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# FUNDAMENTAL REDUCIBILITY OF SELFADJOINT OPERATORS ON KREĬN SPACE

BRIAN W. McENNIS

## 1. INTRODUCTION

An indefinite inner product space $\mathscr{K}$ with inner product $[\cdot,\cdot]$ is called a Kreĭn space if there exists an operator $J$ on $\mathscr{K}$ such that $J = J^* = J^{-1}$ and the *J-inner product*

$$(x, y)_J = [Jx, y] = [x, Jy]$$

makes $\mathscr{K}$ a Hilbert space. Such an operator $J$ is called a *fundamental symmetry*; it determines, and is determined by, the *fundamental decomposition* $\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{K}_-$, where $\mathscr{K}_+ = (I + J)\mathscr{K}$ and $\mathscr{K}_- = (I - J)\mathscr{K}$ are Hilbert spaces with the inner products $[\cdot,\cdot]$ and $-[\cdot,\cdot]$, respectively. (See [3, Chapter V].) The topology on $\mathscr{K}$ is that given by the *J-norm* $\|x\|_J = [Jx, x]^{1/2}$. A Kreĭn space has many different fundamental symmetries, but the $J$-norms obtained are all equivalent [3, Corollary IV.6.3, Theorem V.1.1].

Throughout this paper $A$ will denote a bounded linear operator on $\mathscr{K}$. Unless otherwise stated, concepts involving an inner product will be defined in terms of the indefinite inner product. Thus $A^*$ is the operator satisfying $[Ax, y] = [x, A^*y]$ for all $x, y \in \mathscr{K}$, and $A$ is selfadjoint if $A = A^*$.

If there is a fundamental decomposition that reduces $A$, then $A$ is called *fundamentally reducible*, and the study of $A$ can be reduced to the study of the Hilbert space operators $A \mid \mathscr{K}_+$ and $A \mid \mathscr{K}_-$. It follows from [3, Lemma VIII.1.1 and Theorem VIII.1.2] that $A$ is fundamentally reducible if and only if $AJ = JA$ for some fundamental symmetry $J$.

Various conditions for an operator to be fundamentally reducible have been given, some of which are referred to in the notes to Chapter VIII of [3]. More recently, Bajasgalan [2] has given a condition for the fundamental reducibility of a positive operator, in terms of its spectral function.

In Theorem 1 below we give necessary and sufficient conditions for the fundamental reducibility of a selfadjoint operator $A$. Conditions (iii) and (iv) of

Theorem 1 involve the following growth condition on the resolvent:

$$\|(A-\lambda)^{-k}\| \leqslant M|\mathrm{Im}\lambda|^{-k} \quad (k=1,2,3,\ldots), \tag{1.1}$$

where $M$ is a constant independent of $k$, and $\|\cdot\|$ is a $J$-norm on $\mathscr{K}$. A proof of the equivalence of conditions (i) and (iv), for $a=0$, also appears in the unpublished thesis of C. Bajasgalan (Budapest, 1980).

THEOREM 1. *Let $A$ be a selfadjoint operator on a Kreĭn space $\mathscr{K}$. Then the following are equivalent.*

(i) *$A$ is fundamentally reducible.*

(ii) *$A$ is similar to a selfadjoint operator on a Hilbert space.*

(iii) *The growth condition* (1.1) *holds for all complex numbers $\lambda$ with* $\mathrm{Im}\lambda \neq 0$.

(iv) *There is a real number $a$ such that the growth condition* (1.1) *holds for all complex numbers $\lambda$ with* $\mathrm{Re}\lambda = a$ *and* $\mathrm{Im}\lambda > 0$.

Sections 2 and 3 below are devoted to the proof of Theorem 1. In Section 4 we discuss spectral functions of selfadjoint operators. Theorem 2 gives a necessary and sufficient condition, in terms of the spectral function, for the fundamental reducibility of a positizable operator $A$, and the spectral function is used to describe the fundamental symmetries that commute with $A$.

## 2. PROOF OF THEOREM 1: THE EQUIVALENCE OF (i) AND (ii)

We present in this section a proof, following along the lines of [4, Theorem 2], of the equivalence of conditions (i) and (ii) of Theorem 1. Although this could be deduced from [4, Theorem 2] and [3, Theorem VIII.1.2], the proof presented here is somewhat simpler than that in [4].

Suppose that the selfadjoint operator $A$ is fundamentally reducible, so that $AJ_0=J_0A$ for some fundamental symmetry $J_0$. If a different fundamental symmetry has initially been used in defining the norm on $\mathscr{K}$, then $\mathscr{K}$ can be renormed with the equivalent $J_0$-norm. With this renorming, $A$ becomes a selfadjoint operator in the Hilbert space sense:

$$(Ax, y)_{J_0} = [J_0Ax, y] = [AJ_0x, y] = [J_0x, Ay] = (x, Ay)_{J_0}.$$

Thus (i) implies (ii).

We now assume that $A$ is similar to a selfadjoint operator on a Hilbert space (as well as being selfadjoint with respect to the indefinite inner product). Then there is a Hilbert space inner product $(\cdot,\cdot)_H$ on $\mathscr{K}$, equivalent to any of the $J$-inner products, such that $(Ax, y)_H=(x, Ay)_H$ for all $x, y \in \mathscr{K}$. Choose any fundamental symmetry $J$ on $\mathscr{K}$. Since the norm $\|x\|_H=(x, x)_H^{1/2}$ and the $J$-norm $\|x\|_J$ are equivalent, there is a constant $\alpha$ such that

$$|(x, y)_H| \leqslant \|x\|_H \|y\|_H \leqslant \alpha \|x\|_J \|y\|_J .$$

Thus it follows that there is a bounded operator $H'$ such that $(x, y)_H = (H'x, y)_J = [JH'x, y]$; i.e., $(x, y)_H = [Hx, y]$, where $H = JH'$. $H$ is uniformly positive, in the sense of [3, p. 151], since it follows from the equivalence of the two norms that there is a constant $\beta > 0$ such that

$$[Hx, x] = \|x\|_H^2 \geqslant \beta \|x\|_J^2 \quad \text{for all } x \in \mathscr{K} .$$

Also, $H$ commutes with $A$:

$$[HAx, y] = (Ax, y)_H = (x, Ay)_H = [Hx, Ay] = [AHx, y]$$

for all $x, y \in \mathscr{K}$. Thus, by [3, Theorem VIII. 1.2], $A$ is fundamentally reducible, and the proof of the equivalence of conditions (i) and (ii) is complete.

## 3. PROOF OF THEOREM 1: THE EQUIVALENCE OF (ii), (iii) AND (iv)

Let us assume that $A$ is similar to a selfadjoint operator on a Hilbert space, so that there is a Hilbert space norm $\|\cdot\|_H$ on $\mathscr{K}$, equivalent to any $J$-norm, for which $A$ is selfadjoint. Consequently, $\|(A - \lambda)^{-1}\|_H \leqslant |\mathrm{Im}\lambda|^{-1}$ for all non-real complex numbers $\lambda$, and it follows that, for each positive integer $k$, $\|(A - \lambda)^{-k}\|_H \leqslant |\mathrm{Im}\lambda|^{-k}$. Let $\|\cdot\|$ denote a $J$-norm on $\mathscr{K}$. Since this norm is equivalent to $\|\cdot\|_H$, there exists a constant $M$ (independent of $k$) such that $\|(A - \lambda)^{-k}\| \leqslant M |\mathrm{Im}\lambda|^{-k}$ ($k = 1, 2, 3, \ldots$, $\mathrm{Im}\lambda \neq 0$). Thus (ii) implies (iii). Since (iii) clearly implies (iv), the proof of Theorem 1 is completed by showing that (iv) implies (ii).

If $A$ satisfies condition (iv) of Theorem 1, then the operator $A - aI$ satisfies the condition (1.1) for all $\lambda$ with $\mathrm{Re}\lambda = 0$ and $\mathrm{Im}\lambda > 0$. Since $A - aI$ is similar to a selfadjoint operator on a Hilbert space if and only if $A$ is, it suffices to consider the case where $\mathrm{Re}\lambda = 0$. Also, by taking adjoints (with respect to the indefinite inner product) and using the fact that $A^* = A$, we can obtain the condition (1.1) for all $\lambda$ with $\mathrm{Re}\lambda = 0$ and $\mathrm{Im}\lambda \neq 0$. Thus we have, setting $\lambda = \mathrm{i}\xi$,

$$\|(\mathrm{i}A + \xi)^{-k}\| = \|(A - \mathrm{i}\xi)^{-k}\| \leqslant M|\xi|^{-k}$$

for all nonzero real numbers $\xi$ and for $k = 1,2,3, \ldots$ . It follows from [6, Section IX.1.4] that $-\mathrm{i}A$ is the generator of the bounded group of operators $\mathrm{e}^{-\mathrm{i}tA}$ with $\|\mathrm{e}^{-\mathrm{i}tA}\| \leqslant M$ for all real numbers $t$. (That a bounded group is obtained, rather than just a semigroup, is proved in the same way as in Example 1.6 of Chapter IX of [6].) By a theorem of Sz.-Nagy [10], $\mathrm{e}^{-\mathrm{i}tA}$ is similar to a unitary group on a Hilbert space. Since $A = i \frac{d}{dt} e^{-itA}|_{t=0}$ (cf. [6, p. 483]), it then readily follows that $A$ is similar to a selfadjoint operator on a Hilbert space. (The above argument is similar to that in [1, p. 116].)

This completes the proof of Theorem 1.

## 4. SPECTRAL FUNCTIONS

Let $A$ be a selfadjoint operator on a Kreĭn space $\mathscr{K}$. A spectral function for $A$ is a function $E(\lambda)$ defined on a subset of the real line and taking values which are selfadjoint projections on $\mathscr{K}$. Any two projections $E(\lambda_1)$ and $E(\lambda_2)$ commute: $E(\lambda_1)E(\lambda_2) = E(\min\{\lambda_1, \lambda_2\})$; and each projection $E(\lambda)$ commutes with every operator commuting with $A$. We also require the following: at points where it is defined, $E(\lambda)$ is strongly continuous from the left; $E(\lambda) = 0$ for $\lambda < -\|A\|$; $E(\lambda) = I$ for $\lambda > \|A\|$; and $\sigma(A|(E(\lambda_2) - E(\lambda_1))\mathscr{K}) \subset [\lambda_1, \lambda_2]$. (See, for example, [3, Section VIII.6], [4, Section 3], [5], [7, Section 2], [8, Section 1], [9].)

A real number $\alpha$ is called a *critical point* of $E$ if the subspace $(E(\lambda_2) - E(\lambda_1))\mathscr{K}$ is indefinite for all $\lambda_1, \lambda_2$ in the domain of $E$ satisfying $\lambda_1 < \alpha < \lambda_2$. (On an indefinite subspace, the expression $[x, x]$ takes both positive and negative values.) We will be considering spectral functions $E$ for which $E(\lambda)$ is defined whenever $\lambda$ is not a critical point; $E(\alpha)$ may even be defined for some or all critical points $\alpha$.

We will call the spectral function $E$ *regular* if the strong limits

$$E(\mu-) = \lim_{\lambda \to \mu-} E(\lambda) \quad \text{and} \quad E(\mu+) = \lim_{\lambda \to \mu+} E(\lambda)$$

exist for all real numbers $\mu$ (cf. [1], [7, Section 2.4]).

If $A$ satisfies the conditions of Theorem 1, so that $A = A^*$ and $AJ = JA$ for some fundamental symmetry $J$, then $A$ is selfadjoint with respect to the Hilbert space $J$-inner product on $\mathscr{K}$. Thus, by the spectral theorem for selfadjoint operators on Hilbert space, $A$ has the spectral decomposition

$$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda), \tag{4.1}$$

where $E(\lambda)$ is a spectral function that is selfadjoint with respect to the $J$-inner product. Since $A$ and $J$ commute, it follows that, for each $\lambda$, $E(\lambda)$ also commutes with $J$, and thus $E(\lambda)$ is selfadjoint with respect to the indefinite inner product. Consequently, (4.1) gives us a spectral decomposition of $A$ in terms of a spectral function of the type described in the preceding paragraphs (see [4, Section 3.1]).

Spectral functions are known to exist for another class of selfadjoint operators on a Kreĭn space, namely the *positizable* (or *definitizable*) operators. The selfadjoint operator $A$ is positizable if there is a real polynomial $f$ such that $[f(A)x, x] \geqslant 0$ for all $x \in \mathscr{K}$. (See, for example, [3, Section VIII.6], [8, Section 1], and [9, Section 2].) If the spectrum of the positizable operator $A$ is real, then there exists a spectral function $E(\lambda)$ such that

$$f(A) = S + \int_{-\infty}^{\infty} f(\lambda)\, \mathrm{d}E(\lambda), \tag{4.2}$$

where $S$ is a positive operator satisfying $S^2 = 0$ and $S(E(\lambda_2) - E(\lambda_1)) = 0$ if the closed interval $[\lambda_1, \lambda_2]$ contains no critical points. Also, at a critical point $\alpha$ of $E$, it is necessarily true that $f(\alpha) = 0$, and thus $E$ has only a finite number of critical points.

Theorem 2 below gives a necessary and sufficient condition for the fundamental reducibility of a positizable operator in terms of its spectral function. The spectral function is also used to describe all fundamental symmetries commuting with the operator. In what follows, if $\mathscr{L}$ is any subspace of $\mathscr{K}$, then $\mathscr{L}^{\perp}$ denotes the subspace of all vectors orthogonal to $\mathscr{L}$ with respect to the indefinite inner product. A subspace $\mathscr{L}$ is *non-degenerate* if $\mathscr{L} \cap \mathscr{L}^{\perp} = \{0\}$, and *regular* if $\mathscr{L} \oplus \mathscr{L}^{\perp} = \mathscr{K}$. (A version of Theorem 2 appears in [2], where $A$ is positive and the word *non-degenerate* in (iii) is replaced by *regular*.)

THEOREM 2. *Let $A$ be a positizable operator on a Kreĭn space $\mathscr{K}$. Then $A$ is fundamentally reducible if and only if the following three conditions all hold:*

(i) *$A$ has real spectrum;*

(ii) *the spectral function $E$ of $A$ is regular;*

*and*

(iii) *for each critical point $\alpha$ of $E$, the null space of $A - \alpha$ is non-degenerate.*

*Suppose $A$ is fundamentally reducible, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of critical points of $E$, listed in increasing order. Let $P_k = E(\alpha_{k+1}-) - E(\alpha_k+)$ for $k = 0,1,2, \ldots, n$ (with $E(\alpha_{n+1}-) = I$, $E(\alpha_0+) = 0$). Then every fundamental symmetry $J$ commuting with $A$ is of the form*

$$J = \sum_{k=1}^{n} J_k(E(\alpha_k+) - E(\alpha_k-)) + \sum_{k=0}^{n} \varepsilon_k P_k\,, \tag{4.3}$$

*where $J_k$ is any fundamental symmetry on the null space of $A - \alpha_k$, and $\varepsilon_k = \pm 1$ (the signs being determined by $A$).*

*Proof.* If $A$ is fundamentally reducible then, by Theorem 1, $A$ is similar to a selfadjoint operator on a Hilbert space. Thus, $A$ has real spectrum and its spectral function is regular. Let $\mathscr{N}$ denote the null space of $A - \alpha$ for a critical point $\alpha$, and suppose $J$ is a fundamental symmetry commuting with $A$. Then $J\mathscr{N} \subset \mathscr{N}$, and so

$\mathcal{N} = J^2\mathcal{N} \subset J\mathcal{N}$. Hence, $J\mathcal{N} = \mathcal{N}$ and it follows that $\mathcal{N}^\perp$ is the same as the orthogonal complement of $\mathcal{N}$ in the $J$-inner product (a Hilbert space inner product). Thus $\mathcal{N}$ is non-degenerate.

Now suppose that conditions (i), (ii), and (iii) are satisfied, and let $\alpha$ denote any critical point. Since $E$ is assumed to be regular, $E(\alpha-)$ and $E(\alpha+)$ exist, and $(E(\alpha+) - E(\alpha-))\mathcal{H}$ is the principal subspace of $A$ corresponding to $\alpha$, i.e. the span of the null spaces of the operators $(A-\alpha)^n$, $n = 1, 2, 3, \ldots$ (cf. [3, Section VIII.6]). Let $\mathcal{N}$ and $\mathcal{R}$ denote the null space and range, respectively, of $A-\alpha$. $A-\alpha$ is selfadjoint, and so it follows by the usual argument that $\mathcal{R} \subset \mathcal{N}^\perp$. Since $\mathcal{N}$ is assumed to be non-degenerate, we have $\mathcal{N} \cap \mathcal{R} = \{0\}$. Thus the operators $(A-\alpha)^n$ $(n = 1, 2, 3, \ldots)$ all have the same null space $\mathcal{N}$, and we deduce that $\mathcal{N} = (E(\alpha+) - E(\alpha-))\mathcal{H}$.

We have shown that $\mathcal{N}$ is the range of a selfadjoint projection, and thus (since $\mathcal{N}^\perp$ is the range of the complementary projection) $\mathcal{N}$ is a regular subspace. Consequently, $\mathcal{N}$ is a Kreĭn space in its own right (see [3, Theorem V.3.4], where the term *ortho-complemented* is used instead of *regular*). We can therefore choose, for each critical point $\alpha_k$, a fundamental symmetry $J_k$ acting on the null space of $A-\alpha_k$. $A$ commutes with $J_k$, since $(A-\alpha_k)J_k = J_k(A-\alpha_k) = 0$.

Let $P_k = E(\alpha_{k+1}-) - E(\alpha_k+)$, for $k = 1, 2, \ldots, n-1$, $P_0 = E(\alpha_1-)$, and $P_n = I - E(\alpha_n+)$. Since the intervals $(\alpha_k, \alpha_{k+1})$, $(-\infty, \alpha_1)$, and $(\alpha_n, \infty)$ contain no critical points, it follows that the subspaces $P_k\mathcal{H}$ are definite, $k = 0, 1, 2, \ldots, n$ (cf. [9, p. 688]), and we can define

$$\varepsilon_k = 1 \quad \text{if } [x, x] \geqslant 0 \text{ for all } x \in P_k\mathcal{H}\,,$$

$$\varepsilon_k = -1 \quad \text{if } [x, x] \leqslant 0 \text{ for all } x \in P_k\mathcal{H}\,.$$

If we define $J$ on $\mathcal{H}$ by (4.3), then it readily follows that $J^2 = I$, $J = J^*$, and $[Jx, x] \geqslant 0$ for all $x \in \mathcal{H}$. Thus, with the help of [3, Theorem V.1.1], we can deduce that $J$ is a fundamental symmetry. Since $A$ commutes with the operators $E(\lambda)$ and the operators $J_k$, it follows that $AJ = JA$, and thus $A$ is fundamentally reducible.

It remains to show that every fundamental symmetry commuting with $A$ is of the form (4.3). Suppose $AJ = JA$, and let $\mathcal{N}_k$ be the null space of $A-\alpha_k$. Then, as shown above, $J\mathcal{N}_k = \mathcal{N}_k$ and we can define $J_k = J|\mathcal{N}_k$, a fundamental symmetry on $\mathcal{N}_k$. We also have $E(\lambda)J = JE(\lambda)$, and thus each of the subspaces $P_k\mathcal{H}$ is invariant for $J$. As we did with the subspaces $\mathcal{N}_k$, we can deduce that $P_k\mathcal{H}$ is a Kreĭn space and that $J|P_k\mathcal{H}$ is a fundamental symmetry on $P_k\mathcal{H}$. But $P_k\mathcal{H}$ is a definite subspace, and thus is a Hilbert space with one of the inner products $[\cdot,\cdot]$ or $-[\cdot,\cdot]$. Therefore, $P_k\mathcal{H}$ admits only a trivial fundamental decomposition, and $J|P_k\mathcal{H} = \pm I$. The representation (4.3) follows. ∎

Let $f(\lambda)$ be a positizing polynomial for $A$. The numbers $\varepsilon_k$ in (4.3) can be determined by using the fact that, for all $x \in \mathcal{H}$, the function $[E(\lambda)x, x]$ is monotone nondecreasing when $f(\lambda)$ is positive and monotone nonincreasing when $f(\lambda)$ is negative (see [8, p. 6]). Thus $\varepsilon_k$ is equal to the value of sgn$f(\lambda)$ on the interval $(\alpha_k, \alpha_{k+1})$ (with $\alpha_0 = -\infty$, $\alpha_{n+1} = \infty$). Note that if $A$ is fundamentally reducible, then in the spectral decomposition (4.2) of $f(A)$ we have $S = 0$, and (4.2) can be replaced by the simpler decomposition (4.1) of $A$. If $A - \alpha_k$ is injective for each critical point $\alpha_k$, then Theorem 2 shows that there is only one fundamental symmetry $J$ commuting with $A$, and it follows from (4.3) and the above observations that

$$J = \operatorname{sgn} f(A) = \int_{-\infty}^{\infty} \operatorname{sgn} f(\lambda)\, \mathrm{d}E(\lambda).$$

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*BRIAN W. McENNIS*
*Department of Mathematics,*
*The Ohio State University,*
*Marion, OH 43302,*
*U.S.A.*

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