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ON THE REFLEXIVITY OF $C_{1\cdot}$ CONTRACTIONS AND WEAK CONTRACTIONS

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For a bounded linear operator T on a complex, separable Hilbert space, let $\{T\}'$, $\{T\}''$ and Alg$T$ denote, respectively, its commutant, double commutant and the weakly closed algebra generated by T and I. Let AlgLat$T$ be the (weakly closed) algebra consisting of those operators which leave invariant every invariant subspace of T. Recall that T is *reflexive* if AlgLat$T$ = Alg$T$.

It has been shown in [10] and [14] that any C_{11} contraction with finite defect indices is reflexive. In Section 1 of this paper we generalize this to $C_{1\cdot}$ contractions. More precisely, we show that any $C_{1\cdot}$ contraction with at least one defect index finite is always reflexive. The problem of the reflexivity of weak contractions is taken up in Section 2. We are able to characterize reflexive weak contractions in terms of their characteristic functions and C_0 parts. In particular, this problem is reduced to that of the reflexivity of $C_0(N)$ contractions, which was studied before in [12]. Our proof will be based on the dilation theory of contractions as developed by Sz.-Nagy and Foiaş. The main reference is their book [5].

Let T be a contraction on the Hilbert space $\mathscr{H}$. We use Θ_T to denote its characteristic function. T is *completely non-unitary* (*c.n.u.*) if there is no non-trivial reducing subspace on which the restriction of T is unitary. The *defect indices* of T are

$$d_T = \operatorname{rank}(I - T^*T)^{1/2}$$

and

$$d_{T*} = \operatorname{rank}(I - TT^*)^{1/2}.$$

T is *of class* $C_{1\cdot}$ (resp. $C_{\cdot 1}$) if $T^n x \not\rightarrow 0$ (resp. $T^{*n}x \not\rightarrow 0$) as $n \to \infty$ for any $x \neq 0$ in $\mathscr{H}$. T is *of class* $C_{0\cdot}$ (resp. $C_{\cdot 0}$) if $T^n x \to 0$ (resp. $T^{*n}x \to 0$) as $n \to \infty$ for any $x \in \mathscr{H}$. $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$, $\alpha, \beta = 0, 1$. For operators T_1 and T_2 on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, $T_1 \prec T_2$ denotes that T_1 is a *quasi-affine transform* of T_2, that is, there exists an injection $X: \mathscr{H}_1 \to \mathscr{H}_2$ with dense range (called *quasi-affinity*) such that $T_2 X = X T_1$. $T_1 \overset{\mathrm{ci}}{\prec} T_2$ denotes that there exists a family $\{X_\alpha\}$ of injections

$X_\alpha : \mathscr{H}_1 \to \mathscr{H}_2$ such that $\mathscr{H}_2 = \bigvee_\alpha X_\alpha \mathscr{H}_1$ and $T_2 X_\alpha = X_\alpha T_1$ for all α. T_1 and T_2 are *quasi-similar* ($T_1 \sim T_2$) if $T_1 \prec T_2$ and $T_2 \prec T_1$. For $n \geq 1$, let S_n denote the unilateral shift on H_n^2. For any Borel subset E of the unit circle $\mathbf{T}$, let M_E denote the operator of multiplication by e^{it} on $L^2(E)$.

1. $C_1.$ CONTRACTIONS

In this section we show the reflexivity of $C_1.$ contractions. We start with c.n.u. ones.

LEMMA 1.1. *Let T be a c.n.u. $C_1.$ contraction with $d_T < \infty$ and let $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ be the triangulation of type $\begin{pmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{pmatrix}$. Then T is reflexive if and only if $T_1 \oplus T_2$ is.*

Proof. It suffices to consider the case when $d_T \neq d_{T^*}$, for otherwise $T = T_1$ is itself of class C_{11} (cf. [15], Lemma 3.1). Assume that $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ is acting on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Since T_1 is a C_{11} contraction with finite defect indices, we have $T \sim T_1 \oplus T_2$ (cf. [15], Theorem 2.1). Moreover, there are quasi-affinities $X : \mathscr{H} \to \mathscr{H}_1 \oplus \mathscr{H}_2$ and $Y : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{H}$ which intertwine T and $T_1 \oplus T_2$ and such that $XY = \delta(T_1 \oplus T_2)$ and $YX = \delta(T)$ for some outer function δ. Assume that $T_1 \oplus T_2$ is reflexive. Let $W \in \mathrm{AlgLat}T$. Note that any invariant subspace for $T_1 \oplus T_2$ is of the form $X\mathscr{K}$, where $\mathscr{K}$ is some invariant subspace for T (cf. [15], Corollary 2.2). Since $W\mathscr{K} \subseteq \mathscr{K}$, we have

$$\overline{XWYX\mathscr{K}} = \overline{XW\delta(T)\mathscr{K}} = \overline{XW\mathscr{K}} \subseteq \overline{X\mathscr{K}},$$

where we use the fact that $\delta(T|\mathscr{K})$ is a quasi-affinity for outer δ. This implies that $XWY \in \mathrm{AlgLat}(T_1 \oplus T_2) = \mathrm{Alg}(T_1 \oplus T_2)$. Hence $XWY = \varphi(T_1 \oplus T_2)$ for some $\varphi \in H^\infty$ (cf. [15], Theorem 3.13). Applying Y and X from the left and right of this equation, we obtain

$$YXWYX = Y\varphi(T_1 \oplus T_2)X = YX\varphi(T).$$

It follows that $W\delta(T) = \varphi(T)$. For any $V \in \{T\}'$, we have

$$WV\delta(T) = W\delta(T)V = \varphi(T)V = V\varphi(T) = VW\delta(T),$$

whence $WV = VW$. We conclude that $W \in \{T\}'' = \mathrm{Alg}T$ (cf. [15], Theorem 3.13). Hence T is reflexive as asserted. The converse can be proved similarly.

LEMMA 1.2. *A c.n.u. $C_1.$ contraction T with $d_T < \infty$ is reflexive.*

Proof. We may assume that $d_T \neq d_{T^*}$ for otherwise T is of class C_{11} whence reflexive. Let $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ be the triangulation of type

$\begin{pmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{pmatrix}$. By Lemma 1.1, it suffices to prove the reflexivity of $T_1 \oplus T_2$. Since $d_T < \infty$, T_1 is of class C_{11} and T_2 is of class C_{10} (cf. [15], Lemma 3.2). Let $R \in \mathrm{AlgLat}(T_1 \oplus T_2)$. Then $R = R_1 \oplus R_2$, where $R_1 \in \mathrm{AlgLat}T_1 = \mathrm{Alg}T_1$ and $R_2 \in \mathrm{AlgLat}T_2 = \mathrm{Alg}T_2$ since the C_{11} and C_{10} contractions T_1 and T_2 are both reflexive (cf. [10] and [6]). Consider the functional model of T_1, that is, consider T_1 acting on

$$\mathscr{H}_1 \equiv [H_l^2 \oplus \overline{\Delta_1 L_l^2}] \ominus \{\Theta_{T_1} w \oplus \Delta_1 w : w \in H_l^2\}$$

by

$$T_1(f \oplus g) = P_1(e^{it} f \oplus e^{it} g),$$

where $l = d_{T_1} = d_{T_1^*}$, $\Delta_1 = (I - \Theta_{T_1}^* \Theta_{T_1})^{1/2}$ and P_1 denotes the (orthogonal) projection onto $\mathscr{H}_1$. We have $R_1 = P_1 \begin{pmatrix} A & 0 \\ B & \eta_1 \end{pmatrix}$, where A is a bounded analytic function, B is a bounded measurable function and $\eta_1 \in L^\infty$ satisfying $A\Theta_{T_1} = \Theta_{T_1} A_0$ and $B\Theta_{T_1} + \eta_1 \Delta_1 = \Delta_1 A_0$ for some bounded analytic function A_0 (cf. [10], Lemma 2), and $R_2 = \eta_2(T_2)$ for some $\eta_2 \in H^\infty$ (cf. [6], Theorem 1). Let U be the operator of multiplication by e^{it} on $\overline{\Delta_1 L_l^2}$ and S_{m-n} be the unilateral shift on H_{m-n}^2, where $m = d_{T_2^*}$ and $n = d_{T_2}$. Then $T_1 \sim U$ (cf. [5], p. 72) and $S_{m-n} \overset{ci}{\prec} T_2 \prec S_{m-n}$ (cf. [4], Theorem 3). Moreover, there exist quasi-affinities $X_1 : \mathscr{H}_1 \to \overline{\Delta_1 L_l^2}$ and $Y_1 : \overline{\Delta_1 L_l^2} \to \mathscr{H}_1$ intertwining T_1 and U and such that $X_1 Y_1 = \delta_1(U)$ and $Y_1 X_1 = \delta_1(T_1)$ for some outer δ_1 (cf. [13], Lemma 2.1); there exist operators $X_2 : \mathscr{H}_2 \to H_{m-n}^2$ and $Y_2 : H_{m-n}^2 \to \mathscr{H}_2$ intertwining T_2 and S_{m-n} and such that $X_2 Y_2 = \delta_2(S_{m-n})$ and $Y_2 X_2 = \delta_2(T_2)$ for some $\delta_2 \in H^\infty$, where X_2 is a quasi-affinity and Y_2 is one-to-one (cf. [4], proof of Theorem 3). Since U is an absolutely continuous unitary operator, by spectral theorem $U \cong U' = M_{E_1} \oplus \ldots \oplus M_{E_p}$ on $\mathscr{K}$, where $1 \leqslant p \leqslant l < \infty$ and E_j's are Borel subsets of the unit circle satisfying $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_p$. Let $Z : \overline{\Delta_1 L_l^2} \to \mathscr{K}$ be the unitary operator which implements the unitary equivalence. Let $X_3 = ZX_1 \oplus X_2$ and $Y_3 = \delta_2(T_1) Y_1 Z^{-1} \oplus \delta_1(T_2) Y_2$. Then X_3 and Y_3 intertwine $T_1 \oplus T_2$ and $U' \oplus S_{m-n}$ and $X_3 Y_3 = (\delta_1\delta_2)(U' \oplus S_{m-n})$ and $Y_3 X_3 = (\delta_1\delta_2)(T_1 \oplus T_2)$. Consider

$$\mathscr{M} = \{\chi_{E_1} f \oplus \ldots \oplus \chi_{E_p} f \oplus \underbrace{f \oplus \ldots \oplus f}_{m-n} : f \in H^2\} \subseteq \mathscr{K} \oplus H_{m-n}^2 .$$

Note that $\mathscr{M}$ is a (closed) invariant subspace for $U' \oplus S_{m-n}$. Hence $\overline{Y_3 \mathscr{M}}$ is invariant for $T_1 \oplus T_2$. Thus $R\overline{Y_3 \mathscr{M}} \subseteq \overline{Y_3 \mathscr{M}}$ and therefore $\overline{X_3 R Y_3 \mathscr{M}} \subseteq \overline{X_3 Y_3 \mathscr{M}} = \overline{(\delta_1\delta_2)(U' \oplus S_{m-n})\mathscr{M}} \subseteq \mathscr{M}$. But

$$X_3 R Y_3 = (ZX_1 \oplus X_2)(R_1 \oplus R_2)(\delta_2(T_1) Y_1 Z^{-1} \oplus \delta_1(T_2) Y_2) =$$

$$= ZX_1 R_1 \delta_2(T_1) Y_1 Z^{-1} \oplus X_2 R_2 \delta_1(T_2) Y_2 =$$

$$= (\eta_1 \delta_1 \delta_2)(U') \oplus (\eta_2 \delta_1 \delta_2)(S_{m-n}),$$

where the last equality follows from the expressions of the operators X_1, R_1 and Y_1 (cf. [13], Lemma 2.1). Hence for any $f \in H^2$,

$$(X_3RY_3)(\chi_{E_1}f \oplus \ldots \oplus \chi_{E_p}f \oplus f \oplus \ldots \oplus f) =$$

$$= \eta_1\delta_1\delta_2\chi_{E_1}f \oplus \ldots \oplus \eta_1\delta_1\delta_2\chi_{E_p}f \oplus \eta_2\delta_1\delta_2 f \oplus \ldots \oplus \eta_2\delta_1\delta_2 f$$

is in $\mathcal{M}$. Therefore there exists $g \in H^2$ such that $\eta_1\delta_1\delta_2\chi_{E_j}f = \chi_{E_j}g$ for $j = 1, 2, \ldots, p$ and $\eta_2\delta_1\delta_2 f = g$. In particular, for $f \equiv 1$ we have $\eta_1\delta_1\delta_2 = \eta_2\delta_1\delta_2$ a.e. on E_1. Therefore, $\eta_1 = \eta_2$ a.e. on E_1. Hence $R = R_1 \oplus R_2 = \eta_2(T_1 \oplus T_2) \in \mathrm{Alg}(T_1 \oplus T_2)$. This shows that $T_1 \oplus T_2$, whence T, is reflexive, completing the proof.

Next we consider $C_{1\cdot}$ contractions with a unitary part. The following lemma was proved in [14], Theorem 3. It reduces the problem of the reflexivity of an arbitrary contraction to that of a contraction with an absolutely continuous unitary part.

LEMMA 1.3. *Let $T = U_s \oplus U_a \oplus T'$ be a contraction, where U_s and U_a are singular and absolutely continuous unitary operators and T' is c.n.u. . Then* $\mathrm{Alg}T = \mathrm{Alg}U_s \oplus \mathrm{Alg}(U_a \oplus T')$.

THEOREM 1.4. *A $C_{1\cdot}$ contraction T with $d_T < \infty$ is reflexive.*

Proof. Let $T = U_s \oplus U_a \oplus T'$ be as in Lemma 1.3. Then

$$\mathrm{Alg}T = \mathrm{Alg}U_s \oplus \mathrm{Alg}(U_a \oplus T')$$

implies that

$$\mathrm{AlgLat}T = \mathrm{AlgLat}U_s \oplus \mathrm{AlgLat}(U_a \oplus T')$$

(cf. [1], Proposition 1.3). Since the unitary operator U_s is reflexive, to complete the proof it suffices to show that $U_a \oplus T'$ is reflexive. We may assume that T' is not of class C_{11}, for otherwise T will also be of class C_{11} hence reflexive (cf. [14]). Let $R \in \mathrm{AlgLat}(U_a \oplus T')$. Then $R = R_1 \oplus R_2$, where $R_1 \in \mathrm{AlgLat}U_a = \mathrm{Alg}U_a$ and $R_2 \in \mathrm{AlgLat}T' = \mathrm{Alg}T'$ by Lemma 1.2. Hence there exist $\eta_1 \in L^\infty$ and $\eta_2 \in H^\infty$ such that $R_1 = \eta_1(U_a)$ and $R_2 = \eta_2(T')$ (cf. [15], Theorem 3.13). We proceed as in the proof of Lemma 1.2. Assume that $U_a \oplus T'$ is acting on $\mathcal{H}_a \oplus \mathcal{H}'$. Let $T' = \begin{pmatrix} T_1' & * \\ 0 & T_2' \end{pmatrix}$ on $\mathcal{H}' = \mathcal{H}_1' \oplus \mathcal{H}_2'$ be the triangulation of type $\begin{pmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{pmatrix}$. Then $T' \sim T_1' \oplus T_2'$. As before, let

$$U' = M_{E_1} \oplus \ldots \oplus M_{E_p}$$

on $\mathcal{K}$ be a unitary operator quasi-similar to T_1' and let S_{m-n} on H^2_{m-n} be such that $S_{m-n} \overset{\mathrm{ci}}{\prec} T_2' \prec S_{m-n}$. Also, let

$$U_a' = M_{F_1} \oplus \ldots \oplus M_{F_q}$$

on $\mathscr{H}'_{\mathrm{a}}$ be unitarily equivalent to U_{a}. We deduce, as before, that there are operators $X: \mathscr{H}_{\mathrm{a}} \oplus \mathscr{H}' \to \mathscr{H}'_{\mathrm{a}} \oplus \mathscr{K} \oplus H^2_{m-n}$ and $Y: \mathscr{H}'_{\mathrm{a}} \oplus \mathscr{K} \oplus H^2_{m-n} \to \mathscr{H}_{\mathrm{a}} \oplus \mathscr{H}'$ which intertwine $U_{\mathrm{a}} \oplus T'$ and $U'_{\mathrm{a}} \oplus U' \oplus S_{m-n}$ and satisfy $XY = \delta(U'_{\mathrm{a}} \oplus U' \oplus S_{m-n})$, $YX = \delta(U_{\mathrm{a}} \oplus T')$ and $XRY = (\eta_1\delta)(U'_{\mathrm{a}}) \oplus (\eta_2\delta)(U' \oplus S_{m-n})$ for some $\delta \in H^\infty$. Consider the invariant subspace

$$\mathscr{M} = \{\chi_{F_1} f \oplus \ldots \oplus \chi_{F_q} f \oplus \chi_{E_1} f \oplus \ldots \oplus \chi_{E_p} f \oplus \underbrace{f \oplus \ldots \oplus f}_{m-n} : f \in H^2\}$$

for $U'_{\mathrm{a}} \oplus U' \oplus S_{m-n}$. We deduce as before that $\eta_1 = \eta_2$ a.e. on F_1. Hence $R = \eta_2(U_{\mathrm{a}} \oplus T') \in \mathrm{Alg}(U_{\mathrm{a}} \oplus T')$, which shows that $U_{\mathrm{a}} \oplus T'$, whence T, is reflexive.

2. WEAK CONTRACTIONS

Recall that a contraction T is a *weak contraction* if:

(1) its spectrum $\sigma(T)$ does not fill the open unit disk and

(2) $I - T^*T$ is of finite trace.

For basic properties of weak contractions, readers are referred to [5], Chapter VIII. In this section we will find necessary and sufficient conditions for a c.n.u. weak contraction with finite defect indices to be reflexive. The proof of the next lemma is analogous to that of Lemma 1.1.

LEMMA 2.1. *Let T be a c.n.u. weak contraction with finite defect indices and let $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ be the triangulation of type $\begin{pmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{pmatrix}$. Assume that $\Theta_T(\mathrm{e}^{\mathrm{i}t})$ is not isometric for almost all t. Then T is reflexive if and only if $T_1 \oplus T_2$ is.*

Proof. Follow the same line of arguments as in the proof of Lemma 1.1 and make use of the fact that

$$\mathrm{Alg}(T_1 \oplus T_2) = \{\varphi(T_1 \oplus T_2) : \varphi \in H^\infty\}$$

since

$$\{\mathrm{e}^{\mathrm{i}t}: \Theta_{T_1 \oplus T_2}(\mathrm{e}^{\mathrm{i}t}) \text{ not isometric}\} = \{\mathrm{e}^{\mathrm{i}t}: \Theta_{T_1}(\mathrm{e}^{\mathrm{i}t}) \text{ not isometric}\} =$$

$$= \{\mathrm{e}^{\mathrm{i}t}: \Theta_T(\mathrm{e}^{\mathrm{i}t}) \text{ not isometric}\} = \mathbf{T}$$

a.e. (cf. [11], Theorem 3).

For any inner function φ, let $S(\varphi)$ denote the *compression of the shift* on $H^2 \ominus \varphi H^2$, that is, $S(\varphi)$ is defined by $S(\varphi)f = P(\mathrm{e}^{\mathrm{i}t}f)$ for $f \in H^2 \ominus \varphi H^2$, where P denotes the (orthogonal) projection onto $H^2 \ominus \varphi H^2$. An operator of the form $S(\varphi_1) \oplus \ldots \oplus S(\varphi_q)$ is a *Jordan operator* if $\varphi_{j+1} \mid \varphi_j$ for $j = 1, 2, \ldots, q-1$. The following lemma generalizes [11], Theorem 3, (2).

LEMMA 2.2. *Let T be a c.n.u. weak contraction with finite defect indices. If $\Theta_T(\mathrm{e}^{\mathrm{i}t})$ is not isometric for almost all t, then T is reflexive.*

Proof. Let $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ be the triangulation of type $\begin{pmatrix} C_{\cdot 1} & * \\ 0 & C_{\cdot 0} \end{pmatrix}$. By Lemma 2.1, we have only to show that $T_1 \oplus T_2$ is reflexive. Note that T_1 is of class C_{11} and T_2 is of class $C_0(N)$. Let $R \in \mathrm{AlgLat}(T_1 \oplus T_2)$. Then $R = R_1 \oplus R_2$, where $R_1 \in \mathrm{AlgLat}\,T_1 = \mathrm{Alg}\,T_1$ and $R_2 \in \mathrm{AlgLat}\,T_2$. Since

$$\{e^{it}: \Theta_{T_1}(e^{it}) \text{ not isometric}\} = \{e^{it}: \Theta_T(e^{it}) \text{ not isometric}\} = \mathbf{T}$$

a.e., we have $R_1 = \eta_1(T_1)$ for some $\eta_1 \in H^\infty$ (cf. [10], Theorem 3). Let X_1, Y_1, U, U' and Z be as in the proof of Lemma 1.2. Then $X_1 Y_1 = \delta_1(U)$ and $Y_1 X_1 = \delta_1(T_1)$ for some outer δ_1. On the other hand, let

$$S = S(\varphi_1) \oplus \ldots \oplus S(\varphi_q)$$

on

$$\mathscr{L} = (H^2 \ominus \varphi_1 H^2) \oplus \ldots \oplus (H^2 \ominus \varphi_q H^2)$$

be the Jordan operator quasi-similar to T_2 and let $X_2: \mathscr{H}_2 \to \mathscr{L}$ and $Y_2: \mathscr{L} \to \mathscr{H}_2$ be the quasi-affinities which intertwine T_2 and S and satisfy $X_2 Y_2 = \delta_2(S)$ and $Y_2 X_2 = \delta_2(T_2)$ for some $\delta_2 \in H^\infty$ with $\delta_2 \wedge \varphi_1 = 1$ (cf. [3] and [2]). Let $X_3 = ZX_1 \oplus \oplus X_2$ and $Y_3 = \delta_2(T_1) Y_1 Z^{-1} \oplus \delta_1(T_2) Y_2$. Then X_3 and Y_3 are quasi-affinities which intertwine $T_1 \oplus T_2$ and $U' \oplus S$ and satisfy $X_3 Y_3 = (\delta_1\delta_2)(U' \oplus S)$ and $Y_3 X_3 = (\delta_1\delta_2)(T_1 \oplus T_2)$. Let P_j denote the (orthogonal) projection from H^2 onto $H^2 \ominus \varphi_j H^2$, $j = 1, 2, \ldots, q$. Consider the (closed) invariant subspace for $U' \oplus S$:

$$\mathscr{M} = \{\chi_{E_1} f \oplus \ldots \oplus \chi_{E_p} f \oplus P_1 f \oplus \ldots \oplus P_q f : f \in H^2\} =$$

$$= \{f \oplus \chi_{E_2} f \oplus \ldots \oplus \chi_{E_p} f \oplus P_1 f \oplus \ldots \oplus P_q f : f \in H^2\}.$$

The last equality follows from the fact that

$$E_1 = \{e^{it}: \Theta_{T_1}(e^{it}) \text{ not isometric}\} = \mathbf{T}$$

a.e. . Hence $Y_3\mathscr{M}$ is invariant for $T_1 \oplus T_2$ and therefore $\overline{RY_3\mathscr{M}} \subseteq \overline{Y_3\mathscr{M}}$. We have

$$X_3 R \overline{Y_3 \mathscr{M}} \subseteq X_3 \overline{Y_3 \mathscr{M}} = \overline{(\delta_1\delta_2)(U' \oplus S)\mathscr{M}} \subseteq \mathscr{M}.$$

But

$$X_3 R Y_3 = Z X_1 R_1 \delta_2(T_1) Y_1 Z^{-1} \oplus X_2 R_2 \delta_1(T_2) Y_2 =$$

$$= (\eta_1\delta_1\delta_2)(U') \oplus X_2 R_2 \delta_1(T_2) Y_2 .$$

Since any invariant subspace for S is of the form $X_2\mathscr{N}$, where $\mathscr{N}$ is some invariant subspace for T_2 (cf. [12], Theorem 3), it can be easily verified that

$X_2R_2\delta_1(T_2)Y_2 \in \mathrm{AlgLat}S$. Hence

$$X_2R_2\delta_1(T_2)Y_2 = V_1 \oplus \dots \oplus V_q ,$$

where $V_j \in \mathrm{AlgLat}S(\varphi_j)$ for $j = 1, 2, \dots, q$. Since

$$\overline{((\eta_1\delta_1\delta_2)(U') \oplus V_1 \oplus \dots \oplus V_q)\mathcal{M}} \subseteq \mathcal{M},$$

we conclude that for any $f \in H^2$,

$$\eta_1\delta_1\delta_2 f \oplus \eta_1\delta_1\delta_2\chi_{E_2} f \oplus \dots \oplus \eta_1\delta_1\delta_2\chi_{E_p} f \oplus V_1P_1 f \oplus \dots \oplus V_qP_q f =$$

$$= g \oplus \chi_{E_2} g \oplus \dots \oplus \chi_{E_p} g \oplus P_1 g \oplus \dots \oplus P_q g$$

for some $g \in H^2$. In particular, we have $g = \eta_1\delta_1\delta_2 f$ and $V_jP_j f = P_j g$ for $j = 1,2,\dots,q$. It follows that $V_j = (\eta_1\delta_1\delta_2)(S(\varphi_j))$ for all j, whence $X_3RY_3 = (\eta_1\delta_1\delta_2)(U' \oplus S)$. Applying Y_3 and X_3 from left and right, respectively, we obtain

$$Y_3X_3RY_3X_3 = Y_3(\eta_1\delta_1\delta_2)(U' \oplus S)X_3 =$$

$$= Y_3X_3(\eta_1\delta_1\delta_2)(T_1 \oplus T_2) = Y_3X_3\eta_1(T_1 \oplus T_2)Y_3X_3 .$$

It follows that $R = \eta_1(T_1 \oplus T_2) \in \mathrm{Alg}(T_1 \oplus T_2)$. Hence $T_1 \oplus T_2$, together with T, is reflexive as asserted.

Now we are ready for our main result in this section.

THEOREM 2.3. *Let T be a c.n.u. weak contraction with finite defect indices and let*

$$E_1 = \{e^{it} : \Theta_T(e^{it}) \text{ not isometric}\}.$$

Then the following statements are equivalent:

(1) *T is reflexive;*

(2) *either $E_1 = \mathbf{T}$ a.e. or $E_1 \neq \mathbf{T}$ a.e. and the C_0 part of T is reflexive.*

Here we use the convention that if the C_0 part of T is acting on $\{0\}$, then it is reflexive.

Proof. By Lemma 2.2, it suffices to show that if $E_1 \neq \mathbf{T}$ a.e. then T is reflexive if and only if its C_0 part is. Let T_0 and T_1 be the C_0 and C_{11} parts of T. Assume that T, T_0 and T_1 are acting on $\mathcal{H}$, $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively.

Assume that T is reflexive. Let $V_0 \in \mathrm{AlgLat}T_0$ and $S \in \{T\}''$ be such that $\mathcal{H}_0 = S\mathcal{H}$ (cf. [9], Theorem 1). Since $E_1 \neq \mathbf{T}$ a.e., we have $\{T\}'' = \mathrm{Alg}T$ (cf. [11], Theorem 3). Hence $S \in \mathrm{Alg}T$. The reflexivity of T_0 follows from [12], Lemma 4 and the fact that $\mathrm{AlgLat}T_0 \cap \{T_0\}' = \mathrm{Alg}T_0$ (cf. [7], Theorem 3.3).

Conversely, if T_0 is reflexive, let $V \in \mathrm{AlgLat}T$. Then $V\mathscr{H}_0 \subseteq \mathscr{H}_0$ and $V\mathscr{H}_1 \subseteq \mathscr{H}_1$. Let $V_0 = V|\mathscr{H}_0$ and $V_1 = V|\mathscr{H}_1$. We have $V_0 \in \mathrm{AlgLat}T_0 = \mathrm{Alg}T_0$ and $V_1 \in \mathrm{AlgLat}T_1 = \mathrm{Alg}T_1$ since T_1, being of class C_{11}, is reflexive. Hence $V_0T_0 = T_0V_0$ and $V_1T_1 = T_1V_1$. It follows that $VT = TV$ on $\mathscr{H}_0 \vee \mathscr{H}_1 = \mathscr{H}$ (cf. [5], p. 332). Therefore, $V \in \mathrm{AlgLat}T \cap \{T\}' = \mathrm{Alg}T$ (cf. [11], Theorem 3). This proves the reflexivity of T.

Since reflexive $C_0(N)$ contractions have been characterized in [12], the preceding theorem reduces the reflexivity of a weak contraction to that of $S(\varphi)$. The next corollary generalizes [12], Theorem 2 for $C_0(N)$ contractions.

COROLLARY 2.4. *Let T_1 and T_2 be c.n.u. weak contractions with finite defect indices. Assume that T_1 is quasi-similar to T_2. Then T_1 is reflexive if and only if T_2 is.*

Proof. The quasi-similarity of T_1 and T_2 implies that of their C_0 parts (cf. [8], Corollary 1). The conclusion now follows from Theorem 2.3 and [12], Theorem 2.

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