and bounded linear operators on them, $\mathscr{L}(X, Y)$ the set of linear bounded operators between X and Y and by $\mathscr{K}(X, Y)$ the set of compact operators.

DEFINITION 2.1. An *essential complex* of Banach spaces is a sequence of Banach spaces X^p and operators δ^p:

$$0 \to X^0 \xrightarrow{\delta^0} X^1 \xrightarrow{\delta^1} \dots \to X^{n-1} \xrightarrow{\delta^{n-1}} X^n \to 0$$

such that $\delta^p \delta^{p-1}$ is a compact operator for every $p \in \mathbf{Z}$.

We shall use only finite complexes, i.e. $X^p = 0$ for $p < 0$ or for large p.

The Fredholm property for such an essential complex will mean the exactness modulo compacts:

DEFINITION 2.2. An essential complex $X^\bullet$ is *Fredholm* if for any Banach space Y, the complex

$$\mathscr{L}(Y, X^\bullet)/\mathscr{K}(Y, X^\bullet)$$

is exact.

This is equivalent with the existence of a homotopy between the identity of the complex and the zero map, modulo compacts:

PROPOSITION 2.3. *An essential complex $X^\cdot$ is Fredholm if and only if there exist operators $\varepsilon^p : X^p \to X^{p-1}$ such that:*

$$\delta^{p-1}\varepsilon^p + \varepsilon^{p+1}\delta^p = I + \text{compact}, \quad p \in \mathbf{Z}. \tag{2.1}$$

Moreover if $\varepsilon_1^\cdot$ and $\varepsilon_2^\cdot$ satisfy (2.1), *then they are homotopic in the class of essentially trivial homotopies.*

This characterization of Fredholm complexes agrees with a particular case of the definition of Fredholm complexes of vector bundles given by G. Segal [14].

Proof: The sufficiency is clear because for each Banach space Y, the complex $\mathscr{L}(Y, X^\cdot)/\mathscr{K}(Y, X^\cdot)$ is homotopy-trivial.

To prove the necessity, we shall construct by decreasing induction on p the operators ε^p. The first step is clear because the complex is finite. Suppose that there exists $\varepsilon^{q,s}$ for $q \geqslant p + 1$. Then the complex

$$\mathscr{L}(X^p, X^{p-1})/\mathscr{K}(X^p, X^{p-1}) \xrightarrow{\delta_*^{p-1}} \mathscr{L}(X^p, X^p)/\mathscr{K}(X^p, X^p) \xrightarrow{\delta_*^p}$$

$$\to \mathscr{L}(X^p, X^{p+1})/\mathscr{K}(X^p, X^{p+1})$$

is exact, so that if we choose the class of $I - \varepsilon^{p+1}\delta^p$ in the middle term, there is $\varepsilon^p \in \mathscr{L}(X^p, X^{p-1})$ such that

$$I - \varepsilon^{p+1}\delta^p = \delta^{p-1}\varepsilon^p + \text{compact}.$$

Indeed,

$$\delta^p(I - \varepsilon^{p+1}\delta^p) = \delta^p - \delta^p\varepsilon^{p+1}\delta^p =$$

$$= \delta^p - (I - \varepsilon^{p+2}\delta^{p+1})\delta^p + \text{compact} = \text{compact}.$$

Suppose that we have two essential trivial homotopies $\varepsilon_1^\cdot$ and $\varepsilon_2^\cdot$ of $X^\cdot$. Then for each $\lambda \in [0, 1]$, $\lambda\varepsilon_1^\cdot + (1 - \lambda)\varepsilon_2^\cdot$ is still an essential trivial homotopy, which proves the last affirmation of the proposition.

Let us remark that if we replace $\varepsilon^\cdot$ by $\varepsilon^\cdot\delta^\cdot\varepsilon^\cdot$, then we obtain an essential complex

$$0 \leftarrow X^0 \xleftarrow{\varepsilon^1} X^1 \xleftarrow{\varepsilon^2} \ldots \leftarrow X^{n-1} \xleftarrow{\varepsilon^n} X^n \leftarrow 0 \tag{2.2}$$

which is Fredholm by the symmetry of the relations (2.1).

Let $X^\cdot$ be an essential Fredholm complex, homotopiqually trivial by an essential homotopy $\varepsilon^\cdot$. Then the operator

$$T = \begin{pmatrix} \delta^0 & \varepsilon^2 & & 0 \\ & \delta^2 & \varepsilon^4 & \\ & & \delta^4 & \\ 0 & & & \ddots \end{pmatrix} : \bigoplus_p X^{2p} \to \bigoplus_p X^{2p+1} \tag{2.3}$$

is a Fredholm operator. Indeed, let us define

$$S = \begin{pmatrix} \varepsilon^1 & & & 0 \\ \delta^1 & \varepsilon^3 & & \\ & \delta^3 & \varepsilon^5 & \\ 0 & & & \ddots \end{pmatrix} : \bigoplus_p X^{2p+1} \to \bigoplus_p X^{2p}.$$

Then the operators TS and ST are essential invertible:

$$ST = \begin{pmatrix} I & \varepsilon^1\varepsilon^2 & & 0 \\ & I & \varepsilon^3\varepsilon^4 & \\ & & I & \\ 0 & & & \ddots \end{pmatrix} + \text{compact},$$

$$TS = \begin{pmatrix} I & \varepsilon^2\varepsilon^3 & & 0 \\ & I & \varepsilon^4\varepsilon^5 & \\ & & I & \\ 0 & & & \ddots \end{pmatrix} + \text{compact}.$$

By the last assertion of Proposition 2.3, the index of T does not depend on the choice of the essentially trivial homotopy $\varepsilon^{\cdot}$, so we give the following

DEFINITION 2.4. Let $X^{\cdot}$ be an essential Fredholm complex and let T be the operator defined by 2.3. Then the *index* of $X^{\cdot}$ is:

$$\operatorname{ind} X^{\cdot} = \operatorname{ind} T.$$

The notion of essential Fredholm complex and associated index are obviously stable under compact perturbations of the coboundary operators $\delta^{\cdot}$.

Now we shall prove the stability under small perturbations. Let $(X^{\cdot}, \delta^{\cdot})$ be an essential Fredholm complex and let $\tilde{\delta}^{\cdot}$ be a sufficiently small norm perturbation of $\delta^{\cdot}$, such that $\tilde{\delta}^{\cdot 2} =$ compact. Then for each Banach space Y, the complex

$$(\mathscr{L}(Y, X^{\cdot})/\mathscr{K}(Y, X^{\cdot}), \tilde{\delta}^{\cdot}_{*})$$

is exact. Indeed this holds by the stability theorem for exact complexes (for example [17, Theorem 2.11]).

In order to prove the invariance of the index, let $\tilde{\varepsilon}^{\cdot}$ be a trivial essential homotopy for $\tilde{\delta}^{\cdot}$. By the classical theorem of stability for Fredholm operators, the operator

$$T_0 = \begin{pmatrix} \tilde{\delta}^0 & \varepsilon^2 & & 0 \\ & \tilde{\delta}^2 & \varepsilon^4 & \\ & & \tilde{\delta}^4 & \\ 0 & & & \ddots \end{pmatrix}$$

is Fredholm as a perturbation of T, and $\operatorname{ind} T_0 = \operatorname{ind} T$.

Let us remark that for each $\lambda \in [0, 1]$, the operator

$$T_\lambda = \begin{pmatrix} \tilde{\delta}^0 & \lambda\tilde{\varepsilon}^2 + (1-\lambda)\varepsilon^2 & & 0 \\ & \tilde{\delta}^2 & \lambda\tilde{\varepsilon}^4 + (1-\lambda)\varepsilon^4 & \\ & & \tilde{\varepsilon}^4 & \\ 0 & & & \ddots \end{pmatrix}$$

is Fredholm. Indeed, if we take

$$S_\lambda = \begin{pmatrix} \lambda\tilde{\varepsilon}^1 + (1-\lambda)\varepsilon^1 & & 0 \\ \tilde{\delta}^1 & \lambda\tilde{\varepsilon}^3 + (1-\lambda)\varepsilon^3 & \\ & \tilde{\delta}^3 & \\ 0 & & \ddots \end{pmatrix}$$

then

$$T_\lambda S_\lambda = \begin{pmatrix} \lambda + (1-\lambda)(\tilde{\delta}^0\varepsilon^1 + \varepsilon^2\tilde{\delta}^1) & & * \\ & \lambda + (1-\lambda)(\tilde{\delta}^2\varepsilon^3 + \varepsilon^4\tilde{\delta}^3) & \\ 0 & & \ddots \end{pmatrix}$$

and the diagonal blocks are essentially invertible, because $\tilde{\delta}^0\varepsilon^1 + \varepsilon^2\tilde{\delta}^1, \ldots$ are essentially small perturbations of the identity, hence $-\lambda(1-\lambda)^{-1}$ is not an element of the essential spectrum of them.

Similarly one proves that $S_\lambda T_\lambda$ is essentially invertible. Because the index is invariant to homotopy, we obtain:

$$\operatorname{ind} T_0 = \operatorname{ind} T_1 = \operatorname{ind} T,$$

hence

$$\operatorname{ind}\tilde{X}^\cdot = \operatorname{ind} T_1 = \operatorname{ind} T_0 = \operatorname{ind} T = \operatorname{ind} X^\cdot.$$

Concluding, we have proved the following

THEOREM 2.5. *The Fredholm property and the associated index of an essential complex of Banach spaces are invariant under small norm or compact perturbations of the boundary operators.*

The next proposition relates our index to the Euler characteristic $\chi(X^\cdot)$ of a complex $X^\cdot$.

PROPOSITION 2.6. *Let $X^\cdot$ be a Fredholm complex (i.e. essential Fredholm complex and $\delta^p\delta^{p-1} = 0$, $p \in \mathbf{Z}$).*

Then $H^p(X^\cdot)$, $p \in \mathbf{Z}$, are finite dimensional spaces and

$$\operatorname{ind} X^\cdot = \chi(X^\cdot). \tag{2.4}$$

Proof. Let $\varepsilon^\cdot$ be a homotopy of $X^\cdot$ and $k^p \in \mathscr{K}(X^p)$, such that

$$\varepsilon^{p+1}\delta^p + \delta^{p-1}\varepsilon^p = I + k^p, \quad p \in \mathbf{Z}.$$

Then $\delta^{p-1}\varepsilon^p|\mathrm{Ker}\delta^p = (I + k^p)|\mathrm{Ker}\delta^p \colon \mathrm{Ker}\delta^p \to \mathrm{Ker}\delta^p$, so that $k^p(\mathrm{Ker}\delta^p) \subset \mathrm{Ker}\delta^p$. But $(I + k^p)|\mathrm{Ker}\delta^p$ is a Fredholm operator on $\mathrm{Ker}\delta^p$, hence

$$\dim H^p(X^\cdot) = \dim\mathrm{Ker}\delta^p/\mathrm{Im}\delta^{p-1} \leqslant \dim\mathrm{Coker}(I + k^p)\mathrm{Ker}\delta^p$$

and the last number is finite. As a consequence, the ranges of operators δ^p are closed.

We shall prove the equality (2.4) by induction on the length of the complex $X^\cdot$. If the length of the complex is one, then (2.4) is the definition of the index. Suppose the affirmation is true for complexes of the length $n - 1$.

Let $X^\cdot$ be a Fredholm complex of the length n:

$$0 \to X^0 \to X^1 \to X^2 \to \ldots \to X^n \to 0.$$

Let $\varepsilon^\cdot$ be a homotopy of the complex $X^\cdot$ and let denote $\tilde{X}^\cdot$ the complex:

$$0 \to X^1/\mathrm{Im}\delta^0 \to X^2 \to X^3 \to \ldots \to X^n \to 0.$$

Then $\chi(X^\cdot) = \dim\mathrm{Ker}\delta^0 - \mathrm{ind}\tilde{X}^\cdot$ and $\varepsilon^\cdot$ induces a homotopy $\tilde{\varepsilon}^\cdot$ for $\tilde{X}^\cdot$: $\tilde{\varepsilon}^2 = \pi\circ\varepsilon^2$, $\tilde{\varepsilon}^q = \varepsilon^q$ for $q \geqslant 3$, where π is the projection map $\pi : X^1 \to X^1/\mathrm{Im}\delta^0$.

A short computation shows that

$$\mathrm{ind}T = \dim\mathrm{Ker}\delta^0 + \mathrm{ind}\begin{pmatrix} \tilde{\varepsilon}^2 & & 0 \\ \delta^2 & \varepsilon^4 & \\ & \delta^4 & \\ 0 & & \ddots \end{pmatrix} =$$

$$= \dim\mathrm{Ker}\delta^0 - \mathrm{ind}\begin{pmatrix} \delta^1 & \varepsilon^3 & & 0 \\ & \delta^3 & \varepsilon^5 & \\ 0 & & & \ddots \end{pmatrix} =$$

$$= \dim\mathrm{Ker}\delta^0 - \mathrm{ind}\tilde{X}^\cdot;$$

because the second matrix operator is the essential inverse of the first, hence by the induction hypothesis $\mathrm{ind}T = \chi(X^\cdot)$.

In the Hilbert spaces case one can prove the following:

PROPOSITION 2.6. *Let $H^\cdot$ be an essential complex of Hilbert spaces. The following assertions are equivalent:*

i) *$H^\cdot$ is Fredholm,*

ii) *$(\delta^{\cdot *})$ is almost an essential trivial homotopy for $H^\cdot$, namely the operator $\delta\delta^* + \delta^*\delta$ is Fredholm.*

iii) *There exist compact modifications of the boundaries such that $H^\cdot$ becomes a complex with cohomology spaces of finite dimension.*

COROLLARY 2.7. *If H_1 and $H_2^\bullet$ are two essential Fredholm complexes of Hilbert spaces and if $f^\bullet : H_1^\bullet \to H_2^\bullet$ is, modulo compacts, a morphism of complexes which is a linear isomorphism, then* $\mathrm{ind} H_1^\bullet = \mathrm{ind} H_2^\bullet$.

Proof. Let $\tilde{H}_1^\bullet$ be a complex which is a compact modification of $H_1^\bullet$. Then transporting by $f^\bullet$ the boundaries of $H_1^\bullet$, one obtains a compact perturbation of $H_2^\bullet$ into an essential Fredholm complex $\tilde{H}_2^\bullet$. Then

$$\mathrm{ind} H_1^\bullet = \mathrm{ind} \tilde{H}_1^\bullet = \chi(\tilde{H}_1^\bullet) = \chi(\tilde{H}_2^\bullet) = \mathrm{ind} \tilde{H}_2^\bullet = \mathrm{ind} H_2^\bullet.$$

The property of fredholmicity for an essential complex is stable also by passing to duals:

THEOREM 2.8. *Let $X^\bullet$ be an essential Fredholm complex of Banach spaces. Then the dual essential complex $X^{\bullet\prime}$ is also Fredholm and*

$$\mathrm{ind} X^{\bullet\prime} = \mathrm{ind} X^\bullet$$

if the zero'th covariant component of $X^{\bullet\prime}$ is the dual of the zero'th contravariant component of $X^\bullet$.

Proof. Let $\varepsilon^\bullet$ be an essential trivial homotopy of $X^\bullet$. Then the homotopy relations for $X^\bullet$ give by duality homotopy relations for $X^{\bullet\prime}$, hence the dual complex is Fredholm. The associated Fredholm operator of $X^{\bullet\prime}$ is

$$S = \begin{pmatrix} \varepsilon^{1\prime} & \delta^{1\prime} & & 0 \\ & \varepsilon^{3\prime} & \delta^{3\prime} & \\ & & \varepsilon^{5\prime} & \\ 0 & & & \ddots \end{pmatrix}$$

so that $\mathrm{ind} S = -\mathrm{ind} \begin{pmatrix} \varepsilon^1 & & 0 \\ \delta^1 & \varepsilon^3 & \\ & \delta^3 & \\ 0 & & \ddots \end{pmatrix} = \mathrm{ind} X^\bullet.$

3. SEMI-FREDHOLM SYSTEMS OF ESSENTIALLY COMMUTING OPERATORS

Let X be a Banach space and $\mathscr{T} = (T_{ij})$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, a system of linear bounded operators with the property:

$$[T_{ij}, T_{hk}] \in \mathscr{K}(X), \quad 1 \leqslant i, h \leqslant m, \ 1 \leqslant j, k \leqslant n. \tag{3.1}$$

We shall denote by $K_i(K^j)$ the essential Koszul complex of the i'th row (j'column) of $\mathscr{T}$. Also one can form the essential complexes

$$K^p(\mathscr{T}, X) = K_\bullet(K^1(p), \ldots, K^n(p))$$

and

$$K_p(\mathscr{T}, X) = K_\bullet(K_1(p), \ldots, K_m(p))$$

where $C(p)$ denotes the complexes C with the degree shifted with p steps, and $K_\bullet$ is the complex constructed in § 1. They are indeed essential complexes because

$$\mathscr{L}(X, K_p(\mathscr{T}, X))/\mathscr{K}(X, K_p(\mathscr{T}, X)) = K_p(\mathscr{T}, \mathscr{L}(X)/\mathscr{K}(X))$$

is a complex.

The zero component of the complex K^p or K_p is the one of degree p in Y and 1 in X.

Let $\mathscr{S} = (S_{ji})$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, be a system of operators, such that

$$[S_{ji}, S_{kh}] \in \mathscr{K}(X), \quad [S_{ji}, T_{hk}] \in \mathscr{K}(X) \tag{3.2}$$

for each i, j, k, h.

The following lemma gives a criterion in order that the essential complexes $K_p(\mathscr{T})$ be Fredholm.

LEMMA 3.1. *Assume that, with the above notations, the system $\mathscr{S}$ is a left or right inverse, modulo compacts, of $\mathscr{T}$.*

Then, for each $p \in \mathbf{Z}$, the essential complex $K^p(\mathscr{T})$, respectively $K_p(\mathscr{T})$, is Fredholm.

Proof. Suppose $\mathscr{T}\mathscr{S} = I +$ compact. Then, by Corollary 1.2, the complex $K_p(\mathscr{T}, \mathscr{L}(X)/\mathscr{K}(X))$ is exact, i.e. the essential complex $K_p(\mathscr{T})$ is Fredholm.

DEFINITION 3.2. Let $\mathscr{T}$ be a system of essentially commuting operators on a Banach space. If the essential complexes $K_p(\mathscr{T})$ or $K^p(\mathscr{T})$ are Fredholm, the *indices* of $\mathscr{T}$ are:

$$\operatorname{ind}_p\mathscr{T} = \operatorname{ind}K_p(\mathscr{T}), \quad \text{respectively}, \quad \operatorname{ind}^p\mathscr{T} = \operatorname{ind}K^p(\mathscr{T}).$$

As a consequence of the Theorem 2.5 one can prove the following

THEOREM 3.3. *Let $\mathscr{T}$ be a system of essentially commuting operators on a Banach space, such that the essential complexes $K_p(\mathscr{T})$ or $K^p(\mathscr{T})$, $p \in \mathbf{Z}$, are Fredholm.*

Then the indices $\operatorname{ind}_p\mathscr{T}$, respectively $\operatorname{ind}^p\mathscr{T}$, $p \in \mathbf{Z}$, are invariant by compact or small norm perturbations of $\mathscr{T}$, such that condition (3.1) holds.

If the system $\mathscr{T}$ has only one row, the essential complexes $K_p(\mathscr{T})$ are translations of the essential Koszul complex $K_\bullet(\mathscr{T})$, so that if one of these is Fredholm, then all are Fredholm, and

$$\operatorname{ind}_p\mathscr{T} = \operatorname{ind}K_\bullet(\mathscr{T})(-1)^p = (-1)^p\operatorname{ind}_0\mathscr{T}.$$

If the system $\mathscr{T}$ is of type (n, n), i.e. a square matrix, then $K_0(\mathscr{T})$ coincides with $K^0(\mathscr{T})$ and both with the complex

$$0 \to \bigoplus_1^n X \xrightarrow{\mathscr{T}} \bigoplus_1^n X \to 0$$

so that

$$\mathrm{ind}^0\mathscr{T} = -\mathrm{ind}_0\mathscr{T} = \text{classical index of } \mathscr{T},$$

if there exist these indices.

Also in this case if the operator $\mathscr{T}$ is Fredholm, then there is an essential bilateral inverse of $\mathscr{T}$ which satisfies (3.2) , so that all the essential complexes $K_p(\mathscr{T})$, $K^p(\mathscr{T})$ are Fredholm. A short computation shows that

$$\mathrm{ind}_{-1}\mathscr{T} = \mathrm{ind}\,\det\mathscr{T}.$$

The indices have a good behaviour by passing to duals:

THEOREM 3.4. *Let $\mathscr{T}$ be a system of essentially commuting operators on a Banach space X, such that the essential complexes $K_p(\mathscr{T})$, $p \in \mathbf{Z}$, be Fredholm.*

Then the dual system $\mathscr{T}'$ on X' has the essential complexes $K^p(\mathscr{T}')$ Fredholm and

$$\mathrm{ind}_p\mathscr{T} = \mathrm{ind}^p\mathscr{T}', \quad p \in \mathbf{Z}. \tag{3.3}$$

Proof. The essential complex $K^p(\mathscr{T}')$ is the dual of $K_p(\mathscr{T})$. Indeed, the components are dual, and the boundary operators agree by the duality

$$\langle xY_{i_1} \wedge \ldots \wedge Y_{i_h} \otimes X_{j_1} \ldots X_{j_k},\ x'Y_{i'_1} \wedge \ldots \wedge Y_{i'_h} \otimes X_{j'_1} \ldots X_{j'_k}\rangle =$$

$$= \langle x, x'\rangle \delta_{i_1 i'_1} \ldots \delta_{i_h i'_h} \delta_{j_1 j'_1} \ldots \delta_{j_k j'_k},$$

where $1 \leqslant i_1 < \ldots < i_h \leqslant m$, $1 \leqslant i'_1 < \ldots < i'_h \leqslant m$, $1 \leqslant j_1 \leqslant \ldots \leqslant j_k \leqslant n$,

$1 \leqslant j'_1 \leqslant \ldots \leqslant j'_k \leqslant n$ and h, k depends on p and q. More precisely

$$D(\mathscr{T})' = D'(\mathscr{T}'), \quad [D'(\mathscr{T})]' = D(\mathscr{T}'),$$

$$[\partial_1(\mathscr{T}) \ldots \partial_m(\mathscr{T})]' = \partial^1(\mathscr{T}') \ldots \partial^m(\mathscr{T}') + \text{compact}.$$

By Theorem 2.8 and by the choice of zero'th components of K_p and K^p we obtain (3.3).

In what follows we shall give some properties of the index of a system of type $(1, n)$ on Hilbert spaces.

THEOREM 3.5. *Let $\mathscr{T}_i$ be a commuting system with one row on the Hilbert space H_i, such that the Koszul complex $K_\cdot(\mathscr{T}_i, H_i)$ be Fredholm, $i = 1, 2$.*

Then the complex $K_\bullet(\mathcal{T}_1 \hat{\otimes} I, I \hat{\otimes} \mathcal{T}_2;\ H_1 \hat{\otimes} H_2)$ *is Fredholm and*

$$\operatorname{ind}(\mathcal{T}_1 \hat{\otimes} I, I \hat{\otimes} \mathcal{T}_2) = \operatorname{ind}\mathcal{T}_1 \cdot \operatorname{ind}\mathcal{T}_2.$$

Proof. Let $K_i = K_\bullet(\mathcal{T}_i, H_i)$. Then by the main theorem of [4], $K_1 \hat{\otimes} K_2$ is a Fredholm complex and ind $K_1 \hat{\otimes} K_2 = \operatorname{ind}K_1 \cdot \operatorname{ind}K_2$, so that

$$\operatorname{ind}(\mathcal{T}_1 \hat{\otimes} I,\ I \hat{\otimes} \mathcal{T}_2) = \operatorname{ind}K_1 \hat{\otimes} K_2 =$$

$$= \operatorname{ind}K_1 \cdot \operatorname{ind}K_2 = \operatorname{ind}\mathcal{T}_1 \cdot \operatorname{ind}\mathcal{T}_2 .$$

Another multiplicativity property which generalizes the multiplicativity of the index of Fredholm operators is the following:

THEOREM 3.6. *Let* $\mathcal{T}$ *be a one row system of essentially commuting operators on a Hilbert space* H, *and let* Q, R *be operators on* H *which essentially commute with* $\mathcal{T}$.

If the essential Koszul complexes of $(\mathcal{T}, Q)$ *and* $(\mathcal{T}, R)$ *are Fredholm, then the essential Koszul complex of* $(\mathcal{T}, RQ)$ *is also Fredholm and*

$$\operatorname{ind}(\mathcal{T}, RQ) = \operatorname{ind}(\mathcal{T}, R) + \operatorname{ind}(\mathcal{T}, Q).$$

We shall present a proof which is more natural than the initial one, and which was communicated to us by A. S. Feinstein:

Remark that the system $(\mathcal{T} \oplus \mathcal{T}, R \oplus Q)$ can be deformed in the class of Fredholm systems in $(\mathcal{T} \oplus \mathcal{T}, RQ \oplus I)$, by using the deformation from [1, Lemma 2.4.6]. Then the statement follows by the aditivity of the index.

In order to prove that the systems

$$(3.4)\qquad \left(\mathcal{T} \oplus \mathcal{T},\ \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\right)$$

are Fredholm for any $t \in \left[0, \frac{\pi}{2}\right]$, we shall use the remark that an essentially commuting system $(\mathcal{T}, S)$ is Fredholm iff the operators induced by S on the cohomology goups $H^i(\mathcal{T}, \mathcal{L}(H)/\mathcal{K}(H))$ are isomorphisms. In the concrete case of (3.4) all the matrices induce isomorphisms on the cohomology of $\mathcal{T} \oplus \mathcal{T}$ with coefficients in the Calkin algebra.

COROLLARY 3.8. *If* $\mathcal{T} = (T_1, \ldots, T_n)$ *is an essentially commuting system of operators on a Hilbert space,* $m = (m_1, \ldots, m_n) \in \mathbf{N}^n$, *and if the essential Koszul complex of* $\mathcal{T}$ *is Fredholm, then* $\mathcal{T}^m = (T_1^{m_1}, \ldots, T_n^{m_n})$ *has the same property and*

$$\operatorname{ind}\mathcal{T}^m = m_1 m_2 \ldots m_n \operatorname{ind}\mathcal{T}.$$

This answers a question raised by R. Curto in [2], where there is a deeper investigation of essentially commuting systems with one row, on Hilbert spaces.

4. A K-THEORETIC APPROACH

The indices defined above can be obtained in a natural way, in certain cases, from some elements of K^1 of a Stiefel manifold.

This part of the paper is devoted to show how the K-theory of the Stiefel manifold can be computed using the complexes defined in § 1. The computations are quite long, but elementary, at the level of introductory texts in algebraic topology.

The Stiefel manifold $V_{m,n-m}$ is the set of m-frames in $\mathbf{C}^n$, or, equivalently, the set of right invertible $m \times n$-matrices over $\mathbf{C}$, where $m \leqslant n$ are positive integers. This is an open subset of $\mathbf{C}^{m \cdot n}$ and we shall denote by ζ a point of $V_{m,n-n}$ with the matrix representation

$$\zeta = \begin{pmatrix} z_{11} & z_{12} \ldots z_{1n} \\ z_{21} & z_{22} \ldots z_{2n} \\ \ldots & \\ z_{m1} & z_{m2} \ldots z_{mn} \end{pmatrix}. \tag{4.1}$$

Let $\Lambda^{\cdot} = \Lambda^{\cdot}(n)$ be the exterior algebra with n generators and let $S^{\cdot} = S^{\cdot}(m)$ be the symmetric algebra with m generators, both over $\mathbf{C}$. We shall de note by $\Lambda^p \otimes S^q$ the trivial vector bundle over $V_{m,n-m}$, with fiber $\Lambda^p \otimes_{\mathbf{C}} S^q$. The construction (1.4) applied to the rows of ζ gives for each $p \in \mathbf{Z}$ a complex of vector bundles $\mathscr{S}^p_{m,n-m}$:

$$\ldots \to \Lambda^{p+m+1} \otimes S^1 \xrightarrow{D} \Lambda^{p+m} \otimes S^0 \xrightarrow{\partial_1 \ldots \partial_m} \Lambda^p \otimes S^0 \xrightarrow{D'} \Lambda^{p-1} \otimes S^1 \xrightarrow{D'} \ldots . \tag{4.2}$$

This is an exact complex by Corollary 1.2, hence it defines an element ν^p of $K^1(V_{m,n-m})$, [1].

The purpose of this section is to prove the following

THEOREM 4.1. *The ring $K^*(V_{m,n-m})$ is isomorphic to the exterior algebra over $\mathbf{Z}$, generated freely by any m consecutive elements in the sequence $(\nu^p)_{p \in \mathbf{Z}}$.*

It will be useful for the proof to give another interpretation of the elements ν^p, related with the realization of the Stiefel manifold as a homogenous space:

$$V_{m,n-m} \cong U(n)/U(n-m). \tag{4.3}$$

Let $\lambda^0, \ldots, \lambda^n$ be the exterior power representations of $U(n)$ and let $\mu^0, \ldots, \mu^{n-m}$ be the exterior power representations of $U(n-m)$ on the subalgebra of $\Lambda^{\cdot}(n)$ generated by the first $n-m$ indeterminates. It is known that the ring $K^*(U(n))$ is isomorphic with the algebra $\Lambda[\lambda^1, \ldots, \lambda^n; \mathbf{Z}]$, [1, Theorem 2.7.17].

Given two representations α, β of $U(n)$ on a vector space $\mathbf{C}^k$, which coincide on $U(n-m)$, then the map $\alpha \cdot \beta^{-1} : U(n)/U(n-m) \to \mathrm{GL}(k, \mathbf{C})$ is well defined and determines an element of $\mathrm{K}^1(V_{m,n-m})$. Thus we obtain a group homomorphism from the kernel I of the restriction map between the representation algebras and K^1:

$$\begin{array}{l} 0 \to I \to R(U(n)) \to R(U(n-m)) \\ \qquad\quad \downarrow \\ \qquad\; \mathrm{K}^1(V_{m,n-m}) \end{array} \qquad . \tag{4.4}$$

We shall prove that the elements $\nu^p \in \mathrm{K}^1(V_{m,n-m})$ corresponding to the exact sequences of vector bundles $\mathscr{L}^p_{m,n-m}$ can be realized by this construction.

To this end let us write $\Lambda^{\cdot}(n)$ as $\Lambda^{\cdot}(n-m) \otimes \Lambda^{\cdot}(m)$ with respect to the first $n-m$ generators on which $U(n-m)$ acts. We shall identify the subspaces $\Lambda^{\cdot}(n-m)$ from the direct sum of even, respectively odd, components of $\mathscr{L}^p_{m,n-m}$. In order to prove that the multiplicities of these subspaces agree, we shall compute the image of λ^l in $R(U(n-m))$. By knowing the restriction law for $m = 1$, derived from the Koszul complex, [1, § 2.7], we obtain by a recursive procedure:

$$\begin{aligned} \lambda^0 &= \mu^0 \\ \lambda^1 &= \mu^1 + C^1_m \mu^0 \\ \lambda^2 &= \mu^2 + C^1_m \mu^1 + C^2_m \mu^0 \\ &\dots \\ \lambda^n &= \mu^n + C^1_m \mu^{n-1} + \dots + C^n_m \mu^{n-m}, \end{aligned} \tag{4.5}$$

where $\mu^k = 0$ for $k > n-m$ and $C^b_a = a!/(b!(a-b)!)$ for a, b integers, $0 \leqslant b \leqslant a$, and zero in rest.

If we shall prove that the element of $R(U(n))$

$$\begin{aligned} \sigma^p = (\lambda^p - C^1_m \lambda^{p-1} + C^2_{m+1} \lambda^{p-2} - \dots) - \\ -(\lambda^{p+m} - C^1_m \lambda^{p+m+1} + C^2_{m+1} \lambda^{p+m+2} - \dots) \end{aligned} \tag{4.6}$$

has image zero in $R(U(n-m))$, then ν^p will be the image of σ^p by the map (4.4).

Let us compute the coefficient of μ^k, $1 \leqslant k \leqslant n-m$, in σ^p. If $k = p$, then μ^p appears only in λ^p and in λ^{p+m} and the total coefficient is zero. Suppose $p > k$, so that only the first bracket of σ^p contains μ^k, precisely $\lambda^k, \lambda^{k+1}, \dots, \lambda^{\min(k+m,p)}$ contain μ^k. The coefficient of μ^k is given by the sum:

$$C^m_m C^{p-k-m}_{p-k-1} - C^{m-1}_m C^{p-k-m+1}_{p-k} + C^{m-2}_m C^{p-k-m+2}_{p-k+1} - \dots \quad . \tag{4.7}$$

In the case $p < k$ a similar sum appears, by the symmetry of σ^p.

LEMMA 4.2. *The sum*

$$C_m^m C_{q+m}^{q+1} - C_m^{m-1} C_{q+m-1}^{q} + C_m^{m-2} C_{q+m-2}^{q-1} - \ldots + (-1)^m C_m^0 C_q^{q-m+1}$$

is zero for all positive integers m and q.

Proof. Induction on m. If $m = 1$ the equality is obvious. Suppose the statement holds true for $m-1$. Then we develop the terms as follows:

$$C_{q+m}^{q+1} = C_{q+m-1}^{q+1} + C_{q+m-2}^{q} + C_{q+m-3}^{q-1} + \ldots + C_{m-2}^{0}$$

$$C_{q+m-1}^{q} = C_{q+m-2}^{q} + C_{q+m-3}^{q-1} + \ldots + C_{m-2}^{0}$$

$$\ldots$$

so that if we regroup the terms, the sum becomes

$$C_m^m C_{q+m-1}^{q+1} + (C_m^m - C_m^{m-1}) C_{q+m-2}^{q} + (C_m^m - C_m^{m-1} + C_m^{m-2}) C_{q+m-3}^{q-1} + \ldots$$

or

$$C_{q+m-1}^{q+1} - C_{m-1}^{m-2} C_{q+m-2}^{q} + C_{m-1}^{m-3} C_{q+m-3}^{q-1} - \ldots$$

which is zero by the induction hypothesis.

Let us remark that $\sigma^p + \sigma^{p-1}$ can be computed in a similar manner and the result is $\sigma^p_{m-1,n-m+1}$. Denoting therefore by $\pi\colon V_{m,n-m} \to V_{m-1,n-m+1}$ the projection map on the first $m-1$ rows, the equality

$$\nu_m^p + \nu_m^{p-1} = \pi^*(\nu_{m-1}^p) \tag{4.8}$$

holds. The integer n, the dimension of the space in which we consider the frames, is supposed fixed and will be omitted.

Proof of Theorem 4.1. We shall use induction on m for the proof of the statement together with the assertion (otherwise redundant) that the Chern character

$$\mathrm{ch} : \mathrm{K}^*(V_m) \xrightarrow{\sim} \mathrm{H}^*(V_m, \mathbf{Z}) \tag{4.9}$$

is an isomorphism.

In the case $m = 1$ the manifold V_1 is topologically equivalent with the sphere S^{2n-1} and all the complexes are isomorphic with the Koszul complex, hence the theorem is true by [1, § 2.7].

We shall use for the proof of the induction step the Gysin sequence, as it is used in [3, VIII.12] for the computation of the cohomology of $V_{m,n-m}$.

Let M be the manifold $V_{m-1,n-m+1} \times \mathbf{C}^n$ and let us denote by N the submanifold of M consisting of the points (ζ, v) with $v \in \langle\zeta\rangle$, the linear subspace generated by ζ in $\mathbf{C}^n$. The manifolds M and N are homotopy-equivalent with V_{m-1} and $M \setminus N = V_m$.

There is the long exact sequence of K-theory associated with the pair $M\setminus N\subset M$:

$$\begin{array}{ccccc} K^0(M, M\setminus N) & \to & K^0(M) & \to & K^0(M\setminus N) \\ \uparrow & & & & \downarrow \\ K^1(M\setminus N) & \leftarrow & K^1(M) & \leftarrow & K^1(M, M\setminus N)\,. \end{array} \tag{4.10}$$

Let us remark that the space $M = V_{m-1} \times \mathbf{C}^n$ can be deformed to the normal vector bundle E of N in M. Indeed, one can define the projection $p : M \to N$, $p(\zeta, v) = (\zeta, \mathrm{pr}_{\langle\zeta\rangle} v)$, whose fibers are exactly the normal directions to N in the point $(\zeta, \mathrm{pr}_{\langle\zeta\rangle} v)$. By the Thom isomorphism in K-theory applied to E we obtain

$$K^*(M, M\setminus N) \cong K^*(N)\cdot\tau_k \tag{4.11}$$

where τ_k is the Thom class corresponding to the exterior algebra of the vector bundle p^*E on $(M, M\setminus N)$, [1].

We shall compare now the K-theory with the cohomology via the Chern character. Let us denote for every finite dimensional topological space X, $\mathscr{H}^0(X) = \oplus_q H^{2q}(X, \mathbf{Z})$ and $\mathscr{H}^1(X) = \oplus_q H^{2q+1}(X, \mathbf{Z})$. Replacing K by $\mathscr{H}$ in (4.10) we obtain also an exact sequence of cohomology with a map, the Chern character, between (4.10) and the new diagram. Then we have the Thom Isomorphism Theorem in cohomology

$$\mathscr{H}^*(M, M\setminus N) \cong \mathscr{H}^*(N)\cdot\tau_c \tag{4.12}$$

and comparing (4.11) with (4.12) we obtain by the induction assumption that $\mathrm{ch}(\tau_k) = \tau_c$ and the isomorphism (4.9).

But the image of τ_c in $\mathscr{H}^\cdot(M)$ is the Euler class χ_N^M, which is an element of $H^{2(n-m+1)}(N) = H^{2(n-m+1)}(V_{m-1}) = 0$, [3, Proposition VIII.12.10], therefore we have the equality of $K^*(N)$-modules

$$K^*(M\setminus N) = K^*(N) \oplus K^*(N)\cdot\tau_k\,. \tag{4.13}$$

We proceed now to the proof of the statement of the theorem. Let $p \in \mathbf{Z}$ and let us denote by $\mathbf{Z}[v_m^{p+1}, \ldots, v_m^{p+m}]$ the subalgebra of $K^*(M\setminus N)$ generated by the elements in brackets. Using (4.8) it follows that this algebra contains $v_{m-1}^{p+2}, \ldots, v_{m-1}^{p+m}$, and hence the sequence $(v_{m-1}^q)_{q\in\mathbf{Z}}$, by the induction hypothesis. Using again (4.8) one can shift the interval $(v_m^{p+1}, \ldots, v_m^{p+m})$ to the left or to the right with an arbitrary step, so that $\mathbf{Z}[v_m^{p+1}, \ldots, v_m^{p+m}] = \mathbf{Z}[v_m^{n-m+1}, \ldots, v_m^n]$.

If we shall prove the equality

$$v_m^{n-m+1} = \pm\tau_k\,, \tag{4.14}$$

then the proof will be complete. Indeed, assuming (4.14), the elements

$(v_m^{n-m+1}, \ldots, v_m^n)$ can be transformed by a Z-linear map in $(\tau_k, v_{m-1}^{n-m+2}, \ldots, v_{m-1}^n)$, hence the originl m-tuple is free over $\mathbf{Z}$ and generates, by the induction assumption and by (4.13), the ring $K^*(V_m)$.

The equality (4.14) means that the image of v_m^{n-m+1} by the map $K^1(M\setminus N) \to K^0(M, M\setminus N)$, is the Thom class τ_k. To compute this image means to take the Euler characteristic of the extension of the complex $\mathscr{L}_{m,n-m}^{n-m+1}$ to M. The extension is obvious, considering the expressions of the coboundary operators. We shall prove that the direct images of the components of $\mathscr{L}_{m,n-m}^{n-m+1}$ on the Thom space $M/(M\setminus N)$ are still vector bundles, so that, the Euler characteristic will be the alternating sum of these bundles in $K^0(M/(M\setminus N))$.

A component of the complex $\mathscr{L}_{m,n-m}^{n-m+1}$ is of the form $\Lambda^p(\mathbf{C}^n)\otimes S^q$, where $\mathbf{C}^n$ is the trivial vector bundle of rank n. Deonting by $\langle\zeta\rangle$ the subbundle of $\mathbf{C}^n$ which has the fiber over $(\zeta, v)\in M = V_{m-1}\times \mathbf{C}^n$, equal with the space generated by ζ in $\mathbf{C}^n$, we have the decomposition $\mathbf{C}^n = \langle\zeta\rangle + \langle\zeta\rangle^\perp$. Let us remark that $\langle\zeta\rangle$ is a trivial vector bundle and that $\langle\zeta\rangle^\perp$ coincides with the pull-back of the "normal" vector bundle E of N in M. Thus the image of $\mathbf{C}^n$ on $M/(M\setminus N)$ is the direct sum $\mathbf{C}^{m-1}\oplus q^*(E)$, where $q : M/(M\setminus N) \to N$ is the natural projection map.

Let us denote by α^i the class of $\Lambda^i(\mathbf{C}^{m-1}\oplus q^*E)$ in $K^0(M/(M\setminus N))$ and similarly by β^j the class of $\Lambda^j(q^*E)$. The image of v_m^{n-m+1} in this group will be the sum:

$$\alpha^{n-m+1} - C_m^1\alpha^{n-m} + C_{m+1}^2\alpha^{n-m-1} - C_{m+2}^3\alpha^{n-m-2} + \ldots \quad . \tag{4.15}$$

Replacing α^i by the corresponding expressions in β^j, the coefficient of β^k, $k\in\mathbf{Z}$, in (4.15) is:

$$(-1)^p(C_{m+p-1}^p - C_{m+p}^{p+1}C_{m-1}^1 + C_{m+p+1}^{p+2}C_{m-1}^2 - \ldots)$$

where $p = n - m + 1 - k$. The last sum is equal to

$$(-1)^p(C_{m+p-2}^p - C_{m+p-1}^{p+1}C_{m-1}^1 + C_{m+p}^{p+2}C_{m-1}^2 - \ldots) +$$

$$+ (-1)^p(C_{m+p-2}^{p-1} - C_{m+p-1}^{p}C_{m-1}^1 + C_{m+p}^{p+1}C_{m-1}^2 - \ldots) .$$

But the first sum is zero by Lemma 4.2, so that, by a recursive procedure, the coefficient of β^k in (4.15) is $(-1)^{n+k}$. Finally, the image of v^{n-m+1} is $(-1)^n\sum(-1)^k\beta^k = (-1)^n\tau_k$.

A consequence of the proof is the following

COROLLARY 4.3. *The* Z*-module generated in* $K^1(V_m)$ *by every* m *consecutive elements of the sequence* $(v_m^p)_{p\in\mathbf{Z}}$ *coincides with the module generated by the whole sequence.*

One can prove with a similar technique also

PROPOSITION 4.4. *Denoting by* $i : V_{m,n-m} \to V_{m,n+1-m}$ *the natural inclusion map, the relations*

$$i^* v^p_{m,n+1-m} = v^p_{m,n-m} + v^p_{m,n-m}, \quad p \in \mathbf{Z},$$

holds.

Let us finally compute all the sequence $(v^p_m)_{p\in\mathbf{Z}}$ in terms of $v^0_m, v^0_{m-1}, \ldots, v^0_1$, by forgetting the projection maps $(v^0_{m-1} = \pi^*(v^0_{m-1}), \ldots)$. By taking (4.8) into account, a recurent computation produces the formula:

$$(4.16) \qquad v^p_m = (-1)^p (C^{p-1}_{m+p-2} v^0_1 + C^{p-1}_{m+p-3} v^0_2 + \ldots + C^{p-1}_{p-1} v^0_m)$$

for $p \geqslant 0$ and

$$(4.17) \qquad v^p_m = (-1)^p (C^0_p v^0_m - C^1_p v^0_{m-1} + C^2_p v^0_{m-2} - \ldots)$$

for $p \leqslant 0$.

For example in the case $m = 1$, $v^p_1 = (-1)^p v^0_1$, and in the case $m = 2$, $v^p_2 = (-1)^p (v^0_2 + |p| v^0_1)$.

COROLLARY 4.5. *The ring* $\mathrm{K}^*(V_m)$ *is isomorphic with the exterior algebra over* $\mathbf{Z}$, *freely generated by* $v^0_m, v^0_{m-1}, \ldots, v^0_1$.

Returning to matrices of operators, let E be a Banach space and let A be an essentially commuting subalgebra of $\mathscr{L}(E)$. Then the unital Banach algebra $B = A/\mathscr{K} \cap A$ is commutative and, denoting by X its maximal spectrum, $\mathrm{K}^1(X)$ is isomorphic with $\mathrm{K}_1(B)$ via the Gelfand transformation $\mathscr{G} : B \to \mathscr{C}(X)$, by a theorem due to Novodvorskii, [15, Theorem 7.5].

Let $\mathscr{T}$ be a $m \times n$-matrix with elements in A, invertible on the right in B, and denote by $\pi(\mathscr{T})$ the image of $\mathscr{T}$ in B. The system $\pi(\mathscr{T})$ determines by its Gelfand transform a map

$$\widehat{\pi(\mathscr{T})} : X \to V_{m,n-m}.$$

The essential complex $K_p(\mathscr{T}, E)$ is Fredholm by Lemma 3.1 and a standard computation shows that

$$\mathrm{ind}_p \mathscr{T} = \mathrm{index}(\mathscr{G}^{-1}_* \circ \widehat{\pi(\mathscr{T})}^*(v^{p+1}_m)).$$

Thus one can prove the following result, by using the universal properties of $v^{p,s}_m$:

PROPOSITION 4.6. a). *With the above notations, each* m *consecutive elements of the sequence* $(\mathrm{ind}_p \mathscr{T})_{p\in\mathbf{Z}}$ *determine all the sequence.*

b). *If we add a column with elements in* A *to* $\mathscr{T}$, *then the new system* $\mathscr{T}'$ *is still right invertible in* B *and*

$$\mathrm{ind}_p \mathscr{T}' = \mathrm{ind}_p \mathscr{T} + \mathrm{ind}_{p-1} \mathscr{T}, \quad p \in \mathbf{Z}.$$

c). *If we delete a row from $\mathcal{T}$, then the new system $\mathcal{T}''$ is still right invertible in B and*

$$\mathrm{ind}_p\mathcal{T}'' = \mathrm{ind}_p\mathcal{T} + \mathrm{ind}_{p-1}\mathcal{T}, \quad p \in \mathbf{Z}.$$

COROLLARY 4.7. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are subsystems of $\mathcal{T}$ obtained by deleting the same number of rows from $\mathcal{T}$, then*

$$\mathrm{ind}_p(\mathcal{T}_1) = \mathrm{ind}_p(\mathcal{T}_2), \quad p \in \mathbf{Z}.$$

The relations (4.16) and (4.17) give the explicit expressions of $\mathrm{ind}_p\mathcal{T}$ in terms of ind_0 of the subsystems of $\mathcal{T}$.

Let us remark finally that for a square matrix $\mathcal{T}$, Markus and Feldman have proved that

$$\mathrm{ind}_0\mathcal{T} + \mathrm{ind}_{-1}\mathcal{T} = 0$$

if the commutators of the elements of $\mathcal{T}$ are trace class, [12].

5. APPLICATIONS

The main part of this section is devoted to pseudodifferential operators. First some notations and terminology:

Let M be a compact manifold and let $L^s_{\rho,\delta}(M)$ be the set of pseudodifferential operators on M, of order $s \in \mathbf{R}$ and of type (ρ, δ), $0 \leqslant \delta < \rho \leqslant 1 - \delta$, in the sense of Hörmander (see for example [9]). An elliptic on the right system of pseudodifferential operators (in the sense of Douglis-Nirenberg) will be a matrix $\mathcal{P} = (P_{ij})$ with

$$P_{ij} \in L^{t_j - s_i}(M) \quad \text{for } 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n \tag{5.1}$$

and some real numbers t_j, s_i, such that for each $(x, \xi) \in T^*M$, the matrix of principal symbols

$$\sigma_0(\mathcal{P})(x, \xi) = (\sigma_{t_j - s_i}(P_{ij})(x, \xi))$$

is invertible on the right, for ξ sufficiently large.

For such an elliptic on the right system $\mathcal{P}$ one can construct, with some modifications, essential complexes like $K_p(*)$, as follows:

Let σ be a real number and let us denote by $H^s(M)$ the Sobolev spaces of M. We shall use the indeterminates X and Y of first section only for simplifications of the boundaries formulas. The essential complex K^σ_{p-1}, $(\mathcal{P})$ will be:

$$\ldots K^\sigma_{p+3} \xrightarrow{D} K^\sigma_{p+2} \xrightarrow{D} K^\sigma_{p+1} \xrightarrow{\partial_1 \ldots \partial_m} K^p_\sigma \xrightarrow{D'} K^\sigma_{p-1} \xrightarrow{D'} \ldots \tag{5.2}$$

where we omit the first index $p-1$ and

$$K^{\sigma}_{p\ q} = \oplus H^{\tau(\sigma,j,i,\alpha)}(M) \otimes Y_{j_1} \wedge \ldots \wedge Y_{j_{p-q}} \otimes X_{i_1}^{\alpha_{i_1}} \ldots X_{i_h}^{\alpha_{i_h}}$$

summing by the indices: $1 \leqslant j_1 < \ldots < j_{p-q} \leqslant n$, $1 \leqslant i_1 < \ldots < i_h \leqslant m$, $\alpha_{i_1} + \ldots + \alpha_{i_h} = q$, all $\alpha_{i_k} > 0$ and the order $\tau(\sigma, j, i, \alpha) = \sigma + t_{j_1} + \ldots + t_{j_{p-q}} + \alpha_{i_1} s_{i_1} + \ldots + \alpha_{i_h} s_{i_h}$, all this for $0 \leqslant q \leqslant p$; moreover

$$K^{\sigma}_{p-q} = 0 \quad \text{for } q > p.$$

$$K^{\sigma}_{p+q} = \oplus H^{\tau(\sigma,j,i,\alpha)}(M) \otimes Y_{j_1} \wedge \ldots \wedge Y_{j_{p+m+q-1}} \otimes X_{i_1}^{\alpha_{i_1}} \ldots X_{i_h}^{\alpha_{i_h}},$$

summing by: $1 \leqslant j_1 < \ldots < j_{p+m+q-1} \leqslant n$, $1 \leqslant i_1 < \ldots < i_h \leqslant m$, $\alpha_{i_1} + \ldots + \alpha_{i_h} = q-1$, all $\alpha_{i_k} > 0$ and $\tau(\sigma, j, i, \alpha) = \sigma + t_{j_1} + \ldots + t_{j_{p+m+q-1}} - (s_1 + \ldots + s_m) - (\alpha_{i_1} s_{i_1} + \ldots + \alpha_{i_h} s_{i_h})$, for $q \geqslant 1$.

The corresponding boundaries are as in the first section:

$$D(xY_{j_1} \wedge \ldots \wedge Y_{j_k} \otimes X_{i_1}^{\alpha_{i_1}} \ldots X_{i_h}^{\alpha_{i_h}}) =$$

$$= \sum_{s=1}^{h} \sum_{t=1}^{k} (-1)^t P_{i_s j_t} x Y_{j_1} \wedge \ldots \wedge \hat{Y}_{j_t} \wedge \ldots \wedge Y_{j_k} \otimes X_{i_1}^{\alpha_{i_1}} \ldots X_{i_s}^{\alpha_{i_s}} \ldots X_{i_h}^{\alpha_{i_h}};$$

$$D'(xY_{j_1} \wedge \ldots \wedge Y_{j_k} \otimes X_{i_1}^{\alpha_{i_1}} \ldots X_{i_h}^{\alpha_{i_h}}) =$$

$$= \sum_{s=1}^{m} \sum_{t=1}^{k} (-1)^t P_{s j_t} x Y_{j_1} \wedge \ldots \wedge \hat{Y}_{j_t} \wedge \ldots \wedge Y_{j_k} \otimes X_s X_{i_1}^{\alpha_{i_1}} \ldots X_{i_h}^{\alpha_{i_h}}$$

and similarly $\partial_1 \ldots \partial_m$.

Using the compactness of the commutator

$$[P_1, P_2] : H^{\sigma+s_1+s_2}(M) \to H^{\sigma}(M),$$

where $P_i \in L^{s_i}(M)$, $i=1, 2$, one can prove that $K^{\sigma}_{p-1,\cdot}(\mathscr{P})$ is an essential complex of Hilbert spaces.

One can prove that there exists for each $s \in \mathbf{R}$, operators $A_s \in L^s(M)$ such that the principal symbol $\sigma_0(A_s)$ would be invertible, real and the extensions $A_s : H^{\sigma}(M) \to H^{\sigma-s}(M)$ would be isomorphisms. Moreover, the inverse A_s^{-1} is a pseudodifferential operator, modulo compacts.

Then the elliptic on the right system $\mathscr{P}$ which satisfies (5.1) can be written like

$$\mathscr{P} = \begin{pmatrix} A_{-s_1} & & 0 \\ & \ddots & \\ 0 & & A_{-s_m} \end{pmatrix} \mathscr{P}_0 \begin{pmatrix} A_{t_1} & & 0 \\ & \ddots & \\ 0 & & A_{t_n} \end{pmatrix}$$

where $\mathscr{P}_0$ is also elliptic on the right and satisfies (5.1) and the ellipticity condition with $t_j = s_i = 0$. With the operators A_s one can define an essential isomorphism in the sense of Corollary 2.7 between $K_p^\sigma(\mathscr{P})$ and $K_p^\sigma(\mathscr{P}_0)$, so that we can suppose that the ellipticity condition holds for all $t_j = s_i = 0$. But in this case $K_p^\sigma(\mathscr{P}_0)$ coincides with $K_p^\sigma(\mathscr{P}_0, H^\sigma(M))$, because all the operators P_{ij}^0 have order zero.

THEOREM 5.1. *Let $\mathscr{P}$ be an elliptic on the right (left) system of pseudodifferential operators on a compact manifold M.*

Then the essential complexes $K_{p\bullet}^\sigma(\mathscr{P})$, respectively $K_\sigma^{p\bullet}(\mathscr{P})$, are Fredholm for all $\sigma \in \mathbf{R}$ and $p \in \mathbf{Z}$, and $\operatorname{ind} K_p^\sigma(\mathscr{P})$, *respectively* $\operatorname{ind} K_\sigma^{p\bullet}(\mathscr{P})$, *does not depend on σ.*

Proof. Assume that the components of $\mathscr{P}$ have order zero. There is by Theorem 6.3.7 of [9] a right parametrix $\mathscr{E}$ of $\mathscr{P}$, also of order zero. If σ is a real number, the extensions of $\mathscr{P}$ and $\mathscr{E}$ to $H^\sigma(M)$ give a system of essentially commuting operators $\mathscr{P}^\sigma$ with a right essential inverse $\mathscr{E}^\sigma$, such that all the components of $\mathscr{P}^\sigma, \mathscr{E}^\sigma$ commute modulo compacts. Then by Lemma 3.1 the essential complex $K_{p\bullet}^\sigma(\mathscr{P})$ is Fredholm.

The independence of the index of σ results by Corollary 2.7. The left invertible case is dual.

Let $\mathscr{P}$ be a system like in Theorem 5.1. Then the indices $\operatorname{ind}_p(\mathscr{P}) = \operatorname{ind} K_p^\sigma(\mathscr{P})$ for some $\sigma \in \mathbf{R}, p \in \mathbf{Z}$, depend only on the principal symbol of $\mathscr{P}$ and they are locally constant in the space of right elliptic symbols. Moreover, if we work only with homogenous symbols, they are also topological invariants.

For example let φ be a smooth function on $\mathbf{R}$, such that $\varphi(\xi) = 1$ for $|\xi| \geqslant 2$ and $\varphi(\xi) = 0$ for $|\xi| \leqslant 1$. The function

$$\sigma : T^*(S^1) \to \mathbf{C},$$

$$\sigma(\xi \mathrm{d}z) = \begin{cases} z\varphi(\xi) & \text{if } \xi \geqslant 0, \\ \varphi(\xi) & \text{if } \xi < 0, \end{cases}$$

is an elliptic symbol of order zero on the sphere S^1. A pseudodifferential operator on S^1 with this symbol is the following [13, XVI.6.2]:

$$P : L^2(S^1) \to L^2(S^1)$$

$$P(z^k) = \begin{cases} z^{k+1} & \text{for } k \geqslant 0, \\ z^k & \text{for } k < 0, \end{cases}$$

where (z^k) is the natural basis of $L^2(S^1)$. The index of P is -1.

Let now σ_k be the symbol $1 \otimes \ldots \otimes^{k\downarrow}\sigma \otimes \ldots \otimes 1$ on the n-dimensional torus $T^n = S^1 \times \ldots \times S^1$, and let $\mathscr{P} = (P_1, \ldots, P_n)$ be an essential commuting system of pseudodifferential operators associated to the symbol $(\sigma_1, \ldots, \sigma_n)$. Then $\mathscr{P}$ is an elliptic to the right system, and by Corollary 3.8

$$\operatorname{ind}\mathscr{P}^m = (-1)^n m_1 m_2 \ldots m_n ,$$

for each $m \in \mathbf{N}^n$.

Another class of operators with compact commutators are Toeplitz operators. One can define the invariants $\operatorname{ind}_p\mathscr{T}$ for each system $\mathscr{T}$ of Toeplitz operators on a manifold M, with right invertible matrix symbol Φ at all points of M. Indeed, if this condition holds, then $\Phi^*(\Phi\Phi^*)^{-1}$ is a right inverse of Φ in the set of continuous matrix symbols on M.

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