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A GENERALIZATION OF KOOSIS-LAX INTERIOR COMPACTNESS THEOREM

S. V. HRUŠČEV and A. L. VOLBERG

1. INTRODUCTION

It follows from von Neumann spectral multiplicity theorem that, given an increasing continuous chain $(G_t)_{t\geqslant 0}$ of closed subspaces of a Hilbert space G, there exists a direct integral of Hilbert spaces

$$H = \int_0^{+\infty} \oplus\, H_t \,\mathrm{d}m(t)$$

and a unitary operator $\mathscr{F}: G \to H$ such that

$$\mathscr{F}G_t = \mathbf{1}_{[0,t]}H \overset{\text{def}}{=} \{\mathbf{1}_{[0,t]}f : f \in H\}.$$

Here $\mathbf{1}_{[0,t]}$ stands for the projection of H onto the direct integral $\int_0^t \oplus\, H_s \mathrm{d}m(s)$ and m denotes a positive measure. In the present paper we treat a special family of chains in the Hardy class H^2. Namely, let θ be an inner function corresponding to a positive singular Borel measure μ on the unit circle $\mathbf{T} = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$,

$$\theta(z) \overset{\text{def}}{=} \exp\left\{-\int_{\mathbf{T}} \frac{\zeta + z}{\zeta - z}\,\mathrm{d}\mu(\zeta)\right\},$$

and let $G_t = K_{\theta^t} \overset{\text{def}}{=} H^2 \ominus \theta^t H^2$, $t \geqslant 0$. It is clear that the family $(G_t)_{t\geqslant 0}$ forms a continuous chain in the following sense:

1) $G_s \subset G_t$, $s \leqslant t$;

2) $G_0 = \{0\}$ and $\bigcup\limits_{t\geqslant 0} G_t$ is dense in H^2;

3) for each $s \geqslant 0$, $\bigcup\limits_{t<s} G_t$ is dense in G_s and $\bigcap\limits_{t>s} G_t = G_s$.

The explicit formulae for the direct integral H and for $\mathscr{F}$ proved in [7] give immediately that

$$H = \int_0^\infty \oplus L^2(\mu)\,dt$$

and that the values of the operator $\mathscr{F}$ on the rational fractions $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$, $|\lambda| < 1$, which span H^2, are given by

$$\mathscr{F}k_\lambda = \sqrt{2}\,\bar{\theta}'(\lambda)\,(1-\bar{\lambda}z)^{-1}.$$

We shall study the interior compactness property for the chains $(K_{\theta^t})_{t>0}$. To be more precise we introduce a semigroup of translation operators $\tau_h: H \to H$,

$$\tau_h f(t,\zeta) \overset{\text{def}}{=} f(t+h,\zeta), \quad h \geqslant 0$$

for $f \in H$. Every open interval J of the half-line $\mathbf{R}_+ = (0, +\infty)$ defines a semi-norm

$$\|f\|_J = \left\{\int_J \|f(t,\cdot)\|^2_{L^2(\mu)}\,dt\right\}^{1/2}.$$

A subspace E in H is called *translation invariant* if $\tau_h E \subset E$ for every h, $h \geqslant 0$. We call such a space E interior compact (abbreviated TIIC-space) if the set $\{f \in E : \|f\|_J \leqslant 1\}$ is precompact with respect to the semi-norm $\|\cdot\|_I$ whenever $\operatorname{clos} I \subset J$. This definition is due to P. D. Lax [8]. In [9], P. D. Lax proved the following remarkable theorem.

THEOREM. (P. D. Lax) *Let E be a translation invariant subspace of H. Then E is interior compact iff $\tau_h: E \to E$ is a compact operator for all h.*

Now the problem of describing the subspaces E satisfying the interior compactness property arises in a natural way. P. Koosis [6] and P. D. Lax [10] gave such a description for the case when μ is a one point measure. We state this theorem in the form which is convenient for us.

Let $L^2(\mathbf{R}_+)$ denote the Hilbert space of all square-summable functions on the half-axis $\mathbf{R}_+ = (0, +\infty)$ and let $\omega(z) = \frac{z-i}{z+i}$ be the conformal mapping of the upper half plane onto the unit disc $\mathbf{D}$. The Blaschke product in the upper half plane $\mathbf{C}_+$ with the zero-sequence $(\lambda_n)_{n\geqslant 1}$ is called a Koosis function if

(K) $$\lim_{n\to\infty} \operatorname{Im}\lambda_n = \infty, \quad \lim_{|x|\to\infty} \sum_{n\geqslant 1} \frac{\operatorname{Im}\lambda_n}{|\lambda_n - x|^2} = 0.$$

Let $\mathscr{K}$ stand for the set of all the Koosis functions.

THEOREM. *The closed subspace E of $L^2(\mathbf{R}_+)$ is a* TIIC-*space iff there is a Blaschke product* $B = \prod_{n\geq 1} \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z}$ *in* $\mathbf{D}$ *satisfying*

$$B \circ \omega \in \mathscr{K}, \quad E = \mathscr{F}(H^2 \ominus BH^2).$$

In this paper we shall give an analogous description for some TIIC-spaces in the direct integral $H = \int_0^\infty \oplus L^2(\mu) dt$ with an arbitrary singular measure μ. Namely, we shall suppose from the very beginning that the space E is of the form $E = \mathscr{F} K_B$. Here B is some inner function in the unit disc $\mathbf{D}$ and $K_B \overset{\text{def}}{=} H^2 \ominus BH^2$. Such an E is automatically translation invariant. It should be noted that in the vector valued case (i.e. μ is not one-point measure) not all the TIIC-spaces are the images of K_B-spaces (see [9]).

By Fatou's lemma the algebra H^∞ of bounded analytic functions on $\mathbf{D}$ may be considered as a closed subalgebra of $L^\infty(\mathbf{T})$, namely, the algebra of essentially bounded Lebesgue measurable functions on $\mathbf{T}$. For g in L^∞ let $H^\infty[g]$ denote the uniform subalgebra of L^∞ generated by H^∞ and the function g. These algebras play an important role in the spectral theory of Toeplitz operators. The most interesting example is perhaps the algebra $H^\infty[\bar{z}]$. It is well-known that $H^\infty[\bar{z}] = H^\infty + C$, where C denotes the space of all continuous functions on $\mathbf{T}$. If A is a Douglas algebra (i.e. a uniform algebra such that $H^\infty \subset A \subset L^\infty$) then $\mathscr{M}(A)$ denotes the maximal ideal space of A. It is a deep result due to Chang and Marshall [2], [11] that every Douglas algebra A is generated by H^∞ and by the inverses of all inner functions invertible in A. Sarason's lectures [14] are the best introduction to the subject. We mention only two facts here:

a) $\mathscr{M}(H^\infty + C) = \mathscr{M}(H^\infty)\setminus\mathbf{D}$;

b) if $\{\theta_\lambda\}_{\lambda\in\Lambda}$ is a family of inner functions then

$$\mathscr{M}(H^\infty[\{\bar{\theta}_\lambda\}_{\lambda\in\Lambda}]) = \{\varphi \in \mathscr{M}(H^\infty) : |\hat{\theta}_\lambda(\varphi)| = 1, \quad \forall \lambda \in \Lambda\}.$$

DEFINITION. If $E = \mathscr{F} K_B$ is a TIIC-space in the direct integral $H = \int_0^\infty \oplus L^2(\mu) dt$, where μ is a singular measure on $\mathbf{T}$, and

$$\theta(z) = \exp\left\{-\int_{\mathbf{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\right\}$$

then the pair of inner functions (B, θ) is called a *Koosis pair.*

Now we are in a position to formulate our main result.

THEOREM. *The following statements are equivalent:*

a) (B, θ) *is a Koosis pair;*

b) $\mathcal{M} \stackrel{\text{def}}{=} \mathcal{M}(H^\infty + C) = \{\varphi \in \mathcal{M}(H^\infty) : \hat{\theta}(\varphi) = 0\} \cup \{\varphi \in \mathcal{M}(H^\infty) : |\hat{B}(\varphi)| = 1\}$;

c) $\mathcal{M} = \{\varphi : \hat{\theta}(\varphi) = 0\} \cup \{\varphi : |\hat{B}(\varphi)| \geqslant \eta\}$ *for some* $\eta > 0$.

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2. PROOF OF THE THEOREM

The proof of the theorem is based on the following two lemmas.

LEMMA 1. (L. Carleson [1]) *There is a function* $\sigma(\varepsilon) > 0$ *defined for* ε *positive and sufficiently small,* $\lim_{\varepsilon\to 0} \sigma(\varepsilon) = 0$, *such that for every function* $h \in H^\infty$, $\|h\|_\infty = 1$, *there is a system of closed rectifiable curves* Γ_i *in* clos**D**, *with disjoint interiors, having the following properties:*

1) $\{z \in \mathbf{D} : |h(z)| < \varepsilon\} \subset \bigcup_{i=1}^{\infty} \operatorname{int}(\Gamma_i)$,

2) *if* $z \in \Gamma \cap \mathbf{D}$ *then* $\varepsilon \leqslant |h(z)| \leqslant \sigma(\varepsilon)$,

3) *arc-length measure on* $\Gamma \cap \mathbf{D}$ *is a Carleson measure with Carleson constant* $c(\varepsilon)$ *which depends only on* ε.

LEMMA 2. (S.-Y. Chang, J. Garnett [3]) *Let* B *be an inner function,* $\varepsilon > 0$, *and* Γ *is the contour from Lemma* 1 *constructed for the function* B *and the number* ε. *Then for every function* f *in the Hardy class* H^1 *we have*

$$\int_{\mathbf{T}} \bar{B} f \, dz = \int_{\Gamma} \frac{f}{B} \, dz.$$

LEMMA 3. *If there is a number* $\eta > 0$ *such that*

$$\mathcal{M} = \{\varphi : \hat{\theta}(\varphi) = 0\} \cup \{\varphi : |\hat{B}(\varphi)| \geqslant \eta\}$$

then $\bar{B}\theta \in H^\infty + C$.

Proof. It is sufficient to prove that

$$\lim_{N\to\infty} \operatorname{dist}(z^N \bar{B}\theta, H^\infty) = 0.$$

Now let ε be so small that $\sigma(\varepsilon) \leqslant \frac{\eta}{2}$. Let Γ be the contour from Lemma 1 for B and ε. It is clear that

$$\lim_{z\in\Gamma, |z|\to 1} |\theta(z)| = 0.$$

Indeed, if it is not the case, then there is a sequence $(z_n)_{n\geqslant 1}$, $z_n \in \Gamma$, $|z_n| \to 1$ such that $|\theta(z_n)| \geqslant \delta > 0$ and $|B(z_n)| \leqslant \frac{\eta}{2}$. Let $\varphi \in \mathcal{M}$ be a weak-star limit point for the sequence $(z_n)_{n\geqslant 1}$. Then $\hat{\theta}(\varphi) \neq 0$, $|\hat{B}(\varphi)| \leqslant \frac{\eta}{2}$ and this contradicts to the statement of the lemma.

By the standard duality argument we have

$$\begin{aligned}\operatorname{dist}(z^N\bar{B}\theta, H^\infty) &= \frac{1}{2\pi} \sup_{f\in H^1, \|f\|\leqslant 1} \left\{\left|\int_{\mathbf{T}} z^N \bar{B}\theta f\, dz\right|\right\} = \\ &= \frac{1}{2\pi} \sup_{f\in H^1, \|f\|\leqslant 1} \left|\int_\Gamma \frac{\theta f z^N}{B}\, dz\right|.\end{aligned} \tag{1}$$

It follows from the definition of Γ that

$$\varepsilon \leqslant \inf\{|B(z)| : z \in \Gamma\},$$

and therefore

$$\left|\int_\Gamma \frac{\theta f z^N}{B}\right| \leqslant \int_\Gamma \frac{|\theta|\cdot|f|\,|z^N|}{B}\,|dz| \leqslant \frac{1}{\varepsilon}\int_\Gamma |\theta|\cdot|f|\cdot|z|^N|dz|.$$

Let δ be a small positive number. There is a number $R < 1$ such that $|\theta(z)| \leqslant \delta\varepsilon(2c(\varepsilon))^{-1}$ for $z \in \Gamma$, $|z| \geqslant R$. If a number N is so large that $R^N \leqslant \delta\varepsilon(2c(\varepsilon))^{-1}$, then

$$\frac{1}{\varepsilon}\int_\Gamma |\theta|\cdot|f|\cdot|z|^N|dz| = \frac{1}{\varepsilon}\int_{\Gamma\cap\{|z|>R\}} + \frac{1}{\varepsilon}\int_{\Gamma\cap\{|z|\leqslant R\}} \leqslant \delta.$$

The last inequality implies that $\lim_{N\to\infty} \operatorname{dist}(z^N\bar{B}\theta, H^\infty) \leqslant \delta$ for every $\delta > 0$ (see (1)). ▨

To state Lemma 4 we introduce some additional notation. Let $\mathbf{P}_+$ be the orthogonal projection of L^2 onto H^2 and let $\mathbf{P}_- = I - \mathbf{P}_+$, I being the identity operator. For φ in $L^\infty(\mathbf{T})$, the Toeplitz operator with the symbol φ is the operator T_φ on H^2 defined by $T_\varphi h = \mathbf{P}_+\varphi h$ and the Hankel operator with the same symbol is defined by the formula $H_\varphi h = \mathbf{P}_-\varphi h$, $h \in H^2$. Let P_B be the orthogonal projection of H^2 onto $K_B = H^2 \ominus BH^2$ and let $\mathbf{T}_B$ stand for the Nagy-Foiaş operator: $\mathbf{T}_B h \overset{\text{def}}{=} P_B zh$, $h \in K_B$. It is well-known that $(T_{\bar\theta} | K_B)^* = \theta(\mathbf{T}_B)$ for every θ in H^∞. By N. K. Nikolskii's formula (see [11])

$$\theta(\mathbf{T}_B) \oplus \mathbf{0}_{BH^2} = BH_{\bar B\theta}. \tag{2}$$

LEMMA 4. *Let b be an interpolating Blaschke product and let θ be an inner function. Then $\bar b\theta \in H^\infty + C$ iff $\lim_{n\to\infty} \theta(z_n) = 0$.*

Proof. By the Hartman theorem [14] $\bar b\theta \in H^\infty + C$ iff $H_{\bar b\theta}$ is a compact operator. The facts mentioned above give now that $H_{\bar b\theta}$ is compact iff $T_{\bar\theta} | K_B$ is compact. But the vectors

$$k_{z_n} \overset{\text{def}}{=} \frac{(1-|z_n|^2)^{1/2}}{1-\bar z_n e^{i\varphi}}$$

form a Riesz basis in the space K_b (see [12] for the proof; we use here the fact that b is an interpolating Blaschke product). Now it is obvious that $T_{\bar\theta} | K_B$ is compact iff $\|T_{\bar\theta} k_{z_n}\|_2 \to 0$. But $T_{\bar\theta} k_{z_n} = \overline{\theta(z_n)} k_{z_n}$ and therefore $\|T_{\bar\theta} k_{z_n}\|_2 = |\theta(z_n)|$. ▨

PROPOSITION 5. *Let A be a Douglas algebra and let θ be an inner function. Then $A\cdot\bar\theta \subset H^\infty + C$ iff*

$$\mathcal{M} = \{\varphi : \hat\theta(\varphi) = 0\} \cup \mathcal{M}(A).$$

Proof. Let $\mathcal{M} = \{\varphi : \hat\theta(\varphi) = 0\} \cup \mathcal{M}(A)$. It is clear that $\mathcal{M}(A) \subset \{\varphi : |\hat b(\varphi)| = 1\}$ for every inner function b invertible in A (i.e. $\bar b \in A$) and therefore

$$\mathcal{M} = \{\varphi : \hat\theta(\varphi) = 0\} \cup \{\varphi : |\hat b(\varphi)| = 1\} \tag{3}$$

if $\bar b \in A$. By Lemma 3 $\bar b\theta \in H^\infty + C$. By the Chang-Marshall theorem ([2], [11]) every function f in A has a norm-converged series expansion

$$f = \sum_{n\geq 1} \bar b_n \cdot h_n,$$

where $h_n \in H^\infty$, and b_n is an invertible inner function in A. It follows that $f\bar\theta = \sum_{n\geq 1} \bar b_n(\bar b_n\bar\theta) \in H^\infty + C$ (see Lemma 3 and the formula (3)). Therefore $A\cdot\bar\theta \subset H^\infty + C$.

Suppose now that $A\cdot\bar\theta \subset H^\infty + C$ for some inner function θ and that there is a homomorphism φ in $\mathcal{M}$ satisfying $\hat\theta(\varphi) \neq 0$, $\varphi \notin \mathcal{M}(A)$. There is, clearly, a closed neighbourhood $V(\varphi)$ of φ such that

$$V(\varphi) \cap (\mathcal{M}(A) \cup \{\varphi : \hat\theta(\varphi) = 0\}) = \emptyset.$$

By the corona theorem (see [1]), $V(\varphi) \subset \mathbf{D} \neq \emptyset$ and therefore it is possible to choose an infinite interpolating sequence $(z_n)_{n\geq 1}$ in $V(\varphi) \cap \mathbf{D}$. Let b be the Blaschke product with zeroes $(z_n)_{n\geq 1}$. It is well-known (see [11], [5]) that

$$\operatorname{clos}_{\mathcal{M}(H^\infty)}\{z_n : n \geq 1\} = \{\psi \in \mathcal{M}(H^\infty) : \hat{b}(\psi)=0\}$$

if b is an interpolating Blaschke product. We may conclude therefore that

$$\{\psi \in \mathcal{M}(H^\infty) : \hat{b}(\psi) = 0\} \subset V(\varphi)$$

and so (because $\hat{b} \mid \mathcal{M}(A) \neq 0$) b is an invertible element of the algebra A. By the assumption ($A \cdot \theta \subset H^\infty + C$) we have $\bar{b}\theta \in H^\infty + C$ and by the choice of (z_n) we may conclude that $|\theta(z_n)| \geq \delta > 0$, $n = 1, 2, \ldots$. The last assertion contradicts to Lemma 4.

Proof of the theorem. The implication b) $\Rightarrow$ c) is obvious.

c) $\Rightarrow$ a) By Lemma 3 we conclude that $\bar{B}\theta^h \in H^\infty + C$ for an arbitrary number h, $h > 0$. Applying again Nikol'skii's formula (2) we see that $\theta^h \cdot T_{\bar{\theta}^h} \mid K_B$ is compact. But the operator $\theta^h \cdot T_{\bar{\theta}^h}$ is the orthogonal projection onto the space $\theta^h H^2$ and it is unitary equivalent to the projection $\mathbf{1}_{[h,\infty)}$ in the direct integral $H = \int_0^\infty \oplus L^2(\mu)\, dt$. Now it is very easy to see that if

$$A_h \overset{\text{def}}{=} \tau_h \mid E \quad (E = \mathcal{F} K_B)$$

then

$$A_h^* A_h = P_E \mathbf{1}_{[h,\infty)} \mid E,$$

P_E being the orthogonal projection in H onto E. So A_h is compact for every $h > 0$.

a) $\Rightarrow$ b)

$$T_{\bar{\theta}^h} \mid K_B = \bar{\theta}^h \mathcal{F}^{-1} \mathbf{1}_{[h,\infty)} \mathcal{F} \mid K_B = \bar{\theta}^h \mathcal{F}^{-1} \tau_h^* \tau_h \mathcal{F} \mid K_B$$

and it is obvious that $T_{\bar{\theta}^h} \mid K_B$ is compact for every $h > 0$ also. Now let

$$G_\gamma \overset{\text{def}}{=} \{z \in \mathbf{D} : |B(z)| < \gamma\}, \quad \gamma < 1,$$

and let φ be a homomorphism in $\mathcal{M} = \mathcal{M}(H^\infty + C)$.

We shall distinguish two cases:

1) there is $\gamma < 1$ such that $\varphi \in \operatorname{clos}_{\mathcal{M}(H^\infty)} G_\gamma$;

2) for every $\gamma < 1$ the homomorphism φ does not belong to the set $\operatorname{clos}_{\mathcal{M}(H^\infty)} G_\gamma$.

If the first case occurs then one can find a sequence (z_k) in G_γ such that $\lim_{k\to\infty}\theta(z_k) = \hat{\theta}(\varphi)$.

Let

$$r_n \stackrel{\text{def}}{=} P_B k_{z_n} = \frac{1 - \bar{B}(z_n)\, B(e^{i\varphi})}{1 - \bar{z}_n e^{i\varphi}} (1-|z_n|^2)^{1/2}.$$

We have $|B(z_n)| < \gamma < 1$ and so $r_n \to 0$ in the weak topology of $L^2(\mathbf{T})$. Since the operator $T_{\bar{\theta}^h} | K_B$ is compact we conclude that $\lim_{m\to\infty} \|T_{\bar{\theta}^h} r_m\|_2 = 0$. In other words

$$\|T_{\bar{\theta}^h} r_m\| = \|T_{\bar{\theta}^h} k_{z_n} - \bar{B}(z_m)\, T_{\bar{\theta}^h} B k_{z_m}\| =$$

$$= \|\bar{\theta}^h(z_m)\cdot k_{z_m} - \bar{B}(z_m)\, T_{\bar{\theta}^h} B k_{z_m}\| \xrightarrow[m\to\infty]{} 0.$$

It means that

$$|\theta^h(z_m)| \leqslant o(1) + |B(z_m)|.$$

But $\lim_{m\to\infty} \theta(z_m) = \hat{\theta}(\varphi)$ and so

$$|\hat{\theta}(\varphi)|^h \leqslant \limsup_{n\to\infty} |B(z_n)| \leqslant \gamma < 1,$$

for all $h > 0 \Rightarrow \hat{\theta}(\varphi) = 0$.

In the second case $\varphi \notin \operatorname{clos}_{\mathcal{M}(H^\infty)} G_\gamma$ for every $\gamma < 1$. Since $\varphi \in \operatorname{clos}_{\mathcal{M}(H^\infty)} \mathbf{D}$ we may conclude that $\varphi \in \operatorname{clos}_{\mathcal{M}(H^\infty)} (\mathbf{D}\setminus G_\gamma)$ for all $\gamma < 1$. But $|B(z)| \geqslant \gamma$ for z in $\mathbf{D}\setminus G_\gamma$ and so $|\hat{B}(\varphi)| = 1$. ▨

3. CONCLUDING REMARKS

REMARK 1. It is very easy to see that the statement b) of the theorem is equivalent to

$$\lim_{|z|\to 1} \min(|\theta(z)|,\ 1 - |B(z)|) = 0.$$

COROLLARY 1. *If (B, θ) is a Koosis pair then* $\operatorname{dist}(\bar{B}\theta^h, H^\infty) < 1$ *for every* $h > 0$.

It is exactly the conclusion of Lemma 1.2 in [9] because $\|\tau_h | E\| = \operatorname{dist}(\bar{B}\theta^h, H^\infty)$, but this fact may be deduced from different consideration also. Suppose on the contrary that

$$\operatorname{dist}(\bar{B}\theta^h, H^\infty) = 1$$

for some positive number h. It follows by Hartman's theorem [14] that

$$\mathrm{dist}(\bar{B}\theta^h, H^\infty + C) = 0.$$

Then (see [12], [4]) we deduce the family of implications

$$\bar{\theta}^h B \in H^\infty[\bar{B}\theta^h] \subset H^\infty[\bar{B}] \Rightarrow \bar{\theta}^h \in H^\infty[\bar{B}] \Rightarrow$$

$$\Rightarrow \{\varphi : |\hat{B}(\varphi)| = 1\} \subset \{\varphi : |\hat{\theta}(\varphi)| = 1\}$$

implying together with the theorem that

$$\mathcal{M} = \mathcal{M}(H^\infty + C) = \{\varphi : \hat{\theta}(\varphi) = 0\} \cup \{\varphi : |\hat{\theta}(\varphi)| = 1\}.$$

But the space $\mathcal{M}$ is connected. ▨

REMARK 2. Let (B, θ) be a Koosis pair. Then the function

$$D(h) = \mathrm{dist}(\bar{B}\theta^h, H^\infty)$$

is strictly decreasing, lower semi-continuous and

$$\lim_{h \to 0+} D(h) = 1, \quad \lim_{h \to \infty} D(h) = 0.$$

Moreover $D(h) \leqslant c_1 e^{-c_2 h}$, $h \to \infty$, $c_1, c_2 > 0$.

Proof. Recall that $E = \mathcal{F} K_B$ and $\| t_h | E \| = D(h)$. It follows from Lemma 1.2 of [9] that the function $D(h)$ decreases. To prove that the function $D(h)$ is lower semi-continuous suppose that $h = \lim_n h_n$. Then $\lim_n \bar{B}\theta^{h_n} = \bar{B}\theta^h$ in the weak-star topology of L^∞/H^∞. This implies

$$D(h) \leqslant \liminf_n D(h_n).$$

It is clear that $D(0) = 1$ and therefore $\lim_{h \to 0+} D(h) = 1$. The exponential rate of decreasing of the function D as $h \to \infty$ may be deduced by analogous reasoning from the theorem. It may be deduced also from the following inequality:

$$\lim_{n \to \infty} \| \tau_n | E \|^{1/n} \leqslant q < 1$$

(see [9]).

REMARK 3. Let $(z_k)_{k \geqslant 1}$ be an interpolating sequence for the algebra H^∞ in $\mathbf{D}$ and let B denote the Blaschke product with the zeroes z_k, $k \geqslant 1$. Suppose that $\lim_{k \to \infty} \theta(z_k) = 0$ for a singular inner function θ. Then for every $\gamma \in (0, 1)$

$$\lim_{z \in G_\gamma, |z| \to 1} \theta(z) = 0,$$

where

$$G_\gamma \stackrel{\text{def}}{=} \{z \in \mathbf{D} : |B(z)| < \gamma\}.$$

The proof follows simply from Lemma 4 and the proof of the main theorem. ▨

REMARK 4. One may suspect that if the condition

$$\operatorname{dist}(\bar{B}\theta^h, H^\infty) < 1 \tag{4}$$

is satisfied for every $h > 0$ then (B, θ) is a Koosis pair. The following example shows that it is not the case.

EXAMPLE. Let

$$\lambda_n = \begin{cases} 2^n + i, & n \geqslant 0 \\ -2^{-n} + i, & n < 0, \end{cases}$$

and let B denote the Blaschke product whose zero sequence is $(z_n)_{n\in Z}$, $z_n \stackrel{\text{def}}{=} \omega(\lambda_n)$, $n \in \mathbf{Z}$. The function θ is defined by

$$\theta(z) = \exp\left\{-\frac{1+z}{1-z}\right\}.$$

It is clear that $\bar{B}\theta^h \notin H^\infty + C$ since

$$\inf_{Z} |\theta(z_n)| \geqslant 1/e > 0.$$

To prove the inequality (4) we observe that (4) is equivalent to the statement that the exponentials $(e^{i\lambda_n x})_{n\in Z}$ form a Riesz basis in their linear span in $L^2(0, h)$ (see [12], Ch. 11). We may assume without loss of generality that $h = \pi\cdot 2^{-k}$, k being a positive integer (see Remark 2 for the properties of the function D). Clearly, the family $(e^{i\lambda_n x})_{|n|\geqslant k+1}$ is orthogonal in $L^2(0, \pi\cdot 2^{-k})$ and the deficiency defect of its span is infinite. But we have:

THEOREM. (Schwartz, see [13]). *The set $(e^{i\lambda_n x})_{n\in Z}$ is complete in $L^2(0, h)$ if and only if it is possible to approximate some function $e^{i\lambda x}$ other than those already present.*

It follows immediately from this theorem that our family $(e^{i\lambda_n x})_{n\in Z}$ is free in $L^2(0, \pi\cdot 2^{-k})$. Obviously, this implies the desired property of $(e^{i\lambda_n x})_{n\in Z}$. ▨

REMARK 5. There is a nice reformulation of the statement of the theorem similar to that of P. Koosis (see the condition (K) in § 1). Let (B, θ) be a Koosis pair, where B is the Blaschke product whose zero sequence is $(z_n)_{n\geqslant 1}$, and let

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \{z \in \mathbf{D} : |\theta(z)| > \varepsilon\}, \quad \gamma_\varepsilon \stackrel{\text{def}}{=} \partial\Omega_\varepsilon \setminus \mathbf{T}.$$

PROPOSITION. *The pair (B, θ) is a Koosis pair if and only if*

1) $\lim_{n\to\infty} \theta(z_n) = 0$;

2) $$\lim_{|z|\to 1,\, z\in\gamma_\varepsilon} \sum_{k\geqslant 1} \frac{(1-|z|^2)(1-|z_k|^2)}{|1-\bar z_k z|^2} = 0$$

for every $\varepsilon > 0$.

Proof. It follows from Remark 1 that (B, θ) is a Koosis pair iff

$$\lim_{|z|\to 1,\, z\in\Omega_\varepsilon} |B(z)| = 1 \tag{5}$$

for every positive number ε. The hypothesis 1) of the proposition shows that

$$\sup\left\{\left|\frac{1-\bar z_k z}{z-z_k}\right| : z\in\Omega_\varepsilon\right\} < c$$

for some positive constant c and for $k \geqslant k_0$. Let B_{k_0} be the Blaschke product

$$B_{k_0}(z) \overset{\text{def}}{=} \prod_{k\geqslant k_0} \frac{\bar z_k}{|z_k|}\cdot\frac{z_k - z}{1-\bar z_k z}.$$

Clearly,

$$\sum_{k\geqslant k_0} \frac{(1-|z|^2)(1-|z_k|^2)}{|1-\bar z_k z|} \leqslant 2\log\frac{1}{|B_{k_0}(z)|} \leqslant \sum_{k\geqslant k_0} \frac{(1-|z|^2)(1-|z_k|^2)}{|z-z_k|^2} \leqslant$$

$$\leqslant c\cdot\sum_{k\geqslant k_0} \frac{(1-|z|^2)(1-|z_k|^2)}{|1-\bar z_k z|^2}.$$

This shows that condition (5) is equivalent to the following one

$$\lim_{|z|\to 1,\, z\in\Omega_\varepsilon} \sum_{k=1}^{\infty} \frac{(1-|z|^2)(1-|z_k|^2)}{|1-\bar z_k z|^2} = 0 \tag{6}$$

for every $\varepsilon > 0$. It suffices now to deduce (6) from the hypothesis 2). To check this, we fix an arbitrary $\delta > 0$ and choose a positive number $r_\delta < 1$ in order that the following inequality holds:

$$\inf\{|B(z)| : z\in\gamma_\varepsilon,\ r_\delta \leqslant |z| < 1\} > 1-\delta.$$

Let Ω be a component of $\Omega_\varepsilon \cap \{z : r_\delta < |z| < 1\}$. It is clear that $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 \subset \gamma_\varepsilon \cap \{z : r_\delta < |z| < 1\},\ \Gamma_2 \subset \mathbf{T}_\delta \overset{\text{def}}{=} \{\zeta : |\zeta| = r_\delta\},\ \Gamma_3 \subset \mathbf{T}.$$

We have for N sufficiently large

$$\inf\{|B_N(z)| : z\in\mathbf{T}_\delta\} \geqslant 1-\delta.$$

For every $M > N$ let $B_N^M = B_N \cdot B_M^{-1}$. By the minimum principle for the harmonic function $\log |B_N^M|$ in the domain Ω we conclude that $|B_N^M(z)| \geqslant 1 - \delta$ if $z \in \Omega$ and $M > N$. This implies that

$$|B_N(z)| = \lim_{M \to \infty} |B_N^M(z)| \geqslant 1 - \delta$$

for every z in Ω. ▨

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S. V. HRUŠČEV
LOMI,
Fontanka 27,
191011 Leningrad
U.S.S.R.

A. L. VOLBERG
Kosmonavtov 21–2–45,
196211 Leningrad,
U.S.S.R.

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