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INTERPOLATION SPACES BETWEEN A VON NEUMANN ALGEBRA AND ITS PREDUAL

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INTRODUCTION

The theory of non-commutative L^p spaces — the analogs of ordinary Lebesgue spaces $L^p(X, \mu)$ with a non-commutative von Neumann algebra playing the role of $L^\infty(X, \mu)$ — was first developed for semifinite von Neumann algebras by J. Dixmier [7], I. E. Segal [20], and R. A. Kunze [18]. Much later, U. Haagerup presented [12] (cf. also [23]) a theory of L^p spaces associated with not necessarily semifinite von Neumann algebras. Using [5], M. Hilsum [14] has given a spatial realization of these spaces as spaces of (in general unbounded) operators on a Hilbert space H on which the von Neumann algebra M acts. This realization depends on the choice of a n.f.s. weight ψ on the commutant M' of M.

Recently, H. Kosaki [17] has shown that one may take still another point of view. Suppose that φ is a normal faithful functional on M. Then one may inject M into M_* via $x \mapsto x \cdot \varphi$. Now the theory of complex interpolation spaces [3] applies and provides interpolation spaces $C_\theta(M, M_*)$, $0 < \theta < 1$. Kosaki shows, directly by interpolation theory, that these spaces have all the properties that one usually requires for L^p spaces and that they are isomorphic to Haagerup's L^p spaces.

In the present paper, we shall investigate this point of view in the case where φ is only supposed to be a *weight* (normal, faithful and semifinite). The first difficulty arising from the more general situation that we consider is this: we have to find a suitable space in which M and M_* are both continuously embedded. We shall find it convenient to start with the definition of the "intersection" L of M and M_*. The subspace m_φ plays a key role in this construction and we have $m_\varphi \subseteq L$. We next inject M and M_* continuously into the dual L^* of L (L being a Banach space when equipped with the maximum of the norms inherited from M and M_*). The Banach spaces M and M_* are now compatible in the sense of [2, Section 2.3], so that we can define complex interpolation spaces as in [3] or [2, Chapter 4]. For later use, we give a characterization of the elements of $M + M_*$ in L^* and show that the sum norm coincides with the dual norm inherited from L^*.

In the second part of the paper we show in an explicit way that the interpolation spaces thus constructed are isomorphic to Hilsum's (and hence also to Haagerup's) L^p spaces of operators. To do so, we embed the latter into $M + M_*$.

This paper is a revised and shortened version of an earlier manuscript with the same title.

1. INTERPOLATION SPACES BETWEEN M AND M_*

Let M be a von Neumann algebra with a distinguished normal faithful semifinite weight φ.

We shall use the standard notation for the usual objects associated with φ in the Tomita-Takesaki theory such as

$$n_\varphi = \{x \in M \mid \varphi(x^* x) < \infty\},$$

$$m_\varphi = \mathrm{span}\{y^* x \mid x,\ y \in n_\varphi\} = \mathrm{span}\{x \in M_+ \mid \varphi(x) < \infty\},$$

Λ_φ (or Λ) the canonical injection of n_φ into its Hilbert space completion H_φ, π_φ (or π) the canonical representation of M on H_φ, Δ_φ (or Δ) the modular operator in H_φ arising from the left Hilbert algebra $n_\varphi \cap n_\varphi^*$, J_φ (or J) the associated isometric involution in H_φ, $(\sigma_t^\varphi)_{t\in\mathbf{R}}$ the modular automorphism group of M associated with φ.

DEFINITION 1. We denote by L the set of $x \in M$ for which there exists a $\varphi_x \in M_*$ such that

$$\forall\, y, z \in n_\varphi : \langle \varphi_x, z^* y \rangle = (J\pi(x)^* J\Lambda(y) \mid \Lambda(z)). \tag{1}$$

For $x \in L$, we put

$$\|x\|_L := \max\{\|x\|, \|\varphi_x\|\}. \tag{2}$$

Here, $\langle \cdot,\cdot \rangle$ denotes the duality between M_* and M, and $(\cdot \mid \cdot)$ is the scalar product in H_φ.

Given $x \in M$, there is at most one $\varphi_x \in M_*$ satisfying (1) (φ_x is determined by its values on the σ-weakly dense subspace m_φ). Hence $\|\cdot\|_L$ is well-defined. Directly from Definition 1, one easily shows

PROPOSITION 2. *L is a Banach space with the norm $\|\cdot\|_L$. The mappings*

$$x \mapsto x : L \to M \quad and \quad x \mapsto \varphi_x : L \to M_*$$

are linear norm-decreasing injections.

For certain $x \in M$, we can reformulate the expression occurring at the right hand side of (1):

LEMMA 3. *Let* $x, y, z, v, w \in n_\varphi$. *Then*

$$(J\pi(x)^*J\Lambda(y) \mid \Lambda(z)) = (\Lambda(z^*y) \mid J\Lambda(x)), \tag{4}$$

and

$$(J\pi(w^*v)^*J\Lambda(y) \mid \Lambda(z)) = (J\pi(z^*y)^*J\Lambda(v) \mid \Lambda(w)). \tag{5}$$

Proof. Since $x \in n_\varphi$, the element $J\Lambda(x)$ is right bounded with $\pi'(J\Lambda(x)) = J\pi(x)J$. Using this, we get (4):

$$(J\pi(x)^*J\Lambda(y) \mid \Lambda(z)) = (\Lambda(y) \mid \pi'(J\Lambda(x))\,\Lambda(z)) = (\Lambda(y) \mid \pi(z)J\Lambda(x)) =$$
$$= (\Lambda(z^*y) \mid J\Lambda(x)).$$

(5) follows easily from (4). ▨

PROPOSITION 4. *Let* $x \in m_\varphi$. *Then* $x \in L$. *If* $x = \sum_{i=1}^{n} w_i^* v_i$, $v_i, w_i \in n_\varphi$, *then*

$$\varphi_x = \sum_{i=1}^{n} (J\pi(\cdot)^*J\Lambda(v_i) \mid \Lambda(w_i)). \tag{6}$$

Proof. Obviously, this expression does define an element $\varphi_x \in M_*$. By (5), this element satisfies (1). ▨

COROLLARY 5. 1) *L is σ-weakly dense in M.*

2) *L is weakly dense and hence norm dense in M_*.*

By Proposition 4, we can restate (5) as

PROPOSITION 6. *For all* $x, y \in m_\varphi$, *we have*

$$\langle \varphi_x, y\rangle = \langle \varphi_y, x\rangle. \tag{7}$$

We can also characterize the elements of L in such terms (the right hand side of (1) in Definition 1 may now be written $\langle \varphi_{z^*y}, x\rangle$):

PROPOSITION 7. *Let* $x \in M$ *and* $\psi \in M_*$. *Then* $x \in L$ *with* $\varphi_x = \psi$ *if and only if*

$$\forall\, y \in m_\varphi : \langle \varphi_y, x\rangle = \langle \psi, y\rangle. \tag{8}$$

NOTE. 1) Let $x \in m_\varphi$. Then $\varphi_x \in M_*$ as defined here is related to the functional $\beta(x) \in \pi(M)'_*$ considered by Haagerup in [9, Lemma 1.1] by the formula $\varphi_x(y) = \beta(x)\,(J\pi(y)^*J)$, $y \in M$. The mapping $x \mapsto \varphi_x : m_\varphi \to M_*$ has also been considered by M. Walter in [25, Section 3].

2) If φ is a trace, Proposition 7 implies that $L = m_\varphi$. In the general case, we may have $m_\varphi \subsetneq L$.

3) If φ is a functional, we have

$$\forall\, x, y \in M : \langle \varphi_x, y\rangle = (x\xi_\varphi \mid Jy\xi_\varphi) =: s(x, y^*)$$

where ξ_φ is the vector associated with φ by the G.N.S.-construction and s is the self-polar form associated with φ by [4, Théorème 1.3]. (Note that we work with the "symmetric" injection $x \mapsto \varphi_x$ instead of the "left" injection $x \mapsto x\cdot\varphi$ considered by Kosaki [17].)

THEOREM 8. *Let $x \in L$. Then there exists a net $(x_i)_{i\in I}$ in m_φ such that*

(i) $\sup_{i\in I}\|x_i\|_L < \infty$,

(ii) $x_i \to x$ *σ-weakly,*

(iii) $\|\varphi_{x_i} - \varphi_x\| \to 0$.

The proof of Theorem 8 requires some lemmas. The "converse" — even in the following weak form — is much easier: suppose that $x \in M$ is such that for some net $(x_i)_{i\in I}$ in L we have $x_i \to x$ σ-weakly and $(\varphi_{x_i})_{i\in I}$ Cauchy in M_*. Then $x \in L$. Indeed, put $\psi = \lim_{i\in I}\varphi_{x_i}$; then $\langle\varphi_y, x\rangle = \lim_{i\in I}\langle\varphi_y, x_i\rangle = \lim_{i\in I}\langle\varphi_{x_i}, y\rangle = \langle\psi, y\rangle$ for all $y \in m_\varphi$ by Proposition 7; again by Proposition 7, we conclude that $x \in L$.

LEMMA 9. *Let $\delta \in \mathbf{R}_+$. There exists a net $(e_j)_{j\in J}$ of analytic elements of M such that*

(i) $\forall\, \alpha \in \mathbf{C}\quad \forall\, j \in J$: $\sigma^\varphi_\alpha(e_j) \in n_\varphi \cap n_\varphi^*$,

(ii) $\forall\, \alpha \in \mathbf{C}\quad \forall\, j \in J$: $\|\sigma^\varphi_\alpha(e_j)\| \leqslant e^{\delta(\mathrm{Im}\,\alpha)^2}$,

and

(iii) $e_j \to 1$ *strongly.*

Proof. Take by Kaplansky's density theorem a net $(f_j)_{j\in J}$ in $n_\varphi \cap n_\varphi^*$ such that all $\|f_j\| \leqslant 1$ and $f_j \to 1$ strongly. For each $j \in J$, put

$$e_j = \sqrt{\delta/\pi}\int e^{-\delta t^2}\sigma^\varphi_t(f_j)\,\mathrm{d}t.$$

Then the e_j are analytic with

$$\sigma^\varphi_\alpha(e_j) = \sqrt{\delta/\pi}\int e^{-\delta(t-\alpha)^2}\sigma^\varphi_t(f_j)\,\mathrm{d}t, \quad \alpha \in \mathbf{C},$$

and

$$\|\sigma^\varphi_\alpha(e_j)\| \leqslant \sqrt{\delta/\pi}\int |e^{-\delta(t-\alpha)^2}|\,\mathrm{d}t = e^{\delta(\mathrm{Im}\,\alpha)^2}.$$

By [21, p. 272, Corollary] applied to the achieved left Hilbert algebra $\Lambda(n_\varphi \cap n_\varphi^*)$, we have $\sigma_\alpha^\varphi(e_j) \in n_\varphi \cap n_\varphi^*$ for all $\alpha \in \mathbf{C}$ and $j \in J$.

Let $\xi \in H$. Then

$$(e_j\xi|\xi) = \langle \omega_{\xi,\xi}, \sqrt{\delta/\pi}\int e^{-\delta t^2}\sigma_t^\varphi(f_j)dt\rangle =$$

$$= \langle \sqrt{\delta/\pi}\int e^{-\delta t^2}(\omega_{\xi,\xi}\circ\sigma_t^\varphi)dt,\ f_j\rangle \to$$

$$\to \langle \sqrt{\delta/\pi}\int e^{-\delta t^2}(\omega_{\xi,\xi}\circ\sigma_t^\varphi)dt,\ 1\rangle = \|\xi\|^2.$$

Using also that all $\|e_j\| \leqslant 1$, we find that

$$\limsup_{j\in J}\|e_j\xi - \xi\|^2 = \limsup_{j\in J}(\|e_j\xi\|^2 - (e_j\xi|\xi) - (\xi|e_j\xi) + \|\xi\|^2) \leqslant 0.$$

This proves (iii). ▨

LEMMA 10. *Let $\psi \in M_*$ and let $(e_j)_{j\in J}$ be a net in M such that all $\|e_j\| \leqslant 1$ and $e_j \to 1$ strongly. For each $j \in J$, define $\psi_j \in M_*$ by*

$$\psi_j(y) = \psi(e_j^* y e_j), \quad y \in M.$$

Then

$$\|\psi_j - \psi\| \to 0.$$

Proof. Since $\psi \in M_*$, there exist $\xi, \eta \in H_\varphi$ such that $\psi = (\pi(\cdot)\xi|\eta)$. Then $\psi_j = (\pi(\cdot)\pi(e_j)\xi|\pi(e_j)\eta)$, and the result follows. ▨

LEMMA 11. *Let $(x_i)_{i\in I}$ be a $\|\cdot\|_L$-bounded net in L, and let $x \in L$. Suppose that $\|\varphi_{x_i} - \varphi_x\| \to 0$. Then $x_i \to x$ σ-weakly.*

Proof. For all $y \in m_\varphi$, we have

$$\langle \varphi_y, x_i\rangle = \langle \varphi_{x_i}, y\rangle \to \langle \varphi_x, y\rangle = \langle \varphi_y, x\rangle.$$

Since the φ_y, $y \in m_\varphi$, are dense in M_* and $(x_i)_{i\in I}$ is bounded, we conclude that $\langle \psi, x_i\rangle \to \langle \psi, x\rangle$ for all $\psi \in M_*$, i.e. $x_i \to x$ σ-weakly. ▨

Proof of Theorem 8. Let $\delta \in \mathbf{R}_+$. Take $(e_j)_{j\in J}$ as in Lemma 9. For each $j \in J$, put

$$x_j = \sigma_{i/2}^\varphi(e_j)x\sigma_{i/2}^\varphi(e_j)^*. \tag{9}$$

Then $x_j \in m_\varphi$ since $\sigma_{i/2}^\varphi(e_j)^* \in n_\varphi$. By Lemma 9, (ii), we have

$$\forall\, j \in J : \|x_j\| \leqslant e^{\delta/2}\|x\|. \tag{10}$$

Now, let $y \in M$. Then, using that $\Lambda(\sigma^{\varphi}_{i/2}(e_j)^*) = J\Lambda(e_j)$, we find that

$$\langle \varphi_{x_j}, y \rangle = \langle \varphi_y, \sigma^{\varphi}_{i/2}(e_j) x \sigma^{\varphi}_{i/2}(e_j)^* \rangle =$$

$$= (J\pi(y)^* J\Lambda(x\sigma^{\varphi}_{i/2}(e_j)^*) \mid \Lambda(\sigma^{\varphi}_{i/2}(e_j)^*)) =$$

$$= (J\pi(y)^* J\pi(x) J\Lambda(e_j) \mid J\Lambda(e_j)) =$$

$$= (J\pi(x)^* J\pi(y)\Lambda(e_j) \mid \Lambda(e_j)) = \langle \varphi_x, e_j^* y e_j \rangle,$$

i.e.

$$\varphi_{x_j}(y) = \varphi_x(e_j^* y e_j), \quad y \in M. \tag{11}$$

By Lemma 10 we then conclude that

$$\|\varphi_{x_j} - \varphi\| \to 0.$$

It also follows that

$$\forall j \in J : \|\varphi_{x_j}\| \leqslant \|\varphi_x\| \, \|e_j\|^2 \leqslant \|\varphi_x\|. \tag{12}$$

In all, we have shown (i) and (iii) of Theorem 8. Finally (ii) follows by Lemma 11. ▨

REMARK. Note that by (10) we have actually proved the following sharpened version of Theorem 8: Let $x \in L$ and $\varepsilon \in \mathbf{R}_+$. Then there exists a net $(x_i)_{i \in I}$ in m_φ satisfying (i)–(iii) and such that all $\|x_i\| \leqslant (1 + \varepsilon)\|x\|$.

An important application of Theorem 8 is this:

COROLLARY 12. *For all $x, y \in L$, we have*

$$\langle \varphi_x, y \rangle = \langle \varphi_y, x \rangle.$$

Proof. Take $(x_i)_{i \in I}$ as in Theorem 8. Now by Proposition 7 we have

$$\langle \varphi_{x_i}, y \rangle = \langle \varphi_y, x_i \rangle$$

for all $i \in I$. The result follows by passing to the limit. ▨

We now pass to a discussion of certain subspaces of the dual L^* of the Banach space $(L, \|\cdot\|_L)$.

By transposition of the norm-decreasing injections $L \hookrightarrow M$ and $L \hookrightarrow M_*$ considered in Proposition 2, we obtain norm-decreasing injections $M \hookrightarrow L^*$ and $M_* \hookrightarrow L^*$ given by

$$\langle x, y \rangle_{L^*, L} = \langle \varphi_y, x \rangle_{M_*, M}, \quad y \in L, \tag{13}$$

for all $x \in M$ and

$$\langle \psi, y \rangle_{L^*, L} = \langle \psi, y \rangle_{M_*, M}, \quad y \in L, \tag{14}$$

for all $\psi \in M_*$. (The injectivity follows from Corollary 5.)

Note that the diagram

$$\text{(15)} \qquad \begin{array}{ccccc} & \nearrow & M & \searrow & \\ L & & & & L^* \\ & \searrow & M_* & \nearrow & \end{array}$$

commutes since for all $x \in L$, we have

$$\forall y \in L : \langle x, y \rangle_{L^*, L} = \langle \varphi_y, x \rangle_{M_*, M} =$$

$$= \langle \varphi_x, y \rangle_{M_*, M} = \langle \varphi_x, y \rangle_{L^*, L}.$$

Also note that L is precisely the intersection of M and M_* when these spaces are considered as subspaces of L^*: if $x \in M$ and $\psi \in M_*$ are identical as elements of L^* we have

$$\forall y \in L : \langle \varphi_y, x \rangle_{M_*, M} = \langle x, y \rangle_{L^*, L} =$$

$$= \langle \psi, y \rangle_{L^*, L} = \langle \psi, y \rangle_{M_*, M},$$

whence $x \in L$ by Proposition 7.

We have now turned (M, M_*) into a compatible pair of Banach spaces in the sense of [2, Section 2.3]. Before we go on to define interpolation spaces in this situation, we shall give a useful characterization of $M + M_*$ (Theorem 14 below).

As a Banach space, L is isomorphic to the closed subspace $\{(x, \varphi_x) \mid x \in L\}$ of the Banach space $(M \times M_*, \|\cdot\|_{\max})$ where $\|(x, \psi)\|_{\max} = \max\{\|x\|, \|\psi\|\}$. On $M \times M_*$ we may also consider the product of the σ-weak topology on M with the norm topology on M_*. The topology on L induced by this will be called the σ-w/$\|\cdot\|$-topology. A net $(x_i)_{i \in I}$ in L converges to $x \in L$, σ-w/$\|\cdot\|$, precisely if $x_i \to x$ σ-weakly and $\varphi_{x_i} \to \varphi_x$ in the norm of M_*.

Note that, with this terminology, L is the σ-w/$\|\cdot\|$-closure of m_φ (by Theorem 8 and the remarks following it).

DEFINITION 13. Denote by V the linear space of linear functionals on L that are σ-w/$\|\cdot\|$-continuous on $\|\cdot\|_L$-bounded subsets of L.

We equip V with the norm $\|\cdot\|_V$ inherited from L^* (that actually $V \subseteq L^*$ follows from the fact that any $\|\cdot\|_L$-convergent sequence in L is automatically $\|\cdot\|_L$-bounded and σ-w/$\|\cdot\|$-convergent). Note that V is a Banach space (this can be proved directly; it also follows from the following characterization of V as the sum of M and M_*, cf. [2, 2.3.1 Lemma]).

THEOREM 14. *Let $\chi \in L^*$. Then the following assertions are equivalent:*

(i) $\chi \in M + M_*$,

(ii) χ *is* σ-w/$\|\cdot\|$-*continuous,*

(iii) $\chi \in V$.

If $\chi \in V$, *we have*

(16) $$\|\chi\|_V = \inf\{\|x\| + \|\psi\| \mid \chi = x + \psi,\ x \in M,\ \psi \in M_*\}.$$

The proof will be based on the following lemmas.

LEMMA 15. *Let* $\chi \in V$ *and* $e \in n_\varphi$. *Define* $\psi : M \to \mathbf{C}$ *by*

$$\psi(y) = \chi(e^*ye), \quad y \in M.$$

Then $\psi \in M_*$.

Proof. First note that $e^*ye \in n_\varphi$ so that the definition of ψ as a linear functional on M makes sense.

To prove that $\psi \in M_*$, it suffices by [8, I, §3, Théorème 1] to show that ψ is σ-strongly continuous on the unit ball of M. So let $y \in M$ with $\|y\| \leqslant 1$, let $(y_i)_{i\in I}$ be a net in M with all $\|y_i\| \leqslant 1$, and suppose that $y_i \to y$ σ-strongly. We claim that then

(17) $$e^*y_ie \to e^*ye \quad \sigma\text{-w}/\|\cdot\|.$$

To see this, first note that

$$\langle \varphi_{e^*y_ie}, x\rangle = (J\pi(x)^*J\Lambda(y_ie) \mid \Lambda(e)) =$$

$$= (J\pi(x)^*J\pi(y_i)\Lambda(e) \mid \Lambda(e)), \quad x \in M,$$

and similarly

$$\langle \varphi_{e^*ye}, x\rangle = (J\pi(x)^*J\pi(y)\Lambda(e) \mid \Lambda(e)), \quad x \in M.$$

Since $\pi(y_i) \to \pi(y)$ strongly, it follows that

$$\|\varphi_{e^*y_ie} - \varphi_{e^*ye}\| \to 0.$$

Since all $\|e^*y_ie\| \leqslant \|e\|^2$ and all $\|\varphi_{e^*y_ie}\| \leqslant \|\Lambda(e)\|^2$, we have

(18) $$\sup_{i\in I}\|e^*y_ie\|_L < \infty.$$

By Lemma 11 it follows that

$$e^*y_ie \to e^*ye \quad \sigma\text{-weakly}.$$

In all, we have proved (17).

Since $\chi \in V$, (17) and (18) imply

$$\chi(e^*y_ie) \to \chi(e^*ye),$$

i.e.

$$\psi(y_i) \to \psi(y). \qquad \blacksquare$$

By M^* we denote the dual Banach space of M. The next lemma shows that a sufficient condition for an element $\chi \in V$ to be in M_* is that it agrees with some $\psi \in M^*$.

LEMMA 16. *Let $\chi \in V$. Suppose that for some $\psi \in M^*$ we have*

$$\forall y \in m_\varphi : \langle \chi, y \rangle = \langle \psi, y \rangle.$$

Then there exists a $\tilde{\psi} \in M_*$ with $\|\tilde{\psi}\| \leqslant \|\psi\|$ such that

$$\forall y \in L : \langle \chi, y \rangle = \langle \tilde{\psi}, y \rangle.$$

Proof. Let $(e_i)_{i \in I}$ be an approximate identity for n_φ contained in $(m_\varphi)_+$ [21, 3.21 Proposition].

For each $i \in I$, define ψ_i by

$$\psi_i(y) = \psi(e_i y e_i), \quad y \in M.$$

Since ψ and χ agree on m_φ, we have $\psi_i(y) = \chi(e_i y e_i)$ for all $y \in M$, and hence $\psi_i \in M_*$ by Lemma 15. Obviously, $\|\psi_i\| \leqslant \|\psi\|$.

Now let ρ be a representation of the C^*-algebra $\overline{m_\varphi}$ (where $^-$ means norm closure) on a Hilbert space H_ρ such that

$$\psi(y) = (\rho(y)\,\xi | \eta)_{H_\rho}, \quad y \in \overline{m_\varphi},$$

for some $\xi, \eta \in H_\rho$. Then

$$\psi_i(y) = (\rho(e_i y e_i)\xi \mid \eta)_{H_\rho} = (\rho(y)\,\rho(e_i)\xi | \rho(e_i)\eta)_{H_\rho}, \quad y \in \overline{m_\varphi}.$$

Since $(e_i)_{i \in I}$ is an approximate identity for the C^*-algebra $\overline{m_\varphi}$, we have $\rho(e_i) \to 1$ strongly in H_ρ. It follows that

$$\psi_i | \overline{m_\varphi} \to \psi | \overline{m_\varphi}$$

with respect to the norm in the dual space of $\overline{m_\varphi}$. Now for functionals in M_*, this norm agrees with $\|\cdot\|_{M_*}$. Thus $(\psi_i)_{i \in I}$ is a Cauchy net in M_*, and hence converges to some $\tilde{\psi} \in M_*$.

For all $y \in m_\varphi$, we have

$$\langle \tilde{\psi}, y \rangle = \lim_{i \in I} \langle \psi_i, y \rangle = \langle \psi, y \rangle.$$

Hence

$$\|\tilde{\psi}\| = \sup\{|\langle \tilde{\psi}, y \rangle| \mid y \in m_\varphi, \|y\| \leqslant 1\} \leqslant \|\psi\|$$

and

$$\forall y \in m_\varphi : \langle \chi, y \rangle = \langle \tilde{\psi}, y \rangle.$$

Finally, let $y \in L$. Take $(y_j)_{j\in J}$ in m_φ such that $(y_j)_{j\in J}$ is $\|\cdot\|_L$-bounded, $y_j \to y$ σ-weakly and $\|\varphi_{y_j} - \varphi_y\| \to 0$ (Theorem 8). Then

$$\langle \chi, y\rangle = \lim_{j\in J}\langle \chi, y_j\rangle = \lim_{j\in J}\langle \tilde{\psi}, y_j\rangle = \langle \tilde{\psi}, y\rangle.$$

This completes the proof. ▨

Lemma 16 in particular applies to χ's of the form $x \in M$ (for such χ, Lemma 15 could be proved easier: obviously $y \mapsto \psi(y) = \langle x, e^*ye\rangle = \langle \varphi_{e^*ye}, x\rangle = (J\pi(x)^*J\pi(y)\Lambda(e) \mid \Lambda(e))$ is in M_*). Using this, we get the following characterization of the elements of L:

COROLLARY 17. *Let $x \in M$. Then $x \in L$ if and only if there exists a constant $C \geqslant 0$ such that*

$$\forall\, y \in m_\varphi : |\langle \varphi_y, x\rangle| \leqslant C\|y\|. \tag{19}$$

Proof. If $x \in L$, then $|\langle \varphi_y, x\rangle| = |\langle \varphi_x, y\rangle| \leqslant \|\varphi_x\|\,\|y\|$ for all $y \in m_\varphi$. Conversely, suppose that (19) holds. Then by the Hahn-Banach theorem there exists a bounded functional ψ on M extending $y \mapsto \langle \varphi_y, x\rangle : m_\varphi \to \mathbf{C}$, i.e. a $\psi \in M^*$ such that

$$\forall\, y \in m_\varphi : \langle \varphi_y, x\rangle = \langle \psi, y\rangle.$$

Applying Lemma 16 to $x \in M \subseteq V$ and $\psi \in M^*$, we get an element $\tilde{\psi} \in M_*$ such that

$$\forall\, y \in m_\varphi : \langle \varphi_y, x\rangle = \langle \tilde{\psi}, y\rangle.$$

Hence $x \in L$ by Proposition 7. ▨

Proof of Theorem 14. (i) $\Rightarrow$ (ii): Suppose that $\chi = x + \psi$, $x \in M$, $\psi \in M_*$, and let $(y_i)_{i\in I}$ be a net in L converging to $y \in L$ with respect to σ-w/$\|\cdot\|$. Then $\langle \varphi_{y_i}, x\rangle \to \langle \varphi_y, x\rangle$ and $\langle \psi, x_i\rangle \to \langle \psi, x\rangle$, whence $\langle \chi, y_i\rangle_{L^*,L} \to \langle \chi, y\rangle$. Hence χ is σ-w/$\|\cdot\|$-continuous. Also note that for all $y \in L$, we have

$$|\langle \chi, y\rangle| \leqslant |\langle \varphi_y, x\rangle| + |\langle \psi, y\rangle| \leqslant \|y\|_L\|x\| + \|\psi\|\,\|y\|_L,$$

so that $\|\chi\|_V \leqslant \|x\| + \|\psi\|$.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (i): Suppose that $\chi \in V$. Then in particular χ is a bounded linear functional on $(L, \|\cdot\|_L)$. We identify L with the closed subspace $\{(x, \varphi_x) \mid x \in L\}$ of $(M \times M_*, \|\cdot\|_{\max})$. Then by the Hahn-Banach theorem, χ extends to a bounded linear functional Φ on $(M \times M_*, \|\cdot\|_{\max})$ with the same norm: $\|\Phi\| = \|\chi\|_V$.

Now the dual of $(M \times M_*, \|\cdot\|_{\max})$ is naturally identified with $(M^* \times M, \|\cdot\|_{\text{sum}})$ where $\|(\psi, x)\|_{\text{sum}} = \|\psi\| + \|x\|$ for all $(\psi, x) \in M^* \times M$. Thus there exist $\psi \in M^*$ and $x \in M$ such that

$$\Phi = (\psi, x) \quad \text{and} \quad \|\Phi\| = \|\psi\| + \|x\|.$$

In particular, we have for all $y \in L$

$$\langle \chi, y \rangle = \langle \Phi, (y, \varphi_y) \rangle = \langle (\psi, x), (y, \varphi_y) \rangle = \langle \psi, y \rangle + \langle \varphi_y, x \rangle.$$

Viewing $x \in M$ as an element of V we can now apply Lemma 16 to $\chi - x \in V$ and $\psi \in M^*$. We obtain a $\tilde{\psi} \in M_*$ with $\|\tilde{\psi}\| \leqslant \|\psi\|$ such that

$$\forall y \in L : \langle \chi, y \rangle - \langle \varphi_y, x \rangle = \langle \tilde{\psi}, y \rangle.$$

Thus $\chi = x + \tilde{\psi} \in M + M_*$ and

$$\|x\| + \|\tilde{\psi}\| \leqslant \|x\| + \|\psi\| = \|\Phi\| = \|\chi\|_V.$$

In all, we have proved the equivalence of (i), (ii), and (iii), and at the same time we have shown that

$$\|\chi\|_V = \min\{\|x\| + \|\psi\| \mid \chi = x + \psi,\ x \in M,\ \psi \in M_*\}$$

for all $\chi \in V$. ▨

Finally, we shall introduce the complex interpolation spaces between M and M_* following [3] and [2, Chapter 4]. First we introduce some notation. We write $\|\cdot\|_\infty$ and $\|\cdot\|_1$ for the norms in M and M_*, respectively, and put $S = \{\alpha \in \mathbf{C} \mid 0 \leqslant \operatorname{Re}\alpha \leqslant 1\}$. We denote by $\mathscr{F}(M; M_*)$ the set of functions $f : S \to V$ such that

(i) f is bounded,

(ii) f is analytic in S^0 and continuous on S,

(iii)$_0$ $\forall t \in \mathbf{R} : f(\mathrm{i}t) \in M$ and
$t \mapsto f(\mathrm{i}t) : \mathbf{R} \to M$ is continuous and bounded,

(iii)$_1$ $\forall t \in \mathbf{R} : f(1 + \mathrm{i}t) \in M_*$ and
$t \mapsto f(1 + \mathrm{i}t) : \mathbf{R} \to M_*$ is continuous and bounded.

For $f \in \mathscr{F}(M; M_*)$, put

(20) $$|||f||| = \max\{\sup_{t \in \mathbf{R}} \|f(\mathrm{i}t)\|_\infty, \sup_{t \in \mathbf{R}} \|f(1 + \mathrm{i}t)\|_1\}.$$

Then $(\mathscr{F}(M; M_*), |||\cdot|||)$ is a Banach space. Note that

(21) $$\forall \alpha \in S : \|f(\alpha)\|_V \leqslant |||f|||.$$

We denote by $\mathscr{F}_0(M; M_*)$ the closed subspaces of $\mathscr{F}(M; M_*)$ consisting of those f for which also

(iii)$'_0$ $\|f(\mathrm{i}t)\|_\infty \to 0$ as $|t| \to \infty$,

(iii)$'_1$ $\|f(1 + \mathrm{i}t)\|_1 \to 0$ as $|t| \to \infty$,

(this is the space considered in [2, Section 4.1]).

DEFINITION 18. For each $p \in]1, \infty[$, we denote by V_p the complex interpolation space corresponding to $1/p \in]0, 1[$, i.e.

$$V_p := \{f(1/p) \mid f \in \mathscr{F}_0(M; M_*)\}$$

with the norm

$$\|\chi\|_{V_p} := \inf\{|||f||| \mid f(1/p) = \chi, f \in \mathscr{F}_0(M; M_*)\}, \quad \chi \in V_p.$$

2. L^p SPACES ASSOCIATED WITH M AS INTERPOLATION SPACES

We still consider a von Neumann algebra M with a distinguished normal faithful semifinite weight φ. From now on, we further assume that M is represented on a Hilbert space H and that we have given a normal faithful semifinite weight ψ on the commutant M' of M. We put

$$d := \frac{d\varphi}{d\psi},$$

i.e. d is the spatial derivative of φ with respect to ψ [5] (or [23]).

2.1. THE SPACES $L^p(\psi)$. We denote by $D(H, \psi)$ the set of ψ-bounded vectors, i.e. the set of $\xi \in H$ for which there exists a bounded operator $R^\psi(\xi) : H_\psi \to H$ satisfying $\forall y \in n_\psi : R^\psi(\xi)\Lambda_\psi(y) = y\xi$. For $\xi, \eta \in D(H, \psi)$, we write $\theta_{\xi,\eta} := R^\psi(\xi)R^\psi(\eta)^* \in M$.

By $L^p(\psi)$, $1 \leqslant p \leqslant \infty$, we denote the *spatial L^p space with respect to ψ*, i.e. $L^p(\psi)$ is the space of closed densely defined operators a on H that are $(-1/p)$-homogeneous with respect to ψ and such that $\int |a|^p d\psi < \infty$, equipped with the norm $\|\cdot\|_p := \int |\cdot|^p d\psi$ (if $p = \infty$, $L^\infty(\psi) = M$ with the usual operator norm). For the properties of the Banach spaces $L^p(\psi)$, we refer to [14].

The definition of homogeneity with respect to ψ given in [5] is equivalent to the following:

DEFINITION 19. A closed densely defined operator a is *γ-homogeneous*, where $\gamma \in \mathbf{R}$, if and only if

$$ya \subseteq a\sigma^\psi_{i\gamma}(y) \tag{22}$$

for all $y \in M'$ analytic with respect to σ^ψ.

The advantage of this characterization, which is similar to [22, Definition at the beginning of Section 2], is that one can easily handle sums, products, and adjoints of homogeneous operators. To prove the equivalence of the two definitions, one can proceed as in the proof of [16, Lemma 2.1]; the main idea for the proof is the use of Carlson's theorem for analytic functions.

We shall need the following criterion for integrability:

Proposition 20. *Let a be a positive self-adjoint (-1)-homogeneous operator. Suppose that for some constant $C \geqslant 0$ we have*

$$\sum_{j=1}^{n} (a\xi_j|\xi_j) \leqslant C \Big\| \sum_{j=1}^{n} \theta_{\xi_j,\xi_j} \Big\| \tag{23}$$

for all $n \in \mathbf{N}$ and $\xi_1, \ldots, \xi_n \in D(H, \psi) \cap D(a)$. Then a is integrable and

$$\int a \, \mathrm{d}\psi \leqslant C.$$

Proof. Let χ be the normal semifinite weight on M such that $a = \frac{\mathrm{d}\chi}{\mathrm{d}\psi}$ (see [5, Theorem 13]). We shall prove that

$$\sum_{j=1}^{n} \chi(\theta_{\xi_j,\xi_j}) \leqslant C \Big\| \sum_{j=1}^{n} \theta_{\xi_j,\xi_j} \Big\| \tag{24}$$

for all $n \in \mathbf{N}$ and all $\xi_1, \ldots, \xi_n \in D(H, \psi)$ (by [5, Corollary 18], this implies the desired result).

By the hypothesis, we know that (24) holds whenever $\xi_1, \ldots, \xi_n \in D(H, \psi) \cap \cap D(a)$. Now let $\xi_1, \ldots, \xi_n \in D(H, \psi)$. For each $k \in \mathbf{N}$, put

$$\xi_j^{(k)} = \int h_k(t) \, (a^{it}\xi_j + (1 - p_a)\, \xi_j) \, \mathrm{d}t, \tag{25}$$

where $h_k(\alpha) = \sqrt{n/\pi} \exp(-k\alpha^2)$, $\alpha \in \mathbf{C}$, and p_a is the projection ($\in M$) onto the support of a. Then $\xi_j^{(k)} \in D(H, \psi) \cap D(a)$ and $\xi_j^{(k)} \to \xi_j$ (cf. [14, Proposition 2]), and we conclude by the lower semicontinuity of $\xi \mapsto \varphi(\theta_{\xi,\xi})$ [5, Lemma 6] that

$$\sum_{j=1}^{n} \chi(\theta_{\xi_j,\xi_j}) \leqslant \liminf_{k \in \mathbf{N}} \sum_{j=1}^{n} \chi(\theta_{\xi_j^{(k)},\xi_j^{(k)}}). \tag{26}$$

On the other hand,

$$\Big\| \sum_{j=1}^{n} \theta_{\xi_j^{(k)},\xi_j^{(k)}} \Big\| \leqslant \Big\| \sum_{j=1}^{n} \theta_{\xi_j,\xi_j} \Big\| \tag{27}$$

for all $k \in \mathbf{N}$. Indeed, for each $t, s \in \mathbf{R}$, put

$$x_{t,s} = \sum_{j=1}^{n} R^{\psi}(a^{it}\xi_j + (1 - p_a)\, \xi_j) \, R^{\psi}(a^{is}\, \xi_j + (1 - p_a)\, \xi_j)^*$$

and calculate

$$x_{t,s} = \sum_{j=1}^{n} (a^{it} R^{\psi}(\xi_j) \Delta_{\psi}^{it} + (1-p_a) R^{\psi}(\xi_j)) (\Delta_{\psi}^{-is} R^{\psi}(\xi_j)^* a^{-is} + R^{\psi}(\xi_j)^*(1-p_a))^* =$$

$$= a^{it}\left(\sum_{j=1}^{n} R^{\psi}(\xi_j)\, \Delta_{\psi}^{it}\Delta_{\psi}^{-is} R^{\psi}(\xi_j)^*\right)a^{-is} +$$

$$+ a^{it}\left(\sum_{j=1}^{n} R^{\psi}(\xi_j)\, \Delta_{\psi}^{it} R^{\psi}(\xi_j)^*\right)(1-p_a) +$$

$$+ (1-p_a)\left(\sum_{j=1}^{n} R^{\psi}(\xi_j)\, \Delta_{\psi}^{-is} R^{\psi}(\xi_j)^*\right)a^{-is} +$$

$$+ (1-p_a)\left(\sum_{j=1}^{n} R^{\psi}(\xi_j)\, R^{\psi}(\xi_j)^*\right)(1-p_a).$$

Now, using the inequality (28) below, we get

$$\left\| p_a\, a^{it}\left(\sum_{j=1}^{n} R^{\psi}(\xi_j)\, \Delta_{\psi}^{it-is} R^{\psi}(\xi_j)^*\right)a^{-is} p_a \right\| \leqslant$$

$$\leqslant \left\| \sum_{j=1}^{n} R^{\psi}(\xi_j)\, \Delta_{\psi}^{it-is}\, R^{\psi}(\xi_j)^* \right\| \leqslant \left\| \sum_{j=1}^{n} R^{\psi}(\xi_j)\, R^{\psi}(\xi_j)^* \right\|$$

and similarly for the other terms, so that in all

$$\|x_{t,s}\| \leqslant \left\| \sum_{j=1}^{n} R^{\psi}(\xi_j)\, R^{\psi}(\xi_j)^* \right\| = \left\| \sum_{j=1}^{n} \theta_{\xi_j, \xi_j} \right\|.$$

Finally, (27) follows:

$$\left\| \sum_{j=1}^{n} \theta_{\xi_j^{(k)}, \xi_j^{(k)}} \right\| = \left\| \sum_{j=1}^{n} R^{\psi}((\xi_j)^{(k)})\, R^{\psi}(\xi_j^{(k)})^* \right\| =$$

$$= \left\| \sum_{j=1}^{n} \int h_k(t) \left(\int h_k(s)\, R^{\psi}(a^{it}\xi_j + (1-p_a)\xi_j)\, R^{\psi}(a^{is}\xi_j + (1-p_a)\,\xi_j)^* \, \mathrm{d}s \right) \mathrm{d}t \right\| \leqslant$$

$$\leqslant \iint h_k(t) h_k(s) \|x_{t,s}\| \,\mathrm{d}s\mathrm{d}t \leqslant$$

$$\leqslant \iint h_k(t) h_k(s) \left\| \sum_{j=1}^{n} \theta_{\xi_j, \xi_j} \right\| \mathrm{d}s\mathrm{d}t = \left\| \sum_{j=1}^{n} \theta_{\xi_j, \xi_j} \right\|.$$

Combining (26) with (27), we get (24) for all $\xi_1, \ldots, \xi_n \in D(H, \psi)$ as wanted.

We have used the operator norm inequality

$$\left\| \sum_{j=1}^{n} x_j y x_j^* \right\| \leqslant \|y\| \left\| \sum_{j=1}^{n} x_j x_j^* \right\|. \tag{28}$$

For $n = 1$, (28) is well-known: $\|xyx^*\| \leqslant \|x\| \|y\| \|x\| = \|y\| \|xx^*\|$. For higher n, we have

$$\left\|\sum_{j=1}^{n} x_j y x_j^*\right\| = \left\|\begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & y \end{pmatrix}\begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^*\right\| \leqslant$$

$$\leqslant \left\|\begin{pmatrix} y & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & y \end{pmatrix}\right\| \left\|\begin{pmatrix} x_1 & \cdots & x_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} x & \cdots & x_n \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^*\right\| = \|y\| \left\|\sum_{j=1}^{n} x_j x_j^*\right\|. \qquad ▨$$

2.2. AN ISOMORPHISM $\mathscr{P} : H_\varphi \to L^2(\psi)$.

LEMMA 21. *Let* $x \in n_\varphi$. *Then*

$$D(H, \psi) \subseteq D(d^{1/2}x^*). \tag{29}$$

Furthermore, $d^{1/2}x^* \in L^2(\psi)$ *with*

$$\|d^{1/2}x^*\|_2 = \|\Lambda_\varphi(x)\|. \tag{30}$$

Proof. Recall that for $\eta \in D(H, \psi)$ we have by the definition of $\frac{d\varphi}{d\psi}$ that $\eta \in D\left(\left(\frac{d\varphi}{d\psi}\right)^{1/2}\right)$ if and only if $\varphi(\theta_{\eta,\eta}) < \infty$. Now let $\xi \in D(H, \psi)$. Then $x^*\xi \in D(H, \psi)$ and

$$\varphi(\theta_{x^*\xi,\, x^*\xi}) = \varphi(x^*\theta_{\xi,\xi}x) \leqslant \|\theta_{\xi,\xi}\| \varphi(x^*x) < \infty.$$

Thus $x^*\xi \in D\left(\left(\frac{d\varphi}{d\psi}\right)^{1/2}\right)$, i.e. $\xi \in D(d^{1/2}x^*)$.

Next, we note that

$$\int \frac{d(x\cdot\varphi\cdot x^*)}{d\psi}\, d\psi = (x\cdot\varphi\cdot x^*)(1) = \varphi(x^*x) = \|\Lambda_\varphi(x)\|^2. \tag{31}$$

Hence $\frac{d(x\cdot\varphi\cdot x^*)}{d\psi}$ is integrable. Now

$$|d^{1/2}x^*| = \left(\frac{d(x\cdot\varphi\cdot x^*)}{d\psi}\right)^{1/2}. \tag{32}$$

Indeed, for all $\xi \in D(H, \psi)$, we have

$$\|\, |d^{1/2}x^*|\xi\|^2 = \left\|\left(\frac{d\varphi}{d\psi}\right)^{1/2} x^*\xi\right\|^2 = \varphi(\theta_{x^*\xi,\, x^*\xi}) =$$

$$= \varphi(x^*\theta_{\xi,\xi}x) = (x\cdot\varphi\cdot x^*)(\theta_{\xi,\xi}) = \left\|\left(\frac{d(x\cdot\varphi\cdot x^*)}{d\psi}\right)^{1/2}\xi\right\|^2;$$

and $d^{1/2}x^*$ is $(-1/2)$-homogeneous ($d^{1/2}$, and hence also $d^{1/2}x^*$, satisfies the hypothesis of Definition 19), whence $d^{1/2}x^*$, and thus also $|d^{1/2}x^*|$, is the closure of its restriction to $D(H, \psi)$ (cf. [14, Proposition 2]). By the definition of spatial derivatives we then have (32).

In all, we have shown that $d^{1/2}x^* \in L^2(\psi)$ and

$$\| d^{1/2}x^* \|_2^2 = \int \frac{\mathrm{d}(x\cdot\varphi\cdot x^*)}{\mathrm{d}\psi}\,\mathrm{d}\psi = \| \Lambda_\varphi(x)\|^2. \quad ▨$$

Recall [5, Theorem 9, (1)] that $\sigma_t^\varphi(x) = d^{it}xd^{-it}$ for all $x \in M$ and $t \in \mathbf{R}$. Using this, one can prove the following

LEMMA 22. *Let x be an element of M, analytic with respect to σ^φ. Then for all $\alpha \in \mathbf{C}$ with* $\mathrm{Re}\alpha \geqslant 0$, *we have*

$$xd^\alpha \subseteq d^\alpha \sigma_{i\alpha}^\varphi(x).$$

Now we are ready to prove the main theorem of this section:

THEOREM 23. 1) *Let $x \in n_\varphi$. Then $xd^{1/2}$ is preclosed, its closure $[xd^{1/2}]$ is in $L^2(\psi)$, and*

$$\| [xd^{1/2}] \|_2 = \| \Lambda_\varphi(x) \|.$$

2) *The mapping $x \mapsto [xd^{1/2}] : n_\varphi \to L^2(\psi)$ extends to a linear isometry*

$$\mathscr{P} : H_\varphi \to L^2(\psi)$$

of H_φ onto $L^2(\psi)$.

3) *For all $\xi \in H_\varphi$, we have*

$$\mathscr{P}(J_\varphi\xi) = \mathscr{P}(\xi)^*.$$

Proof. 1) follows immediately from Lemma 21 and the fact that $(xd^{1/2})^* = d^{1/2}x^*$.

2) The mapping $x \mapsto [xd^{1/2}]$ is linear. Indeed, for all $x, y \in n_\varphi$ we have $[d^{1/2}x^* + d^{1/2}y^*] = d^{1/2}(x^* + y^*)$ since these operators agree on $D(H, \psi)$; and the mapping $a \mapsto a^* : L^2(\psi) \to L^2(\psi)$ is conjugate linear.

Denote by $\mathscr{P} : H_\varphi \to L^2(\psi)$ the unique linear isometric extension of $x \mapsto [xd^{1/2}] : n_\varphi \to L^2(\psi)$. Let us show that $\mathscr{P}$ is surjective. Suppose that for some $a \in L^2(\psi)$, we have

$$\forall\ x \in n_\varphi : \int [xd^{1/2}]\cdot a^*\mathrm{d}\psi = 0.$$

Then for all $\xi, \eta \in D(H, \psi)$, we have $\theta_{\xi,\eta}x \in n_\varphi$ and thus

$$(a^*\xi | d^{1/2}x^*\eta) = \int a^* \cdot \theta_{\xi,\eta} \cdot (d^{1/2}x^*)^* \mathrm{d}\psi = \int a^* \cdot [\theta_{\xi,\eta}xd^{1/2}]\mathrm{d}\psi = 0.$$

Now the set

$$\{d^{1/2}x^*\eta \mid \eta \in D(H, \psi),\ \ x \in n_\varphi\} \tag{33}$$

is dense in H (to see this take a net $(x_i)_{i\in I}$ of analytic elements of M satisfying $\sigma^{\varphi}_{\alpha}(x_i)\in n_\varphi \cap n_\varphi^*$ for all $\alpha\in \mathbf{C}$ such that $x_i^*\to 1$ strongly; then by Lemma 22, we have

$$d^{1/2}\sigma^{\varphi}_{-i/2}(x_i)^*\eta = d^{1/2}\sigma^{\varphi}_{i/2}(x_i^*)\eta = x_i^* d^{1/2}\eta \to d^{1/2}\eta;$$

hence (33) is dense in $\{d^{1/2}\eta \mid \eta\in D(H,\psi)\cap D(d^{1/2})\}$, which is dense in H because $d^{1/2}$ is $(-1/2)$-homogeneous). We conclude that $a^* = [a^*|_{D(H,\psi)}] = 0$, whence $a=0$. We have proved that $\mathscr{P}(n_\varphi)$ is dense in $L^2(\psi)$. Since $\mathscr{P}$ is isometric, it follows that $\mathscr{P}(H_\varphi) = L^2(\psi)$.

3) Since both sides of the equality to be proved are continuous as functions of $\xi\in H_\varphi$, we need only consider ξ having the form $\xi = \Lambda_\varphi(x)$ with $x\in n_\varphi\cap n_\varphi^*$ and analytic. In this case, $J_\varphi \Lambda_\varphi(x) = \Lambda_\varphi(\sigma^{\varphi}_{-i/2}(x^*))$ so that by Lemma 22

$$\mathscr{P}(J_\varphi\Lambda_\varphi(x)) = [\sigma^{\varphi}_{-i/2}(x^*)d^{1/2}] \subseteq d^{1/2}\sigma^{\varphi}_{i/2}(\sigma^{\varphi}_{-i/2}(x^*)) =$$
$$= d^{1/2}x^* = [xd^{1/2}]^* = \mathscr{P}(\Lambda_\varphi(x))^*.$$

Theorem 23 is a generalization of [22, Theorem 3.1].

REMARK. Denote by P_φ the usual self-dual cone in H_φ [10, Section 1]. Since both $(M, H_\varphi, J_\varphi, P_\varphi)$ and $(M, L^2(\psi), {}^*, L^2(\psi)_+)$ are standard forms of M (in the sense of [10, Definition 2.1]) we know by [10, Theorem 2.3] that there is a unique unitary $u\colon H_\varphi\to L^2(\psi)$ carrying $(M, H_\varphi, J_\varphi, P_\varphi)$ onto $(M, L^2(\psi), {}^*, L^2(\psi)_+)$. This unitary is exactly $\mathscr{P}\colon H_\varphi\to L^2(\psi)$ since by Theorem 23, $\mathscr{P}$ has the properties that characterize u.

THEOREM 24. *Let $x\in L$ and $\xi,\eta\in D(H,\psi)\cap D(d^{1/2})$. Then*

$$\varphi_x(\theta_{\xi,\eta}) = (xd^{1/2}\xi \mid d^{1/2}\eta). \tag{34}$$

Proof. For all $y, z\in n_\varphi$, we have

$$\varphi_{z^*y}(\theta_{\xi,\eta}) = (J_\varphi\pi_\varphi(\theta_{\xi,\eta})^*J_\varphi\Lambda_\varphi(y)\mid \Lambda_\varphi(z))_{H_\varphi} =$$
$$= (\pi_\varphi(\theta_{\xi,\eta})J_\varphi\Lambda_\varphi(z)\mid J_\varphi\Lambda_\varphi(y))_{H_\varphi} = (\theta_{\xi,\eta}\cdot\mathscr{P}(J_\varphi\Lambda_\varphi(z))\mid \mathscr{P}(J_\varphi\Lambda_\varphi(y)))_{L^2(\psi)} =$$
$$= (\theta_{\xi,\eta}\cdot[zd^{1/2}]^*\mid [yd^{1/2}]^*)_{L^2(\psi)} = \int [yd^{1/2}]\cdot\theta_{\xi,\eta}\cdot[zd^{1/2}]^*\,\mathrm{d}\psi =$$
$$= (yd^{1/2}\xi\mid zd^{1/2}\eta) = (z^*yd^{1/2}\xi\mid d^{1/2}\eta).$$

Hence (34) holds for all $x\in m_\varphi$. For a general $x\in L$, take $(x_i)_{i\in I}$ in m_φ such that $x_i\to x$ σ-weakly and $\varphi_{x_i}\to\varphi_x$ (this is possible by Theorem 8). Then

$$\varphi_x(\theta_{\xi,\eta}) = \lim_{i\in I}\varphi_{x_i}(\theta_{\xi,\eta}) = \lim_{i\in I}(x_i d^{1/2}\xi\mid d^{1/2}\eta) = (xd^{1/2}\xi\mid d^{1/2}\eta).$$

The following proposition is similar to [22, Proposition 2.3]:

PROPOSITION 25. *Let $p \in [2, \infty[$ and $1/p + 1/q = 1$. Let a be a closed densely defined $(-1/p)$-homogeneous operator. Suppose that $D(H, \psi) \subseteq D(a)$ and that for some constant $C \geqslant 0$, we have*

$$\left| \sum_{i=1}^{n} (a\xi_i \mid b\eta_i) \right| \leqslant C \|b\|_q \left\| \sum_{i=1}^{n} \theta_{\xi_i,\eta_i} \right\| \tag{35}$$

for all $b \in L^q(\psi)$, $n \in \mathbf{N}$, $\xi_1, \ldots, \xi_n \in D(H, \psi)$, and $\eta_1, \ldots, \eta_n \in D(H, \psi) \cap D(|b|^q)$. Then $a \in L^p(\psi)$ and $\|a\|_p \leqslant C$.

Proof. We may suppose that $a \geqslant 0$ (to reduce the general case to this case, consider the right polar decomposition of a: $a = |a^*|v$, and note that $|a^*|$ satisfies (35) if a does, since

$$\left\| \sum_{i=1}^{n} \theta_{v^*\xi_i,\eta_i} \right\| = \left\| v^* \sum_{i=1}^{n} \theta_{\xi_i,\eta_i} \right\| \leqslant \left\| \sum_{i=1}^{n} \theta_{\xi_i,\eta_i} \right\| \Bigg).$$

Suppose we know that $a \in L^p(\psi)$. Then $a^{p/q} \in L^q(\psi)$ with $\|a^{p/q}\|_q = \|a\|_p^{p/q}$ so that by (35)

$$\sum_{i=1}^{n} (a^p\xi_i \mid \xi_i) = \sum_{i=1}^{n} (a\xi_i \mid a^{p/q}\xi_i) \leqslant$$

$$\leqslant C \|a\|_p^{p/q} \left\| \sum_{i=1}^{n} \theta_{\xi_i,\xi_i} \right\|$$

for all $n \in \mathbf{N}$ and $\xi_1, \ldots, \xi_n \in D(H, \psi) \cap D(a^p)$. By Proposition 20, this implies that $\|a^p\|_1 \leqslant C\|a\|_p^{p/q}$, i.e.

$$\|a\|_p = \|a\|_p^p \|a\|_p^{-p/q} \leqslant C.$$

In the general case, take $a_j \in L^p(\psi)_+$ such that $a_j^p \nearrow a^p$ and $\int a^p d\psi = \sup_{j\in J} \|a_j^p\|_1$. Then there exist $x_j \in M$, $\|x_j\| \leqslant 1$, such that $a_j \subseteq ax_j$ (cf. the proof of [22, Proposition 2.3]).

Now each a_j satisfies (35) since

$$\left\| \sum_{i=1}^{n} \theta_{x_j\xi_i,\eta_i} \right\| = \left\| x_j \sum_{i=1}^{n} \theta_{\xi_i,\eta_i} \right\| \leqslant \left\| \sum_{i=1}^{n} \theta_{\xi_i,\eta_i} \right\|.$$

By the first part of the proof, it follows that $\|a_j\|_p \leqslant C$. Hence $\int a^p d\psi = \sup_{j\in J} \|a_j\|_p^p \leqslant C^p < \infty$ so that $a \in L^p(\psi)$ and $\|a\|_p \leqslant C$. ▨

THEOREM 26. *Let* $q \in]2, \infty[$.

1) *Let* $x \in n_\varphi$. *Then* $xd^{1/q}$ *is preclosed, the closure* $[xd^{1/q}]$ *is in* $L^q(\psi)$, *and*

$$\|[xd^{1/q}]\|_q \leqslant \max\{\|\Lambda_\varphi(x)\|, \|x\|\}. \tag{36}$$

2) *The set of operators* $\{[xd^{1/q}] \mid x \in n_\varphi\}$ *is dense in* $L^q(\psi)$.

Proof. 1) First note that $D(H, \psi) \subseteq D(d^{1/2}x^*) \subseteq D(d^{1/q}x^*)$.
We shall prove that

$$\left|\sum_{i=1}^{n} (d^{1/q}x^*\xi_i|a\eta_i)\right| \leqslant \max\{\|\Lambda_\varphi(x)\|, \|x\|\}\, \|a\|_p \left\|\sum_{i=1}^{n} \theta_{\xi_i,\eta_i}\right\| \tag{37}$$

for all $\xi_1, \ldots, \xi_n \in D(H, \psi)$, $a \in L^p(\psi)$ where $1/p + 1/q = 1$, and $\eta_1, \ldots, n_n \in$ $\in D(H, \psi) \cap D(|a|^p)$. Take $\xi_1, \ldots, \xi_n, a, \eta_1, \ldots, \eta_n$ as specified. We may assume that $\|a\|_p = 1$. Let $a = u|a|$ be the polar decomposition of a. For each $\alpha \in \mathbf{C}$ with $0 \leqslant \mathrm{Re}\alpha \leqslant 1/2$, put

$$F(\alpha) = \sum_{i=1}^{n} (d^\alpha x^*\xi_i \mid u|a|^{p(1-\bar{\alpha})}\eta_i).$$

We shall estimate $\sum_{i=1}^{n} (d^{1/q}x^*\xi_i \mid a\eta_i) = F(1/q)$ by use of the 3 lines theorem [26, p. 93]. The mapping F is bounded and continuous on $S_{1/2} = \{\alpha \in \mathbf{C} \mid 0 \leqslant \mathrm{Re}\alpha \leqslant 1/2\}$ and analytic in $S^0_{1/2}$, since by [21, 9.15] this is true for each of the vector functions constituting F. Now, let us estimate the values of F on the boundaries of $S_{1/2}$. First recall that since $|a|^p = \dfrac{\mathrm{d}\chi}{\mathrm{d}\psi}$ for some $\chi \in M_*^+$, we have $|a|^{pit}d^{-it} = (\mathrm{D}\chi : \mathrm{D}\varphi)_t \in M$ for all $t \in \mathbf{R}$ (cf. [5, Theorem 9, (2)]). Using also the — easily established — fact that the mappings $t \mapsto d^{it}(\cdot)d^{-it}$ are isometries of $L^1(\psi)$ and $L^2(\psi)$, we find that

$$|F(\mathrm{i}t)| = \left|\sum_{i=1}^{n} (d^{\mathrm{i}t}x^*\xi_i \mid u|a|^p|a|^{pit}\eta_i)\right| =$$

$$= \left|\sum_{i=1}^{n} (x^*\xi_i \mid d^{-\mathrm{i}t}u|a|^p|a|^{pit}\eta_i)\right| =$$

$$= \left|\int x^* \cdot \sum_{i=1}^{n} \theta_{\xi_i,\eta_i} \cdot d^{-\mathrm{i}t}(u|a|^p)(|a|^{pit}d^{-\mathrm{i}t})d^{\mathrm{i}t}\mathrm{d}\psi\right| \leqslant$$

$$\leqslant \|x^*\| \left\|\sum_{i=1}^{n} \theta_{\xi_i,\eta_i}\right\| \|u|a|^p\|_1 =$$

$$= \|x\| \left\|\sum_{i=1}^{n} \theta_{\xi_i,\eta_i}\right\|$$

and

$$|F(1/2+\mathrm{i}t)| = \left|\sum_{i=1}^{n} (d^{\mathrm{i}t}d^{1/2}x^{*}\xi_i \mid u|a|^{p/2}|a|^{p\mathrm{i}t}\eta_i)\right| =$$

$$= \left|\sum_{i=1}^{n} (d^{1/2}x^{*}\xi_i \mid d^{-\mathrm{i}t}u|a|^{p/2}|a|^{p\mathrm{i}t}\eta_i)\right| =$$

$$= \left|\int d^{1/2}x^{*}\cdot \sum_{i=1}^{n} \theta_{\xi_i,\eta_i}\cdot d^{-\mathrm{i}t}(u|a|^{p/2})(|a|^{p\mathrm{i}t}d^{-\mathrm{i}t})d^{\mathrm{i}t}\mathrm{d}\psi\right| \leqslant$$

$$\leqslant \|d^{1/2}x^{*}\|_2 \left\|\sum_{i=1}^{n} \theta_{\xi_i,\eta_i}\right\| \|u|a|^{p/2}\|_2 =$$

$$= \|\Lambda_\varphi(x)\| \left\|\sum_{i=1}^{n} \theta_{\xi_i,\eta_i}\right\|.$$

By the 3 lines theorem, we finally conclude

$$|F(1/q)| \leqslant \max\{\|\Lambda_\varphi(x)\|, \|x\|\}\cdot \left\|\sum_{i=1}^{n} \theta_{\xi_i,\eta_i}\right\|.$$

Thus (37) is proved.

Since $d^{1/2}x^{*}$ is $(-1/q)$-homogeneous (Definition 19), we conclude from (37) by Proposition 25 that $d^{1/q}x^{*} \in L^q(\psi)$ with $\|d^{1/q}x^{*}\|_q \leqslant \max\{\|\Lambda_\varphi(x)\|, \|x\|\}$. Since $(xd^{1/q})^{*} = d^{1/q}x^{*}$, this completes the proof of the first part of the theorem.

2) Suppose that for some $a \in L^p(\psi)$ $(1/p + 1/q = 1)$ we have $\int [xd^{1/q}]\cdot a\,\mathrm{d}\psi = 0$ for all $x \in n_\varphi$. Then proceeding as in the proof of 2) in Theorem 23, we can show that $a = 0$. Hence $\{[xd^{1/q}] \mid x \in n_\varphi\}$ is dense in $L^q(\psi)$. ▨

2.3. Injections of L into the spaces $L^p(\psi)$. We define

$$\mu_\infty\colon L \to L^\infty(\psi) \quad \text{and} \quad \mu_1\colon L \to L^1(\psi)$$

by

$$\mu_\infty(x) := x, \quad \mu_1(x) := \frac{\mathrm{d}\varphi_x}{\mathrm{d}\psi},$$

for all $x \in L$; by Proposition 2, μ_∞ and μ_1 are linear norm-decreasing injections $\left(\text{recall that } M_* \simeq L^1(\psi) \text{ via } \chi \mapsto \frac{\mathrm{d}\chi}{\mathrm{d}\psi}\right)$. By Theorem 24, we have

$$\forall\, y \in n_\varphi : \mu_1(y^{*}y) = d^{1/2}y^{*}\cdot[yd^{1/2}] \tag{38}$$

(indeed, $d^{1/2}y^{*}\cdot[yd^{1/2}] = \frac{\mathrm{d}\chi}{\mathrm{d}\psi}$ for some $\chi \in M_*^+$; hence by [14, Proposition 5] and Theorem 24 we have

$$\chi(\theta_{\xi,\eta}) = \int d^{1/2}y^{*}\cdot[yd^{1/2}]\cdot\theta_{\xi,\eta}\mathrm{d}\psi = (yd^{1/2}\xi \mid yd^{1/2}\eta) = \varphi_{y^*y}(\theta_{\xi,\eta})$$

for all $\xi, \eta \in D(H,\psi) \cap D(d^{1/2})$; it follows that $\chi = \varphi_{y^*y}$, i.e. (38)).

With each $a \in L^p(\psi)$, $1 < p < \infty$, we associate a *sesquilinear form* v_a on $D(H, \psi)$, defined by

(39) $$v_a(\xi, \eta) = (|a|^{1/2}\xi \mid |a|^{1/2}u^*\eta), \quad \xi, \eta \in D(H, \psi),$$

where $a = u|a|$ is the polar decomposition of a. Note that the mapping $a \mapsto v_a$ is *linear* and *injective* (cf. [14, Proposition 11]) and that

(40) $$\forall a \in L^p(\psi) \ \ \forall \xi, \eta \in D(H, \psi): \quad |v_a(\xi, \eta)| \leqslant \|a\|_p \|\xi\|^{1/q} \|\eta\|^{1/q} \|R^\psi(\xi)\|^{1/p} \|R^\psi(\eta)\|^{1/p}$$

where $1/p + 1/q = 1$.

The rest of this section is devoted to a proof of the following theorem:

THEOREM 27. *Let* $p \in]1, \infty[$.

1) *Let* $x \in L$. *There is a unique element* $\mu_p(x) \in L^p(\psi)$ *such that*

(41) $$\forall \xi, \eta \in D(H, \psi) \cap D(d^{1/2p}): v_{\mu_p(x)}(\xi, \eta) = (xd^{1/2p}\xi \mid d^{1/2p}\eta).$$

This element satisfies the following norm inequality:

(42) $$\|\mu_p(x)\|_p \leqslant \|\varphi_x\|^{1/p} \|x\|^{1/q}$$

where $1/p + 1/q = 1$.

2) *The mapping*

$$\mu_p: L \to L^p(\psi)$$

is linear, norm-decreasing, injective and has dense range.

LEMMA 28. *The mappings*

(43) $$(t, a) \mapsto d^{it}ad^{-it}: \mathbf{R} \times L^1(\psi) \to L^1(\psi)$$

(resp. $\mathbf{R} \times L^2(\psi) \to L^2(\psi)$*) and*

(44) $$(x, a) \mapsto a \cdot x: M_1 \times L^1(\psi) \to L^1(\psi)$$

(resp. $M_1 \times L^2(\psi) \to L^2(\psi)$*) are continuous with respect to the norm topology on* $L^1(\psi)$ *(resp. on* $L^2(\psi)$*) and the σ-strong* topology on the unit ball* M_1 *of* M.

Proof. It is well-known that (44) holds in case of $L^2(\psi)$ (we are considering the usual right action of M on $L^2(\psi)$, cf. [12, Theorem 1.21]). It follows that it also holds for $L^1(\psi)$: for all $x, x_0 \in M_1$ and $a, a_0 \in L^1(\psi)$, we have

$$\|a \cdot x - a_0 \cdot x_0\|_1 \leqslant \|(a - a_0) \cdot x\|_1 + \|a_0 \cdot (x - x_0)\|_1 \leqslant$$
$$\leqslant \|a - a_0\|_1 + \|b_0\|_2 \|c_0 \cdot x - c_0 \cdot x_0\|_2$$

where $b, c \in L^2(\psi)$ are chosen so that $a_0 = b_0 \cdot c_0$.

The mapping $(t,\xi)\mapsto \Delta_\varphi^{it}\xi: \mathbf{R}\times H_\varphi \to H_\varphi$ is continuous; since $\mathscr{P}(\Delta^{it}\xi) = d^{it}\mathscr{P}(\xi)d^{-it}$ for all $\xi\in H_\varphi$ and $t\in\mathbf{R}$, the result on (43) in case of $L^2(\psi)$ follows.

Finally, to prove (43) in case of $L^1(\psi)\simeq M_*$, use the fact that $d^{it}\dfrac{d\chi}{d\psi}d^{-it} = \chi\circ\sigma_{-t}^{\varphi}$ for all $\chi\in M_*$ and $t\in\mathbf{R}$ and recall that $(t,\chi)\mapsto\chi\circ\sigma_{-t}^{\varphi}$ is continuous with respect to the norm topology on M_*. (One easily shows that $t\mapsto\chi\circ\sigma_{-t}^{\varphi}$ is weakly continuous for each $\chi\in M_*$. To get norm continuity, apply [6, Proposition 1.23] or [13, p. 306, Corollary], or the simpler argument in [15, pp. 23—26].) ▨

We write $S = \{\alpha\in\mathbf{C} \mid 0\leqslant \operatorname{Re}\alpha\leqslant 1\}$.

LEMMA 29. *Let $a\in L^p(\psi)$, $1\leqslant p<\infty$, with $\|a\|_p = 1$ and polar decomposition $a = u|a|$. Let $y,z\in n_\varphi$. For each $\alpha\in S$, put*

$$F(\alpha) = \int u|a|^{p\alpha}\cdot d^{(1-\alpha)/2}z^*\cdot[yd^{(1-\alpha)/2}]\mathrm{d}\psi. \tag{45}$$

Then F is bounded and continuous on S and analytic in S^0.

Proof. The notation in (45) is slightly abusive; (45) is to be interpreted as

$$F(\alpha) = \int u|a|^{p/s}|a|^{pit}d^{-it}\cdot d^{it/2}(d^{1/2r}z^*\cdot[yd^{1/2r}])d^{-it/2}\mathrm{d}\psi \tag{46}$$

where $\alpha = \dfrac{1}{s}+it$ and $\dfrac{1}{r} = 1-\dfrac{1}{s}$. Note that $u|a|^{p/s}\in L^s(\psi)$ and $d^{1/2r}z^*$, $[yd^{1/2r}]\in L^{2r}(\psi)$ (Theorem 26) so that the integral (46) exists by Hölder's inequality [14, Proposition 8, (1), and Corollaire 12].

To prove the lemma, we first claim that the mapping $g: S\to L^2(\psi)$ defined by

$$g(\alpha) = [yd^{(1-\alpha)/2}]\cdot u|a|^{p\alpha/2},\quad \alpha\in S, \tag{47}$$
$$(= [yd^{1/2r}]\cdot d^{-it/2}u|a|^{p/2s}|a|^{pit/2})$$

is bounded and continuous on S and analytic in S^0. Indeed,

$$\|g(\alpha)\|_2 \leqslant \|[yd^{1/2r}]\|_{2r}\|d^{-it/2}(u|a|^{p/2s})(|a|^{pit/2}d^{-it/2})d^{it/2}\|_{2s}\leqslant$$
$$\leqslant \max\{\|\Lambda_\varphi(y)\|,\|y\|\}$$

by Theorem 26 (and using $\|u|a|^{p/2s}\|_{2s} = 1$), and for all $\xi,\eta\in D(H,\psi)$, the mapping

$$\alpha\mapsto(g(\alpha)\xi|\eta) = (u|a|^{p\alpha/2}\xi \mid d^{(1-\bar\alpha)/2}y^*\eta)$$

is continuous on S and analytic in S^0 so that g is weakly continuous on S and (weakly) analytic in S^0. Furthermore, by Lemma 28 the boundary mappings

$$t\mapsto g(it) = [yd^{1/2}]\cdot d^{-it/2}(u|a|^{pit/2}d^{-it/2})d^{it/2}$$

and

$$t \mapsto g(1 + it) = y \cdot d^{-it/2}(u|a|^{p/2}|a|^{pit/2}d^{-it/2})d^{it/2}$$

are continuous (indeed, $t \mapsto d^{-it/2}(u|a|^{pit/2}d^{-it/2})\,d^{it/2}$ is σ-strongly* continuous by e.g. [11, Lemma 2.2]). We conclude that g has the desired properties (cf. the remark following this proof).

Now, using [14, Proposition 7] and the easily established fact that

$$\forall\, b \in L^1(\psi) \quad \forall\, t \in \mathbf{R} : \int d^{it}bd^{-it}\mathrm{d}\psi = \int b\mathrm{d}\psi \tag{48}$$

we find that

$$F(\alpha) = ([yd^{(1-\alpha)/2}] \cdot u|a|^{p\alpha/2} \mid [zd^{(1-\bar{\alpha})/2}] \cdot |a|^{p\bar{\alpha}/2})_{L^2(\psi)}\,. \tag{49}$$

The result follows. ▨

Remark. We have use the following theorem: Let $f : S \to X$ be a function on the strip S with values in a Banach space X. Suppose that (i) f is bounded, (ii) f is w^*-continuous on S and analytic in S^0, and (iii) $t \mapsto f(it): \mathbf{R} \to X$ and $t \mapsto f(1 + it): \mathbf{R} \to X$ are continuous. Then $f: S \to X$ is continuous. This theorem follows e.g. from the reasoning in [3, Section 29].

Proof of Theorem 27. First note that for a given $x \in L$, there is at most one $\mu_p(x) \in L^p(\psi)$ satisfying (41). Indeed, if for some $a \in L^p(\psi)$ we have $v_a(\xi, \eta) = 0$ for all $\xi, \eta \in D(H, \psi) \cap D(d^{1/2p})$, then actually $v_a(\xi, \eta) = 0$ for all $\xi, \eta \in D(H, \psi)$ (by (40) and the fact that every $\xi \in D(H, \psi)$ may be approximated by $\xi^{(n)} \in D(H, \psi) \cap \cap D(d^{1/2p})$, $n \in \mathbf{N}$, satisfying $\|R^\psi(\xi^{(n)})\| \leqslant \|R^\psi(\xi)\|$ (cf. [14, Proposition 2] or [22, Lemma 2.5])), whence $a = 0$.

Now let us prove the existence. We first assume that $x \in m_\varphi$. Then we can write $x = \sum_{j=1}^{n} \lambda_j y_j^* y_j$ for some $n \in \mathbf{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$, and $y_1, \ldots, y_n \in n_\varphi$. Put

$$\mu_p(x) = \sum_{j=1}^{n} \lambda_j d^{1/2p} y_j^* \cdot [y_j d^{1/2p}]\,; \tag{50}$$

then by Theorem 26 and [14, Corollaire 12] we have $\mu_p(x) \in L^{2p}(\psi) \cdot L^{2p}(\psi) \subseteq L^p(\psi)$, and $\mu_p(x)$ satisfies (41):

$$v_{\mu_p(x)}(\xi, \eta) = \sum_{j=1}^{n} \lambda_j v_{d^{1/2p}y_j^* \cdot [y_j d^{1/2p}]}(\xi, \eta) =$$

$$= \sum_{j=1}^{n} \lambda_j ([y_j d^{1/2p}]\xi \mid [y_j d^{1/2p}]\eta) =$$

$$= \left(\left(\sum_{j=1}^{n} \lambda_j y_j^* y_j \right) d^{1/2p}\xi \,\middle|\, \eta \right).$$

Let us show that (41) holds for all $x \in m_\varphi$. We will do this by showing that

$$\left|\int b\cdot\mu_p(x)\mathrm{d}\psi\right| \leqslant \|\varphi_x\|^{1/p}\|x\|^{1/q} \tag{51}$$

for all $b \in L^q(\psi)$ such that $\|b\|_q = 1$. Then (41) follows by [14, Proposition 8, (2)]. As above, we write $x = \sum_{j=1}^n \lambda_j y_j^* y_j$. Let $b \in L^q(\psi)$ with polar decomposition $b = v|b|$ and $\|b\|_q = 1$. Now for each $\alpha \in S$, put

$$F(\alpha) = \sum_{j=1}^n \lambda_j \int v|b|^{q\alpha}\cdot d^{(1-\alpha)/2}y_j^*\cdot[y_j d^{(1-\alpha)/2}]\mathrm{d}\psi. \tag{52}$$

Then by Lemma 29, F is bounded and continuous on S and analytic in S^0, and for all $t \in \mathbf{R}$ we have, using (38),

$$|F(\mathrm{i}t)| = \left|\int v|b|^{q\mathrm{i}t}d^{-\mathrm{i}t}\cdot d^{\mathrm{i}t/2}\left(\sum_{j=1}^n \lambda_j d^{1/2}y_j^*\cdot[y_j d^{1/2}]\right)d^{\mathrm{i}t/2}\mathrm{d}\psi\right| =$$

$$= \left|\int v|b|^{q\mathrm{i}t}d^{-\mathrm{i}t}\cdot d^{\mathrm{i}t/2}\mu_1\left(\sum_{j=1}^n \lambda_j y_j^* y_j\right)d^{-\mathrm{i}t/2}\mathrm{d}\psi\right| \leqslant$$

$$\leqslant \|\mu_1(x)\|_1 = \|\varphi_x\|$$

and

$$|F(1+\mathrm{i}t)| = \left|\int v|b|^q|b|^{q\mathrm{i}t}d^{-\mathrm{i}t}\cdot d^{\mathrm{i}t/2}\left(\sum_{j=1}^n \lambda_j y_j^* y_j\right)d^{-\mathrm{i}t/2}\mathrm{d}\psi\right| \leqslant$$

$$\leqslant \|v|b|^q\|_1\|x\| \leqslant \|x\|.$$

Since $F(1/q) = \int b\cdot\mu_p(x)\mathrm{d}\psi$, (51) now follows from the 3 lines theorem.

Now we shall prove the existence of $\mu_p(x)$ for a general $x \in L$. Take $(x_i)_{i\in I}$ in m_φ as in Theorem 8. Then

$$\|\mu_p(x_i - x_j)\|_p \leqslant \|\varphi_{x_i} - \varphi_{x_j}\|^{1/p}(2\sup_{i\in I}\|x_i\|)^{1/q}$$

for all $i, j \in I$. Hence $(\mu_p(x_i))_{i\in I}$ is a Cauchy net in $L^p(\psi)$. Put

$$\mu_p(x) = \lim_{i\in I}\mu_p(x_i). \tag{53}$$

Then by (40)

$$v_{\mu_p}(x_i)(\xi,\eta) \to v_{\mu_p(x)}(\xi,\eta)$$

for all $\xi, \eta \in D(H, \psi)$. On the other hand, since $x_i \to x$ σ-weakly, we have

$$v_{\mu_p(x_i)}(\xi, \eta) = (x_i d^{1/2p}\xi \mid d^{1/2p}\eta) \to (x d^{1/2p}\xi \mid d^{1/2p}\eta)$$

for all $\xi, \eta \in D(H, \psi) \cap D(d^{1/2p})$. We conclude that $\mu_p(x)$ satisfies (41).

To show (42), let $\varepsilon \in \mathbf{R}_+$ and take $(x_i)_{i\in I}$ as above and such that all $\|x_i\| \leqslant \leqslant (1 + \varepsilon)\|x\|$ (this is possible by the remark following the proof of Theorem 8). Then

$$\|\mu_p(x)\|_p = \lim_{i\in I}\|\mu_p(x_i)\|_p \leqslant$$

$$\leqslant \limsup_{i\in I}\|\varphi_{x_i}\|^{1/p}\|x_i\|^{1/q} \leqslant$$

$$\leqslant \|\varphi_x\|^{1/p}(1 + \varepsilon)^{1/q}\|x\|^{1/q}$$

since $\varphi_{x_i} \to \varphi_x$. This inequality holds for all $\varepsilon \in \mathbf{R}_+$. Then (42) follows. This completes the proof of the first part of the theorem.

2) We have $\|\mu_p(x)\|_p \leqslant \max\{\|\varphi_x\|, \|x\|\} = \|x\|_L$ for all $x \in L$. Theorem 26, 2), implies that $\mu_p(m_\varphi)$ is dense in $L^{2p}(\psi) \cdot L^{2p}(\psi) = L^p(\psi)$. ▨

2.4. Injections of the spaces $L^p(\psi)$ into V.

Theorem 30. *Let* $p \in]1, \infty[$. *Define* $q \in]1, \infty[$ *by* $1/p + 1/q = 1$.

1) *Let* $a \in L^p(\psi)$. *Then there is a unique element* $v_p(a) \in V$ *such that*

$$\forall x \in L : \langle v_p(a), x\rangle_{V,L} = \int a \cdot \mu_q(x) \mathrm{d}\psi. \tag{54}$$

2) *The mapping*

$$v_p : L^p(\psi) \to V$$

is linear, norm-decreasing, and injective.

Proof. Define $v_p : L^p(\psi) \to L^*$ to be the transpose of $\mu_q : L \to L^q(\psi)$ (we identify $L^p(\psi)$ with the dual of $L^q(\psi)$ [14, Théorème 10, (2)]). Then v_p satisfies 1) and 2) with L^* instead of V. That actually v_p maps $L^p(\psi)$ into V follows from the inequality

$$|\langle v_p(a), x\rangle_{L^*,L}| = \left|\int a \cdot \mu_q(x) \mathrm{d}\psi\right| \leqslant$$

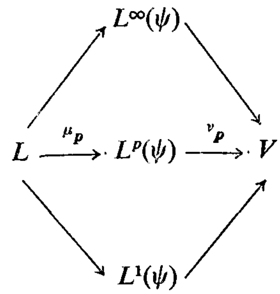
$$\leqslant \|a_p\| \|\mu_q(x)\|_q =$$

$$= \|a\|_p \|\varphi_x\|^{1/q} \|x\|^{1/p}, \quad x \in L,$$

showing that $x \mapsto \langle v_p(a), x\rangle$ is σ-w/$\|\cdot\|$-continuous on bounded subsets of L. ▨

For each $p \in]1, \infty[$, the diagram

(55)



is commutative. This follows from the fact that

$$\forall x, y \in L : \int \mu_p(x) \cdot \mu_q(y) \mathrm{d}\psi = \langle \varphi_x, y \rangle. \tag{56}$$

(To show (56), note that we may assume that $x, y \in m_\varphi$, and verify (using (38)) the formula

$$\int \mu_p(x_2^* x_1) \cdot \mu_q(y_2^* y_1) \mathrm{d}\psi = \int d^{1/2} x_2^* \cdot [x_1 d^{1/2}] \cdot y_2^* y_1 \mathrm{d}\psi = \langle \varphi_{x_2^* x_1}, y_2^* y_1 \rangle$$

for all $x_1, x_2, y_1, y_2 \in n_\varphi$.)

The following lemmas serve as a preparation for the identification of the $L^p(\psi)$ with complex interpolation spaces in the next section.

LEMMA 31. *Let* $a \in L^p(\psi)$, $1 \leqslant p < \infty$, *with* $\|a\|_p \leqslant 1$ *and polar decomposition* $a = u|a|$.

1) *For each* $\alpha \in S$, *there is a unique* $f(\alpha) \in V$ *such that*

$$\langle f(\alpha), x \rangle = \int u|a|^{p\alpha} \cdot d^{(1-\alpha)/2} x d^{(1-\alpha)/2} \mathrm{d}\psi \tag{57}$$

for all $x \in L$.

2) *The mapping*

$$f : S \to V$$

thus defined satisfies

(i) $\forall \alpha \in S : \|f(\alpha)\|_V \leqslant 1$,

(ii) *for all* $x \in m_\varphi$, *the function*

$$\alpha \mapsto \langle f(\alpha), x \rangle_{V,L}$$

is continuous on S *and analytic in* S^0;

$(\mathrm{iii})_0$ $\forall t \in \mathbf{R} : f(it) \in M$ *with* $\|f(it)\|_\infty \leqslant 1$, *and the mapping* $t \mapsto f(it) : \mathbf{R} \to M$ *is* σ*-weakly continuous;*

$(\mathrm{iii})_1$ $\forall t \in \mathbf{R} : f(1 + it) \in M_*$ *with* $\|f(1 + it)\|_1 \leqslant 1$, *and the mapping* $t \mapsto f(1 + it) : \mathbf{R} \to M_*$ *is* $\| \cdot \|_1$*-continuous.*

Proof. The right hand side of (57) is to be interpreted as

$$\int u|a|^{p/s}|a|^{pit}d^{-it}\cdot d^{it/2}\mu_r(x)d^{-it/2}\mathrm{d}\psi \tag{58}$$

where $\alpha := \frac{1}{s} + it$ and $\frac{1}{r} = 1 - \frac{1}{s}$. By Hölder's inequality this integral exists and

$$\begin{aligned}\left|\int u|a|^{p\alpha}\cdot d^{(1-\alpha)/2}xd^{(1-\alpha)/2}\mathrm{d}\psi\right| &\leqslant \|u|a|^{p/s}\|_s\,\|\mu_r(x)\|_r \leqslant \\ &\leqslant \|\varphi_x\|^{1/r}\,\|x\|^{1/s}.\end{aligned} \tag{59}$$

If $\mathrm{Re}\alpha \neq 1$ $\left(\text{so that } \frac{1}{r} \neq 0\right)$, this implies that the linear functional

$$x \mapsto \int u|a|^{p\alpha}\cdot d^{(1-\alpha)/2}xd^{(1-\alpha)/2}\mathrm{d}\psi \tag{60}$$

is indeed an element $f(\alpha)$ of V (by Definition 13). Since $\|\varphi_x\|^{1/r}\|x\|^{1/s} \leqslant \|x\|_L$ for all $x \in L$, we have also shown that $\|f(\alpha)\|_V \leqslant 1$, so that (i) holds for all $\alpha \in S$ with $\mathrm{Re}\alpha \neq 1$.

In case $\mathrm{Re}\alpha = 1$, we simply put

$$f(\alpha) = f(1 + it) = d^{-it/2}(u|a|^p|a|^{pit}d^{-it})d^{it/2} \in L^1(\psi) \simeq M^* \subseteq V. \tag{61}$$

Then $\|f(1 + it)\|_1 \leqslant \|u|a|^p\|_1 \leqslant 1$ for all $t \in \mathbf{R}$, and it follows from Lemma 28 that the mapping $t \mapsto f(1 + it)\colon \mathbf{R} \to L^1(\psi)$ is $\|\cdot\|_1$-continuous. This proves $(\mathrm{iii})_1$. Since $\|\cdot\|_V \leqslant \|\cdot\|_1$, (i) holds also for $\mathrm{Re}\alpha = 1$. Note that

$$\begin{aligned}\langle f(1 + it), x\rangle_{V,L} &= \langle f(1 + it), x\rangle_{L^1(\psi),L^\infty(\psi)} = \\ &= \int d^{-it/2}(u|a|^p|a|^{pit}d^{-it})d^{it/2}\cdot x\mathrm{d}\psi = \\ &= \int u|a|^p|a|^{pit}d^{-it}\cdot d^{it/2}xd^{-it/2}\mathrm{d}\psi\end{aligned}$$

so that $f(\alpha)$ is characterized by the equation (57) also when $\mathrm{Re}\alpha = 1$.

In all, we have now defined $f: S \to V$ as required in 1) and shown (i) and $(\mathrm{iii})_1$.

As for $(\mathrm{iii})_0$, note that

$$\langle f(\mathrm{i}t), x\rangle = \int u|a|^{p\mathrm{i}t}d^{-\mathrm{i}t}\cdot d^{\mathrm{i}t/2}\mu_1(x)d^{-\mathrm{i}t/2}\mathrm{d}\psi =$$

$$= \int d^{-\mathrm{i}t/2}(u|a|^{p\mathrm{i}t}d^{-\mathrm{i}t})d^{\mathrm{i}t/2}\cdot\mu_1(x)\mathrm{d}\psi$$

for all $x \in L$ and $t \in \mathbf{R}$. Hence

(62) $$f(\mathrm{i}t) = d^{-\mathrm{i}t/2}(u|a|^{p\mathrm{i}t}d^{-\mathrm{i}t})d^{\mathrm{i}t/2} \in L^\infty(\psi) \simeq M$$

and $\|f(\mathrm{i}t)\|_\infty \leqslant 1$ for all $t \in \mathbf{R}$. Furthermore, $t \mapsto f(\mathrm{i}t)\colon \mathbf{R} \to M$ is $\sigma(M, M_*)$-continuous (by e.g., [11, Lemma 2.2]).

To prove (ii), we may assume that $x = \sum_{j=1}^{n} \lambda_j y_j^* y_j$, $y_1, \ldots, y_n \in n_\varphi$. Then the desired result follows from (50) and Lemma 29. ▨

LEMMA 32. *Let $f\colon S \to V$ be a function satisfying* (i), (ii), $(\mathrm{iii})_0$ *and* $(\mathrm{iii})_1$ *in the conclusion of Lemma* 31. *Then for each $n \in \mathbf{N}$, we can define*

$$f_n\colon S \to V$$

by

(63) $$\forall x \in m_\varphi : \langle f_n(\alpha), x\rangle = \sqrt{n/\pi}\int \mathrm{e}^{-nt^2}\langle f(\alpha - \mathrm{i}t), x\rangle \mathrm{d}t$$

for all $\alpha \in S$. We have

(64) $$f_n \in \mathscr{F}(M; M_*) \ \textit{with}\ |||f_n||| \leqslant 1$$

for all $n \in \mathbf{N}$, and

(65) $$\|f_n(\alpha) - f(\alpha)\|_V \to 0 \quad \textit{as } n \to \infty$$

for each $\alpha \in S^0$.

Proof. We put $h_n(\alpha) = \sqrt{n/\pi}\,\mathrm{e}^{-n\alpha^2}$ for all $n \in \mathbf{N}$ and $\alpha \in \mathbf{C}$.

Let $n \in \mathbf{N}$. For each $\alpha \in S$, there is at most one element $f_n(\alpha) \in V$ satisfying (63) (since V is the dual of the σ-w/$\|\cdot\|$-closure of m_φ). We shall prove the existence.

First note that (i) and (ii) imply that f is actually norm-analytic in S^0. (To see this, apply [19, Appendix 5, Theorem 1]. This is possible since, using Theorem 8, we have $\|\chi\|_V = \sup\{|\langle\chi, x\rangle| \mid x \in L, \|x\|_L \leqslant 1\} = \sup\{|\langle\chi, x\rangle| \mid x \in m_\varphi, \|x\|_L \leqslant 1\}$ for all $\chi \in V$.) Hence f is also norm-continuous in S^0. In particular, for each $\alpha \in S^0$ the function

$$t \mapsto f(\alpha - \mathrm{i}t)\colon \mathbf{R} \to V$$

is $\|\cdot\|_V$-continuous. Since it is also bounded, we may form the usual V-valued integral (as in [1, Proposition 1.2])

$$f_n(\alpha) = \int h_n(t) f(\alpha - it)\mathrm{d}t,$$

and this integral satisfies (63).

If $\mathrm{Re}\alpha = 1$, the function

$$t \mapsto f(\alpha - it): \mathbf{R} \to M_*$$

is bounded and continuous by $(\mathrm{iii})_1$, and therefore $f_n(\alpha)$ exists as an M_*-valued integral; we have

$$\langle f_n(\alpha), x\rangle = \int h_n(t)\langle f(\alpha - it), x\rangle\mathrm{d}t$$

for all $x \in M$, hence in particular (63).

To deal with the case $\mathrm{Re}\alpha = 0$, note that by $(\mathrm{iii})_0$, the function

$$t \mapsto f(\alpha - it): \mathbf{R} \to M$$

is bounded and $\sigma(M, M_*)$-continuous. Therefore by (a simple case of) [1, Proposition 1.2], there is a unique element $f_n(\alpha) \in M$ satisfying

$$\langle \psi, f_n(\alpha)\rangle = \int h_n(t)\langle \psi, f(\alpha - it)\rangle\mathrm{d}t$$

for all $\psi \in M_*$; in particular, putting $\psi = \varphi_x$ where $x \in m_\varphi$, it follows that (63) holds.

We have now defined $f_n(\alpha) \in V$ for all values of $\alpha \in S$. Let us show (64). First, we note that by the definition of f_n, we have $\|f_n(\alpha)\|_V \leqslant 1$ for all $\alpha \in S^0$, $\|f_n(\alpha)\|_1 \leqslant 1$ for all α such that $\mathrm{Re}\alpha = 1$ and $\|f_n(\alpha)\|_\infty \leqslant 1$ for all α such that $\mathrm{Re}\alpha = 0$. Hence f is bounded and $|||f_n||| \leqslant 1$.

Next, f_n is continuous on S. Let $\alpha, \alpha_0 \in S$. Then for all $x \in m_\varphi$, we have, using (ii),

$$\langle f_n(\alpha), x\rangle = \int h_n(t)\langle f(\alpha_0 - i(t + i(\alpha - \alpha_0))), x\rangle\mathrm{d}t =$$

$$= \int h_n(t - i(\alpha - \alpha_0))\langle f(\alpha_0 - it), x\rangle\mathrm{d}t$$

and thus

$$\langle f_n(\alpha) - f_n(\alpha_0), x\rangle = \int (h_n(t - i(\alpha - \alpha_0)) - h_n(t))\langle f(\alpha_0 - it), x\rangle\mathrm{d}t.$$

Now

$$\int |h_n(t - i(\alpha - \alpha_0)) - h_n(t)|\mathrm{d}t \to 0 \quad \text{as } \alpha \to \alpha_0;$$

it follows that $\|f_n(\alpha) - f_n(\alpha_0)\|_V \to 0$ as $\alpha \to \alpha_0$, i.e. f_n is continuous.

By similar arguments, $t \mapsto f_n(it): \mathbf{R} \to M$ and $t \mapsto f_n(1+it): \mathbf{R} \to M_*$ are $\|\cdot\|_\infty$-, resp. $\|\cdot\|_1$-continuous.

That f_n is analytic in S^0 follows from the fact that f has this property. Hence $f_n \in \mathscr{F}(M; M_*)$.

Finally, let $\alpha \in S^0$. For all $n \in \mathbf{N}$ and $x \in m_\varphi$, we have

$$\langle f_n(\alpha) - f(\alpha), x\rangle = \int h_n(t)\langle f(\alpha - it), x\rangle \mathrm{d}t - \int h_n(t)\langle f(\alpha), x\rangle \mathrm{d}t =$$

$$= \int h_n(t)\langle f(\alpha - it) - f(\alpha), x\rangle \mathrm{d}t.$$

Since f is bounded and $(h_n)_{n\in\mathbf{N}}$ is an approximate identity, (65) follows. ∎

2.5. THE SPACES $L^p(\psi)$ AS INTERPOLATION SPACES. In the preceding section, we constructed injections v_p of the spaces $L^p(\psi)$ into V. Now put

$$L^p = v_p(L^p(\psi))$$

and

$$\|v_p(a)\|_p = \|a\|_p, \quad a \in L^p(\psi).$$

Then $(L^p, \|\cdot\|_p)$ is a Banach space continuously embedded in V and isomorphic to $L^p(\psi)$.

PROPOSITION 33. *Let $p \in]1, \infty[$. Then V_p is contained in L^p and*

$$\forall \chi \in V_p : \|\chi\|_p \leqslant \|\chi\|_{V_p}.$$

Proof. By definition of the interpolation spaces V_p (Definition 18), we have to prove that

(66) $$g(1/p) \in L^p \quad \text{and} \quad \|g(1/p)\|_p \leqslant |||g|||$$

for all $g \in \mathscr{F}_0(M; M_*)$.

Denote by $\mathscr{F}_0(L; L)$ the set of functions $f: S \to L$ satisfying

(i) f is bounded,
(ii) f is analytic in S^0 and continuous on S,
(iii) $\|f(\alpha)\|_L \to 0$ uniformly in $\operatorname{Re}\alpha$ as $|\operatorname{Im}\alpha| \to 0$.

By [2, Lemma 4.2.3], $\mathscr{F}_0(L; L)$ is dense in $\mathscr{F}_0(M; M_*)$. Hence we need only prove (66) for $g \in \mathscr{F}_0(L; L)$. (Indeed, suppose that we have proved the lemma in case of $g \in \mathscr{F}_0(L; L)$. Then the mapping $g \mapsto g(1/p): \mathscr{F}_0(L; L) \to L^p$ extends by continuity to a mapping $\Phi_p: \mathscr{F}_0(M; M_*) \to L^p$ satisfying $\forall g \in \mathscr{F}_0(M; M_*): \|\Phi_p(g)\|_p \leqslant |||g|||$. We claim that $\Phi_p(g) = g(1/p)$ for all $g \in \mathscr{F}_0(M; M_*)$. To see this, take $g_n \in \mathscr{F}_0(L; L)$ such that $|||g_n - g||| \to 0$. Then by (21) also $g_n(1/p) \to g(1/p)$ in V. On the other hand, $g_n(1/p) = \Phi_p(g_n) \to \Phi_p(g)$ in L^p and hence in V. Thus $\Phi_p(g) = g(1/p)$, so that $g(1/p) \in L^p$ and $\|g(1/p)\|_p \leqslant |||g|||$.)

Now let $g \in \mathscr{F}_0(L; L)$. Then $g(1/p) \in L$. We shall prove that

$$(67) \qquad |\langle v_q(b),\ g(1/p)\rangle_{V,L}| \leqslant |||g|||\,\|b\|_q$$

for all $b \in L^q(\psi)$ (where $1/p + 1/q = 1$). This will imply (66). (Indeed, by the duality theorem, the validity of (67) for all $b \in L^q(\psi)$ implies the existence of an element $a \in L^p(\psi)$ with $\|a\|_p \leqslant |||g|||$ such that

$$\forall\, b \in L^q(\psi) : \langle v_q(b), g(1/p)\rangle_{V,L} = \int b \cdot a\mathrm{d}\psi.$$

In particular this holds for all $\mu_q(x)$, $x \in L$, so that

$$\langle g(1/p), x\rangle_{V,L} = \langle v_q(\mu_q(x)), g(1/p)\rangle_{V,L} =$$

$$= \int \mu_q(x) \cdot a\mathrm{d}\psi = \int x \cdot v_p(a)\mathrm{d}\psi.$$

It follows that $g(1/p) = v_p(a) \in L^p$ and $\|g(1/p)\|_p \leqslant |||g|||$.)

Now let us prove (67). We may assume that $\|b\|_q = 1$. Define f: $S \to V$ corresponding to $b \in L^q(\psi)$ as in Lemma 31 (with b instead of a, q instead of p) and the corresponding $f_n \in \mathscr{F}(M; M_*)$, $n \in \mathbf{N}$, as in Lemma 32.

For each $n \in \mathbf{N}$, put

$$F_n(\alpha) = \langle f_n(\alpha), g(1-\alpha)\rangle_{V,L}, \quad \alpha \in S.$$

Then F_n is bounded and continuous on S and analytic in S^0. We estimate F_n on the boundaries of S: for all $t \in \mathbf{R}$, we have, using $|||f_n||| \leqslant 1$,

$$|F_n(\mathrm{i}t)| = |\langle f_n(\mathrm{i}t), g(1-\mathrm{i}t)\rangle_{V,L}| =$$

$$= |\langle f_n(\mathrm{i}t), g(1-\mathrm{i}t)\rangle_{M,M_*}| \leqslant \|f_n(\mathrm{i}t)\|_\infty \|g(1-\mathrm{i}t)\|_1 \leqslant$$

$$\leqslant |||f_n|||\ |||g||| \leqslant |||g|||$$

and

$$|F_n(1+\mathrm{i}t)| = |\langle f_n(1+\mathrm{i}t), g(-\mathrm{i}t)\rangle_{V,L}| =$$

$$= |\langle f_n(1+\mathrm{i}t), g(-\mathrm{i}t)\rangle_{M_*,M}| \leqslant \|f_n(1+\mathrm{i}t)\|_1 \|g(-\mathrm{i}t)\|_\infty \leqslant$$

$$\leqslant |||f_n|||\ |||g||| \leqslant |||g|||.$$

By the 3 lines theorem, we conclude that

$$|\langle f_n(1/q), g(1/p)\rangle_{V,L}| = |F_n(1/q)| \leqslant |||g|||.$$

Since $f_n(1/q) \to f(1/q) = v_q(b)$ in V it follows that

$$|\langle v_q(b), g(1/p)\rangle_{V,L}| \leqslant |||g|||,$$

i.e. (67) is proved. ▨

LEMMA 34. *Let $p \in]1, \infty[$. Let $\chi \in L^p$ with $\|\chi\|_p \leqslant 1$. Then there exists a sequence $(\chi_n)_{n \in \mathbf{N}}$ in V_p such that all $\|\chi_n\|_{V_p} \leqslant 1$ and*

$$\|\chi_n - \chi\|_V \to 0.$$

Proof. We have $\chi = v_p(a)$ where $a \in L^p(\psi)$ with $\|a\|_p \leqslant 1$. Define $f\colon S \to V$ corresponding to a as in Lemma 31 and $f_n \in \mathscr{F}(M; M_*)$, $n \in \mathbf{N}$, corresponding to f as in Lemma 32. Then $f_n(1/p) \to f(1/p) = v_p(a)$ in V.

For each $n \in \mathbf{N}$, define $g_n\colon S \to V$ by

$$g_n(\alpha) = \exp(\alpha^2/n) f_n(\alpha), \quad \alpha \in S.$$

Then $g_n \in \mathscr{F}_0(M; M_*)$ and $|||g_n||| \leqslant \exp(1/n)$ so that

$$g_n(1/p) \in V_p \text{ with } \|g_n(1/p)\|_{V_p} \leqslant \exp(1/n).$$

Put

$$\chi_n = \exp(-1/n) g_n(1/p), \quad n \in \mathbf{N}.$$

Then

$$\chi_n \in V_p \text{ with } \|\chi_n\|_{V_p} \leqslant 1$$

and

$$\chi_n = \exp(-1/n)\exp(1/(np^2)) f_n(1/p) \to f(1/p) = v_p(a)$$

in V. ▨

PROPOSITION 35. *The unit ball of V_p is dense in the unit ball of L^p $(1 < p < \infty)$.*

Proof. Let $\chi \in L^p$, $\|\chi\|_p \leqslant 1$. Take χ_n, $n \in \mathbf{N}$, as in Lemma 34. Then the χ_n are in the unit ball of V_p and hence by Proposition 33 also in the unit ball B of L^p. Since $\chi_n \to \chi$ in V, we have in particular $\chi_n \to \chi$ $\sigma(V, V^*)$. Now since L^p is continuously embedded into V, the (Hausdorff) topology on L^p induced by $\sigma(V, V^*)$ is weaker than $\sigma(L^p, L^q)$. Since B is compact in $\sigma(L^p, L^q)$ (L^p is reflexive), these topologies coincide on B. We conclude that $\chi_n \to \chi$ $\sigma(L^p, L^q)$.

Since χ was arbitrary, we have shown that the unit ball of V_p is weakly dense in L^p. Since it is convex, it is also dense with respect to the norm topology on L^p. ▨

Combining Proposition 33 and 35, we can now finally prove the main theorem:

THEOREM 36. *For each $p \in]1, \infty[$, we have*

$$V_p = L^p \quad \text{and} \quad \|\cdot\|_{V_p} = \|\cdot\|_p.$$

Proof. Denote by j the continuous embedding of V_p into L^p. Denoting B_r, resp. B_r', the closed ball with radius r and center at the origin in L^p, resp. V_p, we have $B_1 \subseteq \overline{j(B_1')}$ and hence

$$B_r = rB_1 \subseteq \overline{rj(B_1')} = \overline{j(B_r')}$$

for all $r \in \mathbf{R}_+$. By [24, p. 171, Lemma 17.2] we conclude that then

$$B_1 \subseteq j(B'_{1+\varepsilon})$$

for all $\varepsilon \in \mathbf{R}_+$. In particular B_1, and hence L^p, is contained in V_p. If $\chi \in L^p$ with $\|\chi\|_p = 1$, then $\|\chi\|_{V_p} \leqslant 1 + \varepsilon$ for all $\varepsilon \in \mathbf{R}_+$, whence $\|\chi\|_{V_p} \leqslant 1 = \|\chi\|_p$.

In all, we have proved that $V_p = L^p$ with equal norms. ▨

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