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A NON-COMMUTATIVE GELFAND-NAIMARK THEOREM

PAWEŁ KRUSZYŃSKI and STANISŁAW L. WORONOWICZ

INTRODUCTION

One of the most fundamental facts in the theory of C^*-algebras was discovered in early forties by Gelfand and Naimark. It says that any commutative C^*-algebra with unit is isomorphic to the algebra of all continuous functions on a compact topological space. Moreover, any homomorphism of commutative C^*-algebras is related to a continuous map of the underlying compact spaces. Briefly speaking, the category of commutative C^*-algebras with unit is dual to the category of compact topological spaces.

With this result, any fact dealing with compact topological spaces has a counterpart in the commutative C^*-algebras theory. In many cases these counterparts possess natural generalizations for noncommutative C^*-algebras and may be proved. It shows that even in the noncommutative case consideration of C^*-algebras as algebras of continuous functions is of a great inspiratory value. This point of view provides us with interesting conjectures and sometimes suggest a way of a proof.

To formalize this point of view one may introduce the notion of compact pseudospaces [6]. Briefly, a compact pseudospace is an object of the category dual to the category of C^*-algebras with unit. Then any C^*-algebra can be viewed as the algebra of all "continuous functions" on the corresponding pseudospace. One should stress however that introducing the category of pseudospaces we do not create any new theory, we introduce only a new language to the C^*-algebras theory.

In the present paper we try to develop more constructive approach to the problem. We consider so called domains and operator functions defined on them. To explain better our ideas let us recall two possible formulations of the basic Gelfand--Naimark result. In general the Gelfand-Naimark theorem deals with abstract compact topological spaces and continuous complex valued functions. Therefore the familiarity with the abstract topology is necessary in this case. On the other hand, for finitely generated C^*-algebras it is sufficient to consider closed bounded subsets of the arithmetic spaces $\mathbf{C}^N$ or $\mathbf{R}^N$. Continuous mappings of compact spaces are in this case replaced by N-tuples of continuous $\mathbf{C}^1$-valued functions. The whole theory can be understood without any knowledge of the general topology.

The theory developed in the present paper corresponds to the second, more restrictive setting. For the moment we are not able to construct a general concrete (i.e. not formal) theory of pseudospaces. Instead we introduce compact domains which should be identified with "embedded compact pseudospaces". Morphisms (i.e. "continuous mappings") of domains are defined by N-tuples of "continuous functions". The notion of continuous function that we use in our theory is very natural. It turns out that it is closely related to the notion of operator function introduced in our previous paper [5]. However one should note that domains and operator functions introduced in [5] correspond to Borel subsets of $\mathbf{C}^N$ and Borel functions rather than to compact spaces and continuous functions. We call them measurable domains and measurable operator functions, respectively.

Now we are going to summarize the results of this work. At first we introduce the new notions of compact domains and continuous operator functions defined on the them. We recall the notions of measurable operator functions and their domains introduced before in [5]. Next we investigate relations between these objects. In particular, it turns out that every continuous operator function is a measurable one. Further, we prove boundness of continuous operator functions on compact domains and show that a $*$-algebra of these functions is a C^*-algebra. This algebra is finitely generated. In this way we obtain an analogue of the Stone-Weierstrass theorem. Moreover it appears that each finitely generated C^*-algebra is isomorphic to a C^*-algebra of continuous operator functions on a compact domain. This domain is uniquely defined up to a homeomorphism. This is the analogue of the Gelfand-Naimark theorem which is one of our main results. A generalization of this result for the case of separable C^*-algebras is easy to obtain.

The homeomorphism of compact domains mentioned above is a particular case of morphism of compact domains. Properties of these morphisms reveal a lot of analogies between the category of topological compact spaces and the category of compact domains: for instance we prove that an injective continuous image of a compact domain is a compact domain again, and a bijective continuous morphism has a continuous inverse morphism.

Measurable operator functions invariant under a special action of the group R^1 on their domain appeared in [5]. Presently, we show a result which describes operator functions invariant with respect to the linear action of the compact groups of matrices that is defined on their domains. At last we sketch connections of this topic with the theory of ergodic action of compact groups on C^*-algebras ([3]), and we characterize compact domains that are related to these algebras by the Gelfand-Naimark theorem.

Here is the detailed content of the paper: Section 1 "Domains and Operator Functions" contains definitions of basic notions together with relations between them. Proofs of results formulated there will be presented in Sections 2, 3 and 4.

In Section 2 "Compact Domains" we give a nice characterization of compact domains. Results of this section is applied in Section 3 "Algebras of Continuous Operator Functions", where we prove the main results of this paper — namely Stone-Weierstrass and Gelfand-Naimark theorems.

Section 4 "The Application of a Voiculescu Result" is rather technical and it is devoted to the proof of a theorem formulated in Section 1.

Section 5 "Invariant Operator Functions" contains certain applications of the introduced notions to a special case of operator functions — i.e. invariant operator functions. We indicate also further possibilities of applications.

General terminology and needed facts from the theory of C^*-algebras can be found in the book [1]. Moreover we assume the following convention (we will use it without reminder):

All Hilbert spaces are separable,

All C^*-algebras are unital,

A representation of a C^*-algebra always is a unital $*$-homomorphism into $B(H)$,

The topology on sets of operators (e.g. operator algebras) is the norm topology.

In order to keep simplicity and elegance of the introduced notions we regard the symbol H, which means a Hilbert space, as a variable. It inserts to our considerations the class of all Hilbert spaces which clearly is not a set. Hence, such fundamental notions as domains and operator functions appear to be families of sets and mappings respectively, indexed by the class of Hilbert spaces. This may lead to some set-theoretical problems. This problem is apparent however. It follows easily from the basic definitions that one could restrict oneself to only one separable infinite dimensional Hilbert space. In particular operator functions defined on a fixed domain constitute a set.

Despite we consider functions of a finite number of variables, our results can be easily generalized for the case of denumerable amounts of arguments. In this case we use the Tichonov topology in underlying infinite Cartesian products. Then it would be easy to prove our version of Gelfand-Naimark theorem for non--commutative separable C^*-algebras.

1. DOMAINS AND OPERATOR FUNCTIONS

The notions introduced here need certain preparatory explanation. "Operator functions" are understood in a double sense. The first one is that of the notion of a mapping from some set of operators in a given Hilbert space into the set of operators in it. The latter is an idealization of such a mapping, independent of its actual realization in a Hilbert space. This abstract notion can be figured as a prescription for computing a value of a map in a general point.

For example the function $(a, b) \to a + b$ can be understood as the instruction: take two operators a and b acting on an arbitrary Hilbert space and add each to other. The sense of this prescription is independent of underlying Hilbert space. However, even in this simple example, some reasonable restrictions must be imposed on the set of arguments of the function. Namely, the admissible pairs (a, b) are those for which the operation "+" is well defined.

We can go one step further: ask what kind of restrictions must be necessarily imposed on a set to be a domain of functions, independently of a particular Hilbert space and operator function. At the moment we assume only that the set of arguments of operator functions is contained in the set $\mathscr{C}(H)$ of closed operators in a Hilbert space H ("measurable case") or in the set $B(H)$ of bounded operators in H ("continuous case"). Clearly, domains of functions of many variables should be subsets of $\mathscr{C}(H)^N$ or $B(H)^N$ respectively. Remembering that we have to consider any (separable) Hilbert space, we see that a domain is a family $\{D(H) : H \text{ is a Hilbert space}\}$, where for every Hilbert space H, $D(H) \subset \mathscr{C}(H)^N$. In what follows, such a family will be denoted by the single letter D. To formulate conditions imposed on D we use the following notation:

Let N be a natural number. For given $a^1, a^2, \ldots, a^N \in \mathscr{C}(H)$ by a we denote the N-tuple $(a^1, a^2, \ldots, a^N) \in \mathscr{C}(H)^N$. For

$$H = \int_{\Lambda}^{\oplus} H(\lambda) \mathrm{d}\mu(\lambda)$$

we write

$$a = \int_{\Lambda}^{\oplus} a(\lambda) \mathrm{d}\mu(\lambda)$$

if and only if $\{a^k(\lambda)\}_{\lambda \in \Lambda}$ is a measurable field of closed operators and

$$a^k = \int_{\Lambda}^{\oplus} a^k(\lambda) \mathrm{d}\mu(\lambda) \quad \text{for all } k = 1, 2, \ldots, N$$

(see [5] for the definition of direct integral of closed operators).

If Λ is denumerable set (say $\Lambda = \mathbf{N}$) and μ is the counting measure then we write $\bigoplus_{r \in \Lambda} H(r)$ and $\bigoplus_{r \in \Lambda} a(r)$ instead of $\int_{\Lambda}^{\oplus} H(\lambda) \mathrm{d}\mu(\lambda)$ and $\int_{\Lambda}^{\oplus} a(\lambda) \mathrm{d}\mu(\lambda)$ respectively.

If Φ is an operation defined on $a^1, a^2, \ldots, a^N$ then $\Phi(a)$ denotes the N-tuple $(\Phi(a^1), \Phi(a^2), \ldots, \Phi(a^N))$. In particular if $a \in \mathscr{C}(H)^N$ and U is a unitary operator from the Hilbert space H onto the Hilbert space K, then

$$UaU^* = (Ua^1U^*, Ua^2U^*, \ldots, Ua^NU^*) \in \mathscr{C}(K)^N.$$

Similarly if π is a representation of a C^*-algebra $A \subset B(H)$ and $a^1, a^2, \ldots, a^N \in A$ then

$$\pi(a) = (\pi(a^1), \pi(a^2), \ldots, \pi(a^N)) \in B(H_\pi)^N,$$

where H_π is the carrier Hilbert space of π.

If $a \in B(H)^N$ then by $C^*(a)$ we denote the unital C^*-algebra generated in $B(H)$ by $a^1, a^2, \ldots, a^N$.

DEFINITION 1.1. The family D of sets $D(H) \subset \mathscr{C}(H)^N$ is called a *measurable domain* (more precisely *N-dimensional measurable domain*) if the following two conditions hold:

1) For every Hilbert spaces H, K, any $a \in D(H)$ and any unitary operator $U\colon H \to K$ we have $UaU^* \in D(K)$;

2) For every measure space (Λ, μ) and any measurable field of Hilbert spaces $\{H(\lambda)\}_{\lambda\in\Lambda}$ we have

$$\int_\Lambda^\oplus a(\lambda)\mathrm{d}\mu(\lambda) \in D\left(\int_\Lambda^\oplus H(\lambda)\mathrm{d}\mu(\lambda)\right)$$

iff $a(\lambda) \in D(H(\lambda))$ for μ-a.a. $\lambda \in \Lambda$.

The above definition is the same as in [5]. In many cases when we deal with bounded operators another concept of domain is more interesting.

DEFINITION 1.2. The family D of sets $D(H) \subset B(H)^N$ is called a *compact domain* (more precisely *N-dimensional compact domain*) if for every Hilbert space H the following three conditions are fulfilled:

1) For every $a \in D(H)$ and every $*$-representation

$$\pi : C^*(a) \to B(K)$$

we have $\pi(a) \in D(K)$;

2) If for a given $a \in B(H)^N$ there exists a family $\{\pi_\lambda\}_{\lambda\in\Lambda}$ of representations of $C^*(a)$ such that $\bigcap_{\lambda\in\Lambda} \mathrm{Ker}\pi_\lambda = \{0\}$ and $\pi_\lambda(a) \in D(H_\lambda)$ for all $\lambda \in \Lambda$ (where H_λ denotes the carrier Hilbert space of π) then $a \in D(H)$;

3) There exists a constant $M > 0$, independent of H such that $\sup_{a\in D(H)} \|a^i\| \leqslant M$.

It turns out that the notion of a measurable domain is more general. The following result is proven in Section 2.

THEOREM 1.1. *Every compact domain is a measurable domain.*

It means that if a family D fulfils conditions 1), 2), 3) of Definition 1.2 then it fulfils also conditions 1), 2) of Definition 1.1.

One can ask when the convers relation is true. A necessary condition is given by:

PROPOSITION 1.2. *If D is a compact domain then for every Hilbert space H, $D(H)$ is closed in the uniform topology (i.e. the topology of $B(H)^N$ induced by the Cartesian product norm $\|a\| = \max_i \|a^i\|$).*

The proof is given in Section 2.

It turns out that this necessary condition is also a sufficient one.

THEOREM 1.3. *Let D be a measurable domain such that for each Hilbert space H, $D(H)$ is a closed subset of $B(H)^N$. Then D is a compact domain.*

A stronger version of this result is proven in Section 4.

Now we are ready to specify the notion of operator functions.

DEFINITION 1.3. Let D be a measurable domain. A family $F = \{F_H : H$ i Hilbert space$\}$ of maps $F_H : D(H) \to \mathscr{C}(H)$ is called a *measurable operator functiosn* if the following two conditions hold:

1) For every Hilbert spaces H, K, every $a \in D(H)$, and every unitary $U : H \to K$ awe have

$$F_K(UaU^*) = UF_H(a)U^*;$$

2) For every measure space (Λ, μ), every measurable field of Hilbert spaces $\{H(\lambda)\}_{\lambda\in\Lambda}$, and every decomposable

$$a = \int_\Lambda^\oplus a(\lambda)\mathrm{d}\mu(\lambda) \in D\left(\int_\Lambda^\oplus H(\lambda)\mathrm{d}\mu(\lambda)\right)$$

the field of operators $\{F_{H(\lambda)}(a(\lambda))\}_{\lambda\in\Lambda}$ is measurable and

$$F_{\int_\Lambda^\oplus H(\lambda)\mathrm{d}\mu(\lambda)}(a) = \int_\Lambda^\oplus F_{H(\lambda)}(a(\lambda))\mathrm{d}\mu(\lambda).$$

For compact domains the notion of continuous operator function is more natural:

DEFINITION 1.4. Let D be a compact domain. A family $F = \{F_H : H$ is a Hilbert space$\}$ of maps $F_H : D(H) \to B(H)$ is called a *continuous operator function* if for every Hilbert space H the following two conditions are fulfilled:

1) For every $a \in D(H)$ we have $F_H(a) \in C^*(a)$;

2) If $a \in D(H)$ and $\pi : C^*(a) \to B(K)$ is a representation then $F_K(\pi(a)) = \pi(F_H(a))$.

In the following we will drop the index H whenever it does not lead to misunderstanding.

The terminology used here is justified by the following theorem, proven in Section 3:

THEOREM 1.4. *If F is a continuous operator function then for each Hilbert space H the map $F_H: D(H) \to B(H)$ is continuous in the sense of the norm topologies.*

In the traditional mathematics, functions may be defined by means of their graphs. The analogous description takes place for operator functions.

THEOREM 1.5. 1) *Let F be a measurable (respectively continuous) operator function defined on a measurable (respectively compact) N-dimensional domain D; then the family D' introduced by*

$$D'(H) = \{(a^1, a^2, \ldots, a^N, F(a^1, \ldots, a^N)) : a \in D(H)\} \tag{1.1}$$

is the measurable (resp. compact) $(N+1)$-dimensional domain.

2) *Let D' be a measurable (respectively compact) $(N+1)$-dimensional domain such that for every Hilbert space H and every $a \in \mathscr{C}(H)^N$ there exists at most one $b \in \mathscr{C}(H)$ for which $(a^1, a^2, \ldots, a^N, b) \in D'(H)$. Then denoting by $D(H)$ the set of all $a \in \mathscr{C}(H)^N$ for which the operator b exists and setting*

$$F_H(a) = b \tag{1.2}$$

we have:

a) *D is a measurable (respectively compact) N-dimensional domain,*

b) *F is a measurable (respectively continuous) operator function defined on D.*

Clearly the proof of the above theorem should be divided into two parts: the one concerning the measurable case, the second concerning the case of continuous functions. The first case is almost obvious and is left to the reader. The second case will be proved in Section 3.

In [5] operator functions were introduced in the measurable sense. Hence it is very useful for further applications to find connections between measurable and continuous functions. Now we consider this relation.

THEOREM 1.6. 1) *Every continuous operator functions is a measurable one.*

2) *Every measurable operator function F defined on a compact domain D and such that the maps*

$$F_H: D(H) \to B(H)$$

are continuous in the norm topologies, is a continuous operator function.

Proof. The proof of both parts of the theorem is based on Theorem 1.5.

1) Let F be a continuous operator function of N variables. Then in virtue of the Theorem 1.5.1) the domain D' defined by $D'(H) = \{(a, F(a)) : a \in D(H)\}$, where D is the N-dimensional domain of F, is a compact $(N+1)$-dimensional domain; hence by the Theorem 1.1 it is a measurable domain. Thus by the Theorem 1.5.2) F is a measurable operator function.

2) Suppose now that the maps F_H are continuous in norm topologies. Then by Proposition 1.2 the sets $D'(H) = \{(a, F(a)) : a \in D(H)\}$ are closed in norm and at the same time they form a measurable $(N+1)$-dimensional domain by Theorem 1.5.1). Thus by Theorem 1.3 D' is a compact domain and in virtue of Theorem 1.5.2) b) F is a continuous operator function. Q.E.D.

2. COMPACT DOMAINS

At first we derive an obvious property of compact domains implied by Definition 1.2; this property will be used very frequently in the sequel.

PROPOSITION 2.1. *Let D be a compact N-dimensional domain and Λ be a denumerable set. Then for every family $a_i \in D(H_i)$, $i \in \Lambda$, we have*

$$\bigoplus_{i\in\Lambda} a_i \in D(\bigoplus_{i\in\Lambda} H_i).$$

Proof. At first we note that due to the condition 3) of Definition 1.2, $a = \bigoplus_{i\in\Lambda} a_i \in B(\bigoplus_{i\in\Lambda} H_i)^N$. Clearly any element $x \in C^*(a)$ is of the form $x = \bigoplus_{i\in\Lambda} x_i$, where $x_i \in B(H_i)$. Denoting by $\pi_i(x)$ the "i-th" component of x, $\pi_i(x) = x_i$, we introduce the faithful family $\{\pi_i\}_{i\in\Lambda}$ of representations of $C^*(a)$. Obviously $\pi_i(a) = a_i \in D(H_i)$. Therefore using the condition 2) of Definition 1.2 we get the desired result: $a \in D(\bigoplus_{i\in\Lambda} H_i)$. Q.E.D.

In what follows we shall use the free $*$-algebra spanned by N generators $X^1, X^2, \ldots, X^N$. This algebra will be denoted by P_N. Any element $w \in P_N$ is of the form:

$$w = \sum w_{i_1\ldots i_M} X^{i_1\#} X^{i_2\#} \ldots X^{i_M\#} \tag{2.1}$$

where $X^{i\#}$ denotes either X^i or X^{i*}, $w_{i_1\ldots i_M}$ are complex coefficients and the sum contains only a finite number of summands.

Let $w \in P_N$ and $a \in B(H)^N$. Replacing in w all X^i by a^i $(i = 1, 2, \ldots, N)$ we obtain an operator acting on H. This operator will be denoted by $w(a)$. For example, if w is given by (2.1) then

$$w(a) = \sum w_{i_1\ldots i_M} a^{i_1\#} \ldots a^{i_M\#}.$$

Clearly for any $a \in B(H)^N$ the mapping $P_N \ni w \mapsto w(a) \in B(H)$ is a representation of the $*$-algebra P_N. We also have

$$X^i(a) = a^i \quad \text{for} \quad i = 1, 2, \ldots, N. \tag{2.2}$$

PROPOSITION 2.2. *Let φ be a function on P_N with non-negative real values. For any Hilbert space H we set*

$$D_\varphi(H) := \{a \in B(H)^N : \|w(a)\| \leqslant \varphi(w) \text{ for all } w \in P_N\}.$$

Then D_φ is a compact domain.

Proof. Assume that $a \in D_\varphi(H)$. Then for any representation $\pi : C^*(a) \to B(K)$ we have:

$$\|w(\pi(a))\| = \|\pi(w(a))\| \leqslant \|w(a)\| \leqslant \varphi(w)$$

for all $w \in P_N$. It means that $\pi(a) \in D_\varphi(K)$, i.e. the condition 1) of Definition 1.2 holds.

Assume now that $a \in B(H)^N$ and that $\{\pi_i\}_{i \in \Lambda}$ is a faithful family of representations of $C^*(a)$. Then $\|x\| = \sup_{i \in \Lambda} \|\pi_i(x)\|$ for any $x \in C^*(a)$. If $\pi_i(a) \in D(H_i)$ (where H_i denotes the carrier Hilbert space of π_i) for all $i \in \Lambda$, then $\|w(\pi_i(a))\| \leqslant \varphi(w)$ for all $w \in P_N$ and $i \in \Lambda$ and

$$\|w(a)\| = \sup_{i \in \Lambda} \|\pi_i(w(a))\| = \sup_{i \in \Lambda} \|w(\pi_i(a))\| \leqslant \varphi(w)$$

for all $w \in P_N$. It means that $a \in D_\varphi(H)$ and the condition 2) of Definition 1.2 is satisfied.

If $a \in D_\varphi(H)$ then in virtue of (2.2)

$$\|a^i\| = \|X^i(a)\| \leqslant \varphi(X^i) \leqslant M$$

where $M := \max\{\varphi(X^1), \ldots, \varphi(X^N)\}$, i.e. the condition 3) of Definition 1.2. Q.E.D.

PROPOSITION 2.3. *Let H, K be Hilbert spaces and $a \in B(H)^N$, $b \in B(K)^N$. Then the following conditions are equivalent:*

1) *For every $w \in P_N$*

$$\|w(a)\| \geqslant \|w(b)\|; \tag{2.3}$$

2) *There exists a representation $\pi : C^*(a) \to B(K)$ such that $\pi(a) = b$;*

3) *For every N-dimensional compact domain D if $a \in D(H)$ then $b \in D(K)$.*

We say that b is *subordinated* to a if and only if one (all) of these conditions is satisfied. In this case we write $a \gg b$.

Proof. We shall show that 3) $\Rightarrow$ 1) $\Rightarrow$ 2).

The implication 2) $\Rightarrow$ 3) is obvious.

Assume that the condition 1) is false i.e. that $\|w_0(a)\| < \|w_0(b)\|$ for some $w_0 \in P_N$. Let φ be a function on P_N introduced by $\varphi(w) = \|w(a)\|$ for all $w \in P_N$ and let D_φ be the compact domain considered in Proposition 2.2. Obviously $a \in D_\varphi(H)$ and $b \notin D_\varphi(K)$, i.e. the condition 3) is false. This way we showed that 3) $\Rightarrow$ 1).

Let $C_0^*(a)$ denote the $*$-algebra generated by $a^1, a^2, \ldots, a^N$. Clearly $C_0^*(a)$ is dense in $C^*(a)$ and any element $x \in C_0^*(a)$ is of the form $x = w(a)$ where $w \in P_N$.

Assume now that the condition 1) is fulfilled. For any $x \in C_0^*(a)$ we set

$$\pi_0(x) \stackrel{\text{def}}{=} w(b) \tag{2.4}$$

where w is an element of P_N such that $w(a) = x$. If $w_1, w_2 \in P_N$ and $w_1(a) = w_2(a) = x$, then $(w_1 - w_2)(a) = 0$ and using (2.3) we get $(w_1 - w_2)(b) = 0$ i.e. $w_1(b) = w_2(b)$. This proves that the definition (2.4) is meaningful. Clearly π_0 is a representation of the $*$-algebra $C_0^*(a)$ and in virtue of (2.3)

$$\|\pi_0(x)\| \leqslant \|x\|$$

for any $x \in C_0^*(a)$. Let $\pi: C^*(a) \to B(K)$ denote the continuous extension of π_0. π is a representation of the C^*-algebra $C^*(a)$ and using (2.2) and (2.4) we have

$$\pi(a^i) = \pi_0(a^i) = \pi_0(X^i(a)) = X^i(b) = b^i$$

i.e. $\pi(a) = b$. Q.E.D.

REMARK. The representation π satisfying 2) of Proposition 2.3 is unique, because every representation π of the algebra A generated by $a^1, \ldots, a^N$ is uniquely determined by the images $\pi(a^1), \ldots, \pi(a^N)$.

It turns out that every compact domain D contains an element a which is maximal in the sense of the relation $\gg$. This property of compact domains will be very useful in the future.

THEOREM 2.4. *Let D be a compact domain. Then there exists a Hilbert space K and an element $b \in D(K)$ such that $b \gg a$ for any $a \in D(H)$ and any Hilbert space H.*

Proof. Let P_N^0 be the denumerable subset of P_N composed of all polynomials of the form (2.1), where all coefficients $w_{i_1 \ldots i_N}$ are complex rational.

It follows immediately from the condition 3) of Definition 1.2 that the numbers

$$\varphi_D(w) = \sup\{\|w(a)\| : a \in D(H),\ H \text{ is a Hilbert space}\}$$

are finite for all $w \in P_N$. Therefore for each natural number n and each $w \in P_N^0$ there exists a Hilbert space $K_{n,w}$ and an element $b_{n,w} \in D(K_{n,w})$ such that

$$\| w(b_{n,w}) \| \geqslant \varphi_D(w) - 1/n .$$

Let

$$K := \bigoplus_{\substack{n=1,2\dots \\ w \in P_N^0}} K_{n,w}$$

$$b := \bigoplus_{\substack{n=1,2\dots \\ w \in P_N^0}} b_{n,w} .$$

In virtue of Proposition 2.1 we have $b \in D(K)$.

Let H be a Hilbert space and $a \in D(H)$. Then for every $w \in P_N^0$ and any natural n we have

$$\| w(b) \| \geqslant \| w(b_{n,w}) \| \geqslant \varphi_D(w) - 1/n \geqslant \| w(a) \| - 1/n.$$

Setting $n \to \infty$ we obtain $\| w(b) \| \geqslant \| w(a) \|$ for every $w \in P_N^0$. By a continuity argument the same inequality holds for all $w \in P_N$. It means (cf. Proposition 2.3) that $b \gg a$. Q.E.D.

REMARK 2.5. Let D be a N-dimensional compact domain and b be a maximal element of D. Then according to Proposition 2.3 and Theorem 2.4:

$$D(H) = \{a \in B(H)^N : b \gg a\}. \tag{2.5}$$

Conversely, if K is a Hilbert space and b is an element of $B(K)^N$, then the formula (2.5) introduces a compact domain D. This fact follows from Proposition 2.2.

Assume that $a_n \in B(H)^N$ and $b \in B(K)^N$, $a_n \ll b$, $a_n \xrightarrow[n\to\infty]{} a \in B(H)^N$. It follows immediately from Proposition 2.2 that $a \ll b$. This way we obtain the result announced before in Proposition 1.2:

If D is a compact domain, then for each Hilbert space H the set $D(H)$ is closed in the norm.

To prove Theorem 1.1 we consider a measurable space (Λ, μ) and a measurable field of Hilbert spaces $\{H(\lambda)\}_{\lambda\in\Lambda}$. Assume that $\{a^i(\lambda)\}_{\lambda\in\Lambda}$ are bounded measurable fields of bounded operators.

Let $b \in B(K)^N$. If $a(\lambda) \ll b$ for μ-almost all $\lambda \in \Lambda$ then for the same λ's $\| w(a(\lambda)) \| \leqslant \|w(b)\|$ for all $w \in P_N$. Therefore

$$\left\| w\left(\int_\Lambda^\oplus a(\lambda) \mathrm{d}\mu(\lambda)\right)\right\| = \left\| \int_\Lambda^\oplus w(a(\lambda)) \mathrm{d}\mu(\lambda)\right\| =$$

$$= \operatorname*{ess\,sup}_{\lambda\in\Lambda} \| w(a(\lambda)) \| \leqslant \| w(b) \|$$

for all $w \in P_N$. It means that

$$\int_A^{\oplus} a(\lambda)\mathrm{d}\mu(\lambda) \preccurlyeq b. \tag{2.6}$$

Conversely if (2.6) holds then for all $w \in P_N$ we have:

$$\sup_{\lambda\in A} \operatorname{ess} \| w(a(\lambda)) \| = \left\| \int_A^{\oplus} w(a(\lambda))\mathrm{d}\mu(\lambda) \right\| =$$

$$= \left\| w\left(\int_A^{\oplus} a(\lambda)\mathrm{d}\mu(\lambda)\right) \right\| \leqslant \| w(b) \|.$$

It means that $\| w(a(\lambda)) \| \leqslant \| w(b) \|$ for μ-a.a. $\lambda \in A$. More precisely for every $w \in P_N$ there exists a subset $A_w \subset A$ such that $\mu(A_w) = 0$ and

$$\| w(a(\lambda)) \| \leqslant \| w(b) \| \tag{2.7}$$

for all $\lambda \in A \setminus A_w$.

Let

$$A_0 = \bigcup_{w\in P_N^0} A_w$$

(see the proof of Theorem 2.4 for the meaning of P_N^0). Then $\mu(A_0) = 0$ (P_N^0 is denumerable) and (2.7) holds for all $w \in P_N^0$ and $\lambda \in A \setminus A_0$. By a continuity argument (2.7) holds for all $w \in P_N$ and $\lambda \in A \setminus A_0$, i.e.

$$a(\lambda) \preccurlyeq b \tag{2.8}$$

for μ-a.a. $\lambda \in A$.

This way we have shown that the condition (2.6) and (2.8) are equivalent. Taking into account (2.5) we see that for compact domains D the condition:

$$\int_A^{\oplus} a(\lambda)\mathrm{d}\mu(\lambda) \in D\left(\int_A^{\oplus} H(\lambda)\mathrm{d}\mu(\lambda)\right)$$

and

$$a(\lambda) \in D(H(\lambda)) \quad \text{for } \mu\text{-a.a. } \lambda \in A$$

are equivalent, i.e. for all compact domains the condition 2) of Definition 1.1 is satisfied.

To end the proof of Theorem 1.1 we notice that obviously $UaU^* \preccurlyeq a$ for every unitary operator $U\colon H_1 \to H_2$ and every $a \in B(H_1)^N$. Therefore $UaU^* \preccurlyeq b$ whenever $a \preccurlyeq b$. In virtue of (2.5), compact domains satisfy the condition 1) of Definition 1.1.

Summarizing: compact domains are measurable ones.

3. ALGEBRAS OF CONTINUOUS OPERATOR FUNCTIONS

The well known theorem of Gelfand and Naimark says that there is a one-to--one correspondence between commutative unital C^*-algebras and compact topological spaces. Every commutative C^*-algebra with unit is isomorphic to the algebra of all continuous complex valued functions on the corresponding compact space. We show here an analogous result for the non-commutative case. In this case compact topological spaces and continuous functions are replaced by compact domains and continuous operator functions respectively. We shall show that the set of all continuous operator functions defined on a given domain, endowed with a natural algebraic and topological structure is a C^*-algebra. This C^*-algebra is finitely generated and any finitely generated C^*-algebra arises in this way.

For a given finitely generated C^*-algebra the corresponding compact domain is defined uniquely up to a homeomorphism. We shall also prove an analogue of the Stone-Weierstrass theorem (Theorem 3.4). Theorem 1.4 follows immediately from this result.

We start with the following:

THEOREM 3.1. *Let D be a compact domain and F be a continuous operator function on D. Then there exists a constant $M > 0$ such that $\|F(a)\| \leqslant M$ for every Hilbert space H and all $a \in D(H)$.*

Proof. Let b be a maximal element of D and put $M = \|F(b)\|$. Then for every $a \in D(H)$ there exists a representation $\pi: C^*(b) \to B(H)$ such that $\pi(b) = a$ (cf. Proposition 2.3). Using the property 2) in Definition 1.4 we have:

$$\|F(a)\| = \|F(\pi(b))\| = \|\pi(F(b))\| \leqslant \|F(b)\| = M. \tag{3.1}$$

Q.E.D.

Let D be a compact domain. The set of all continuous operator functions defined on D will be denoted by $C(D)$.

For any $F, G \in C(D)$ and $\lambda \in \mathbf{C}^1$ we set for $a \in D(H)$ where H is a Hilbert space:

$$\begin{aligned} (F+G)(a) &:= F(a) + G(a) \\ (\lambda F)(a) &:= \lambda F(a) \\ (FG)(a) &:= F(a)\, G(a) \\ (F^*)(a) &:= (F(a))^*. \end{aligned} \tag{3.2}$$

One can easily check that $F+G$, λF, FG, $F^* \in C(D)$. Moreover for any $F \in C(D)$ we put

$$\|F\| = \sup\{\|F(a)\| : a \in D(H),\ H \text{ is a Hilbert space}\}. \tag{3.3}$$

The above supremum is finite for all $F \in C(D)$. This fact follows from Theorem 3.1.

THEOREM 3.2. *For every compact domain D, the set C(D) endowed with the algebraic rules* (3.2) *and the norm* (3.3) *is a C^*-algebra.*

Proof. By simple computations one can show that $C(D)$ is a $*$-algebra. It is also obvious that the formula (3.3) defines a C^*-norm on $C(D)$. To end the proof one has to show that $C(D)$ is complete. We can prove this fact directly. Let $F_1, F_2, \ldots$ be a Cauchy sequence in $C(D)$. Then for every a in the domain D, $F_1(a)$, $F_2(a)$, $\ldots$ is a Cauchy sequence in $C^*(a)$. Since $C^*(a)$ is complete, this sequence is convergent. We set

$$F(a) = \lim_{n\to\infty} F_n(a).$$

One can easily check that $F \in C(D)$ and that $F = \lim F_n$. Q.E.D.

To understand better the structure of $C(D)$ we shall prove that this C^*-algebra is isomorphic to $C^*(b)$, where b is a maximal element of D.

PROPOSITION 3.3. *Let D be a compact domain and b be a maximal element of D. Then the mapping:*

$$\varphi : C(D) \ni F \mapsto F(b) \in C^*(b) \tag{3.4}$$

is a C^*-algebra isomorphism of $C(D)$ onto $C^*(b)$.

Proof. It follows immediately from the definitions (3.4) and (3.2) that φ is a C^*-algebra homomorphism. Moreover if $F(b) = 0$ then according to (3.1) $F(a) = 0$ for any a in D and so $F = 0$. It means that φ is injective. To end the proof we have to show that φ is surjective.

Let $x \in C^*(b)$. For every Hilbert space H and every $a \in D(H)$ there exists a unique representation $\pi_a : C^*(b) \to B(H)$ such that $\pi_a(b) = a$ (cf. Proposition 2.3). Let

$$F(a) \overset{\text{def}}{=} \pi_a(x).$$

We shall prove that $F \in C(D)$. Indeed π_a maps $C^*(b)$ into $C^*(a)$, therefore $F(a) = \pi_a(x) \in C^*(a)$. Moreover if ρ is a representation of $C^*(a)$ acting on a Hilbert space H_ρ then $\rho(a) \in D(H_\rho)$ and $\rho(a) = (\rho \circ \pi_a)(b)$. It means that $\pi_{\rho(a)} = \rho \circ \pi_a$ and

$$F(\rho(a)) = \pi_{\rho(a)}(x) = (\rho \circ \pi_a)(x) = \rho(\pi_a(x)) = \rho(F(a)).$$

Moreover noticing that π_b = identity map we have $F(b) = \pi_b(x) = x$ i.e. $\varphi(F) = x$. It means that φ is surjective: for any $x \in C^*(b)$ we have found $F \in C(D)$ such that $\varphi(F) = x$. Q.E.D.

The classical Stone-Weierstrass theorem asserts that for any compact subset $K \subset \mathbf{C}^N$, the C^*-algebra $C(K)$ is generated by the coordinates. In the case of compact domains one can consider coordinate functions X^i introduced by

$$X^i(a) = a^i \tag{3.5}$$

where $i = 1,2, \ldots, N$, N is the dimension of D, $a \in D(H)$, and H is a Hilbert space.

One can easily check that $X^1, X^2, \ldots, X^N \in C(D)$.

THEOREM 3.4. *For every compact domain D, the C^*-algebra $C(D)$ is generated by the coordinate functions $X^1, X^2, \ldots, X^N$, where N is the dimension of D.*

Proof. We use the isomorphism φ (cf. 3.4) described in Proposition 3.3. Obviously

$$\varphi(X^i) = X^i(b) = b^i$$

and because $b^1, b^2, \ldots, b^N$ generate the C^*-algebra $C^*(b)$ we get the desired result. Q.E.D.

COROLLARY 3.5. *For any compact domain D the C^*-algebra $C(D)$ is finitely generated.*

Now we can prove Theorem 1.4.

Let $F \in C(D)$. According to Theorem 3.4 $F = \lim F_n$, where F_n are polynomials of X^i and X^{i*} $(i = 1,2, \ldots, N)$. Obviously for any Hilbert space H, X^i and X^{i*} are continuous maps from $D(H)$ into $B(H)$. Since the algebraic rules in $B(H)$ are continuous, every F_n defines a continuous map from $D(H)$ into $B(H)$. It is also obvious that the norm convergence in $C(D)$ implies the uniform convergence of maps from $D(H)$ into $B(H)$. Since the uniform limit of continuous maps is a continuous map we see that F is a continuous mapping $F: D(H) \to B(H)$. This ends the proof of Theorem 1.4.

In Theorem 1.5 a domain has been extended by enlarging its dimension by one. It has been done by means of operator function F, and in fact we have used the map $\mathrm{id} \times F: D \to D'$ that transformed one domain onto another one. Hence it seems interesting to consider such maps more generally.

Let D be a compact domain of dimension N. We shall consider M-tuple $S = (S^1, S^2, \ldots, S^M)$ where $S^1, S^2, \ldots, S^M \in C(D)$. For any Hilbert space H and any $a \in D(H)$ we set

$$S(a) \overset{\text{def}}{=} (S^1(a), S^2(a), \ldots, S^M(a)) \in B(H)^M.$$

Then we have:

$$C^*(S(a)) \subset C^*(a) \tag{3.6}$$

$$\pi(S(a)) = S(\pi(a)) \tag{3.7}$$

for any representation π of $C^*(a)$. These facts follow directly from Definition 1.4.

Let D and D_1 be compact domains of dimension N and M respectively. Assume that $S \in C(D)^M$. We say that S is a morphism from D into D_1 if and only if for any Hilbert space H and any $a \in D(H)$ we have

$$S(a) \in D_1(H).$$

In this case we write $S: D \to D_1$.

If $S: D \to D_1$ and $F \in C(D_1)$ then for any Hilbert space H and any $a \in D(H)$ we set:

$$S^*(F)(a) = F(S(a)). \tag{3.8}$$

One can easily check that $S^*(F) \in C(D)$.

Assume now that D, D_1, D_2 are compact domains of dimensions N, M_1, M_2 respectively and that $S: D \to D_1$ and $R: D_1 \to D_2$. Then we set $R \circ S =$ $(S^*(R^1), S^*(R^2), \ldots, S^*(R^{M_2}))$. Obviously for any $a \in D(H)$ we have

$$(R \circ S)(a) = R(S(a)).$$

Therefore $R \circ S$ is a morphism from D into D_2. This way we introduced the composition of morphisms. Compact domains together with their morphisms constitute a category. Taking into account the obvious analogy with the category of compact spaces and continuous maps we say that a morphism $S: D \to D_1$ is a homeomorphism if there exists a morphism $R: D_1 \to D$ such that $S \circ R = \mathrm{id}$, $R \circ S = \mathrm{id}$. In this case D and D_1 are said to be homeomorphic.

Let D_1 and D_2 be compact domains and $S: D_1 \to D_2$ be a morphism. Then one can easily check that the inverse image map $S^*: C(D_2) \to C(D_1)$ introduced by (3.8) is a unital homomorphism of C^*-algebras. It turns out that every homomorphism $\varkappa: C(D_2) \to C(D_1)$ is of this form.

PROPOSITION 3.6. *Let D_1 and D_2 be compact domains and $\varkappa: C(D_2) \to C(D_1)$ be a unital C^*-algebra homomorphism. Then there exists a unique morphism $S: D_1 \to D_2$, such that $\varkappa = S^*$.*

Proof. Let $X^1, X^2, \ldots, X^M$ be the coordinate functions on the domain D_2. We put $S^i = \varkappa(X^i) \in C(D_1)$ and $S = (S^1, S^2, \ldots, S^M)$. We shall show that S is a morphism from D_1 into D_2. To this end we have to prove that for any Hilbert space H and for any $a \in D_1(H)$, $S(a) \in D_2(H)$. Let b be a maximal element of D_2. We consider the following composition of C^*-algebra homomorphisms:

$$\rho: C^*(b) \xrightarrow{\varphi^{-1}} C(D_2) \xrightarrow{\varkappa} C(D_1) \xrightarrow{\varphi_a} B(H)$$

where φ^{-1} is the inverse of the isomorphism (3.4) considered in Proposition 3.3 and $\varphi_a(F) \overset{\text{def}}{=} F(a)$ for any $F \in C(D_1)$. Now we have

$$\rho(b^i) = (\varphi_a \circ \varkappa \circ \varphi^{-1})(b^i) = (\varphi_a \circ \varkappa)(X^i) = \varphi_a(S^i) = S^i(a)$$

for $i = 1, 2, \ldots, M$. In other words, $\rho(b) = S(a)$. Therefore (cf. condition 1) of Definition 1.2) $S(a) \in D_2(H)$. Moreover for any $a \in D_1(H)$ we have

$$\varkappa(X^i)(a) = S^i(a) = X^i(S(a)) = (S^*X^i)(a),$$

i.e. $\varkappa(X^i) = S^*(X^i)$ for $i = 1, 2, \ldots, M$ and using Theorem 3.4 we get $\varkappa = S$.

The uniqueness of S is obvious. Q.E.D.

Now we are able to formulate our version of the Gelfand-Naimark theorem:

THEOREM 3.7. *Let A be a finitely generated unital C^*-algebra. Then there exists a compact domain D such that A is isomorphic to $C(D)$. The domain D is defined uniquely up to a homeomorphism.*

Proof. We may assume that $A \subset B(K)$, where K is a separable Hilbert space. Let $b^1, b^2, \ldots, b^N \in A$ be generators of A and $b = (b^1, b^2, \ldots, b^N)$. For any Hilbert space H we set

$$D(H) = \{a \in B(H)^N : a \preccurlyeq b\}.$$

Then D is a compact domain (cf. Remark 2.5), b is a maximal element of D and in virtue of Proposition 3.3 the C^*-algebra $C(D)$ is isomorphic to $C^*(b) = A$. The last statement of the theorem follows directly from Proposition 3.6. Q.E.D.

In the theory of compact spaces any continuous image of a compact space is compact. Moreover if a continuous map is bijective then the inverse map is continuous. We have an analogous result.

THEOREM 3.8. *Let D be a compact domain of dimension N and $S^1, S^2, \ldots, S^M \in C(D)$. We assume that for any Hilbert space H and for any $a_1, a_2 \in D(H)$ we have:*

$$(S(a_1) = S(a_2)) \Rightarrow (a_1 = a_2) \tag{3.9}$$

i.e. that S is injective. For any Hilbert space H we set

$$D_1(H) = \{S(a) : a \in D(H)\}. \tag{3.10}$$

Moreover if $c \in D_1(H)$ then we set $R^k(c) = a^k$, where a is the unique element of $D(H)$ such that $S(a) = c$.

Then D_1 is a compact domain, $R^1, R^2, \ldots, R^N \in C(D_1)$ and $R = (R^1, \ldots, R^N)$ is a morphism from D_1 into D. This morphism is the inverse of S.

To prove this theorem we need the following lemma:

LEMMA 3.9. *Let A be a C^*-algebra and B be a C^*-subalgebra of A. Assume that for any two representations π, π' of A if $\pi|B = \pi'|B$ then $\pi = \pi'$.*

In this case $B = A$.

Proof. It is sufficient to show that any continuous functional $f \in A^*$ such that $f(b) = 0$ for all $b \in B$ vanishes identically on A.

Let f be such a functional. Remembering that any element of A^* is a linear combination of positive functionals and using GNS construction one can find a representation $\pi: A \to B(H)$ and a finite dimensional operator $\rho \in B(H)$ such that

$$f(a) = \mathrm{Tr}(\rho\pi(a))$$

for all $a \in A$.

Let $V \in B(H)$ be a unitary operator commuting with $\pi(b)$ for all $b \in B$. For every $a \in A$ we set

$$\pi'(a) = V^*\pi(a)V.$$

Then π' is a representation of A and $\pi'|B = \pi|B$. We assumed that in this case $\pi = \pi'$. It means that $[V, \pi(a)] = 0$ for any $a \in A$. This fact holds for every unitary operator V commuting with $\pi(B) = \{\pi(b) : b \in B\}$. In virtue of the von Neumann density theorem $\pi(a)$ is contained in the weak closure of $\pi(B)$. Therefore there exists a (generalized) sequence $b_\alpha \in B$ such that

$$\pi(a) = \text{w-lim}\, \pi(b_\alpha).$$

Now we have

$$f(a) = \mathrm{Tr}(\rho\, \text{w-lim}\pi(b_\alpha)) = \lim \mathrm{Tr}(\rho\, \pi(b_\alpha)) = \lim f(b_\alpha) = 0$$

because $f|B = 0$. Q.E.D

Proof of the Theorem 3.8. In virtue of (3.6) we have $C^*(S(a)) \subset C^*(a)$ for any element a of the domain D. It turns out that the injectivity condition (3.9) implies more: for any such a we have:

$$C^*(S(a)) = C^*(a). \tag{3.11}$$

Indeed assume that π, π' are representations of $C^*(a)$ such that they coincide on $C^*(S(a))$. Then using (3.7) we obtain

$$S(\pi(a)) = \pi(S(a)) = \pi'(S(a)) = S(\pi'(a)).$$

Taking into account the assumption (3.9) we get $\pi'(a) = \pi(a)$. Therefore π and π' coincide on the C^*-algebra generated by a, i.e. on $C^*(a)$. Now (3.11) follows immediately from Lemma 3.9.

Now we are able to show that D_1 is a compact domain.

Let b be a maximal element of D. Then for any Hilbert space H and any $a \in B(H)^M$ the following five statements are equivalent:

1) $a \in D_1(H)$;

2) There exists $x \leqslant b$ such that $S(x) = a$;

3) There exists a representation $\pi : C^*(b) \to B(H)$ such that $S(\pi(b)) = a$;

4) There exists a representation $\pi : C^*(S(b)) \to B(H)$ such that $\pi(S(b)) = a$;

5) $a \leqslant S(b)$.

Indeed, equivalence 1) $\Leftrightarrow$ 2) follows directly from definition (3.10) and Remark 2.5.

Using Proposition 2.3 we get 2) $\Leftrightarrow$ 3) and 4) $\Leftrightarrow$ 5).

3) $\Leftrightarrow$ 4) follows from (3.11) and (3.7). Equivalence 1) $\Leftrightarrow$ 5) means that

$$D_1(H) = \{a \in B(H)^M : a \leqslant S(b)\},$$

i.e. D_1 is a compact domain (cf. Remark 2.5).

Assume now that $c \in D_1(H)$. Then $c = S(a)$ for some $a \in D(H)$ and (cf. 3.11):

$$R^k(c) = a^k \in C^*(a) = C^*(S(a)) = C^*(c)$$

i.e. R^k satisfies the condition 1) of Definition 1.4. Moreover if π is a representation of $C^*(c)$, then $\pi(c) = \pi(S(a)) = S(\pi(a))$ and $R^k(\pi(c))$ equals to the k-component of $\pi(a)$ i.e. to the operator $\pi(a^k) = \pi(R^k(c))$. It means that R^k satisfies the condition 2) of Definition 1.4. Therefore $R^k \in C(D_1)$.

To end the proof one notices that for any $c \in D_1(H)$, $R(c) = a$, where a is an element of $D(H)$ such that $S(a) = c$. It means that R is a morphism from D_1 into D, inverse to S. Q.E.D.

Now we are able to present the proof of the more difficult part of Theorem 1.5: the part concerning compact domains and continuous functions. We use the notation introduced in the text of Theorem 1.5.

Proof of Theorem 1.5. Ad. 1) For any element a of D we set

$$S^1(a) = a^1,\ S^2(a) = a^2,\ \ldots,\ S^N(a) = a^N$$

$$S^{N+1}(a) = F(a).$$

Clearly $S^1, S^2, \ldots, S^{N+1} \in C(D)$ and the $(N+1)$-tuple $S = (S^1, S^2, \ldots, S^{N+1})$ satisfies the assumption of Theorem 3.8. Moreover the sets $D'(H)$ introduced by (1.1) coincide with $D_1(H)$ defined by (3.10) and the desired statement follows directly from Theorem 3.8.

Ad. 2) For any $a \in D'(H)$ we set

$$S^1(a) = a^1,\ S^2(a) = a^2,\ \ldots,\ S^N(a) = a^N.$$

The N-tuple $S = (S^1, S^2, \ldots, S^N)$ satisfies the assumption (3.9) of Theorem 3.8. Indeed if $a, a' \in D'(H)$ and $S(a) = S(a')$ then a and a' have the same first N components and according to the assumption of Theorem 1.5.2) in this case the last components must be equal. This shows that $S(a) = S(a')$ implies that $a = a'$.

To end the proof we notice that the sets $D(H)$ considered in the second part of Theorem 1.5 coincide with $D_1(H)$ introduced by (3.10). Moreover $F_H(a)$ introduced by (1.2) equals to $R^{N+1}(a)$. Q.E.D.

We should point out that in general the image of a compact domain needs not be a domain.

COUNTEREXAMPLE. For any Hilbert space H we set

$$D(H) = \{(U, V) \in B(H)^2 : U, V \text{ unitary operatory } UV = e^{i}VU\} \tag{3.12}$$

$$S(U, V) = U, \quad \text{for } (U, V) \in D(H).$$

Clearly D is a compact domain and $S \in C(D)$.

Let for any Hilbert space H

$$D_1(H) = \{S(U, V) : (U, V) \in D(H)\} =$$

$$= \{U \in B(H) : \text{there exists } V \in B(H) \text{ such that } (U, V) \in D(H)\}.$$

It turns out that D_1 is not a domain. Indeed due to the commutation relation with V, for any $U \in D_1(H)$ the spectrum of U must coincide with $S^1 = \{z \in \mathbf{C}^1 : |z| = 1\}$ (it must be closed and invariant under rotation by the angle of 1 radian). On the other hand U may be written as $U_1 \oplus U_2$ with $\mathrm{Sp}U_1 \subsetneq S^1$ and $\mathrm{Sp}U_2 \subsetneq S^1$. Therefore $U_1, U_2 \notin D_1(H)$ and the condition 2) of Definition 1.1 is not satisfied. Hence $D_1 = S(D)$ is not a measurable domain.

4. THE APPLICATION OF A VOICULESCU RESULT

In this section we prove that any measurable domain closed in a certain sense is a compact domain. The following proposition contains the strongest version of such a result. Theorem 1.3 follows immediately from it.

PROPOSITION 4.1. *Assume that for any Hilbert space H we have a given subset $D(H) \subset B(H)^N$ and that the following conditions are satisfied:*

1) UNITARY COVARIANCE. *If U is a unitary operator acting from H onto K (H, K are Hilbert spaces) and $a \in D(H)$ then $UaU^* \in D(K)$.*

2) DIRECT SUM PROPERTY. *If $a(\alpha) \in B(H(\alpha))^N$, $\alpha \in \Lambda$, ($H(\alpha)$ are Hilbert spaces, Λ is a denumerable set) then $\bigoplus_{\alpha\in\Lambda} a(\alpha) \in D(\bigoplus_{\alpha\in\Lambda} H(\alpha))$ if and only if $a(\alpha) \in D(H(\alpha))$ for all $\alpha \in \Lambda$.*

3) CLOSEDNESS. *If $a \in B(H)^N$ and for every $\varepsilon > 0$, there exists $a_\varepsilon \in D(H)$ such that $a^i - a^i_\varepsilon$ are compact operators and $\|a^i - a^i_\varepsilon\| \leqslant \varepsilon$ ($i = 1, 2, \ldots, N$) then $a \in D(H)$.*

Then D is a compact domain.

The proof of the above proposition is based on a result of D. Voiculescu [4]. We say that a representation π of a C^*-algebra A is strongly faithful if for any non-zero element $a \in A$ the operator $\pi(a)$ is not compact. Obviously, for any faithful representation ρ of A the infinite direct sum $\rho \oplus \rho \oplus \ldots$ is strongly faithful. Moreover, if π is strongly faithful then $\rho \oplus \pi$ is strongly faithful for any representation ρ.

The following lemma is a simplified version of Corollary 1.4 of [4].

LEMMA 4.2. *Let A be a unital separable C^*-algebra and π_1, π_2 be strongly faithful representation of A acting on separable Hilbert spaces H_1, H_2 respectively.*

Then for each positive number ε and every finite family of elements $x^1, \ldots, x^N \in A$ there exists a unitary mapping $U : H_2 \to H_1$ such that for all $k = 1, 2, \ldots, N$ the operators $\pi_2(x^k) - U^*\pi_1(x^k)U$ are compact and

$$\|\pi_2(x^k) - U^*\pi_1(x^k)U\| \leqslant \varepsilon.$$

Proof of Proposition 4.1. Assume that $D = \{D(H) : H \text{ is a Hilbert space}\}$ satisfies the three conditions given in Proposition 4.1. Let H be a Hilbert space, $a \in D(H)$ and π be a representation of $C^*(a)$ acting on a separable Hilbert space K. Let us consider the following strongly faithful representations of $C^*(a)$:

$$\pi_1 = \mathrm{id} \oplus \mathrm{id} \oplus \ldots$$

$$\pi_2 = \pi \oplus \pi_1$$

acting on Hilbert spaces $H_1 = H \oplus H \oplus \ldots$ and $H_2 = K \oplus H \oplus H \oplus \ldots$ respectively. It follows from Lemma 4.2 that for any $\varepsilon > 0$ there exists a unitary $U_\varepsilon : H_2 \to H_1$ such that $\pi_2(a^i) - U_\varepsilon^*\pi_1(a^i)U_\varepsilon$ is a compact operator of norm less than ε. We assumed that $a \in D(H)$. Using the direct sum property we see that $\pi_1(a) \in D(H_1)$. Therefore according to the unitary covariance condition

$U_\varepsilon^* \pi_1(a) U_\varepsilon \in D(H_2)$. Now we use the closedness of D. We obtain $\pi_2(a) \in D(H_2)$. Using once more the direct sum property we finally get $\pi(a) \in D(K)$. This way we showed that D satisfies the condition 1) of Definition 1.2.

Now let H be a Hilbert space and $a \in B(H)^N$. Assume that there exists a family $\{\rho_\lambda\}_{\lambda \in \Lambda}$ of representations of $C^*(a)$:

$$\rho_\lambda : C^*(a) \to B(H_\lambda)$$

such that $\rho_\lambda(a) \in D(H_\lambda)$ and $\bigcap_{\lambda \in \Lambda} \mathrm{Ker}\rho_\lambda = \{0\}$. The C^*-algebra $C^*(a)$ is finitely generated hence separable. Therefore one can find a denumerable subset $\Lambda_0 \subset \Lambda$ such that $\bigcap_{\lambda \in \Lambda_0} \mathrm{Ker}\rho_\lambda = \{0\}$. Let

$$\rho = \bigoplus_{\lambda \in \Lambda_0} \rho_\lambda.$$

Then ρ is a faithful representation of $C^*(a)$ and according to the direct sum property $\rho(a) \in D(K)$ where $K = \bigoplus_{\lambda \in \Lambda_0} H$. Clearly ρ^{-1} is a representation of $C^*(\rho(a))$ and $\rho^{-1}(\rho(a)) = a$. Using (already proven) condition 1) of Definition 1.2 we obtain $a \in D(H)$. This shows that D satisfies the condition 2) of Definition 1.2.

To end the proof we have to show that D is bounded. Assume the contrary that for any natural n there exists a Hilbert space H_n and $a_n \in D(H_n)$ such that

$$\sup_{i=1,\ldots,N} \|a_n^i\| \geqslant n.$$

Let $H = \bigoplus_{n=1}^{\infty} H_n$ and $a = \bigoplus_{n=1}^{\infty} a_n$. Then $a \in D(H)$ and $a \notin B(H)^N$. This contradicts the assumed inclusion $D(H) \subset B(H)^N$ for any Hilbert space H. It shows that our conjecture was false, i.e. that D is bounded. Q.E.D.

5. INVARIANT OPERATOR FUNCTIONS

In [5] we considered measurable operator functions homogeneous with respect to a group of variables. Those are functions invariant under a special action of the group R^1. To be more precise we set:

$$\sigma_t(a^1, a^2, \ldots, a^K, b^1, \ldots, b^N) = (a^1, a^2, \ldots, a^K, e^t b^1, \ldots, e^t b^N).$$

A measurable operator function F defined on a measurable domain D is called homogeneous with respect to the last N variables if and only if for any Hilbert space H and any $a \in D(H)$ we have $\sigma_t(a) \in D(H)$ and $F(\sigma_t(a)) = F(a)$.

In [5] we discovered a nontrivial, very interesting result concerning homogeneous operator functions. One may try to obtain similar results considering operator functions invariant under more complicated groups. In the present section we consider compact groups acting on operator domains in the linear way. An application to the theory of ergodic action of compact groups on C^*-algebras is indicated ([2], [3]).

Let G be a compact group of $N \times N$ complex matrices. The group G acts on P_N and $B(H)^N$ in a natural way.

Let $g \in G$ and $w \in P_N$. Then w is of the form (cf. (2.1))

$$w = \sum w_{i_1 \dots i_M} X^{i_1 \#} \dots X^{i_M \#} \tag{5.1}$$

where $X^{i\#}$ denotes either X^i or X^{i*}, $w_{i_1 \dots i_M}$ are complex coefficients and the sum is finite. Replacing in (5.1) X^i by $\sum_{k=1}^{N} g^i_k X^k$ we obtain another element of P_N. This element will be denoted by $g^* w$. Clearly the mapping

$$P_N \ni w \mapsto g^* w \in P_N$$

is an automorphism of P_N. We say that an element $w \in P_N$ is G-invariant if $g^* w = w$ for all $g \in G$. The set of all G-invariant elements of P_N will be denoted by P^G_N.

Let $g \in G$ and $a \in B(H)^N$, where H, is a Hilbert space. Then ga will denote an element of $B(H)^N$ such that $(ga)^i = \sum_{k=1}^{N} g^i_k a^k$.

It follows immediately from the above definitions that

$$w(ga) = (g^* w)(a)$$

for all $w \in P_N$, $g \in G$, $a \in B(H)^N$.

DEFINITION 5.1. 1) Let D be a measurable domain. D is called a *G-invariant domain* if $ga \in D(H)$ for each $g \in G$, each Hilbert space H, and all $a \in D(H)$.

2) Let D be a G-invariant measurable domain and F be a measurable operator function defined on D. F is called an *invariant operator function* if $F(ga) = F(a)$ for each $g \in G$, each Hilbert space H, and all $a \in D(H)$.

The main result of this section is contained in the following theorem.

THEOREM 5.1. *Let G be a compact group of $N \times N$ complex matrices, D be a G-invariant N-dimensional measurable domain, F be a G-invariant measurable function defined on D, H be a Hilbert space, $a \in D(H)$ and $b \in B(H)^N$. Assume that $u(a) = u(b)$ for all $u \in P^G_N$. Then:*

1) $b \in D(H)$,
2) $F(a) = F(b)$.

Before the proof we have to introduce convenient notation.

Let μ be a normalized Haar measure on G. We consider the Hilbert space

$$\tilde{H} = \int_G^{\oplus} H \, d\mu(g) = L^2(G, H).$$

The elements of $\tilde{H}$ are square integrable functions on G with values in H. The constant functions form a subspace of $\tilde{H}$ canonically isomorphic to H. For any $h \in H$, $\tilde{h}$ will denote the corresponding element of $\tilde{H}$: $\tilde{h}(g) = h$ for all $g \in G$.

For any $c \in B(H)^N$ we set

$$\tilde{c} \overset{\text{def}}{=} \int_G^{\oplus} (gc) \, d\mu(g).$$

Obviously $\tilde{c} \in B(\tilde{H})^N$.

Let K be a subspace of $\tilde{H}$. We say that K is $\tilde{c}$-invariant if $\tilde{c}^i k$, $\tilde{c}^{i*} k \in K$ for all $k \in K$ and $i = 1, 2, \ldots, N$. In this case the orthogonal complement $K^\perp = \tilde{H} \ominus K$ is also $\tilde{c}$-invariant and using the obvious notation we have

$$\tilde{c} = \tilde{c}|K \oplus \tilde{c}|K^\perp. \tag{5.2}$$

Let H_c denote the subspace of $\tilde{H}$ spanned by $\{w(\tilde{c})\tilde{h} : w \in P_N, h \in H\}$. One can easily check that H_c is the smallest $\tilde{c}$-invariant subspace of $\tilde{H}$ containing all constant functions.

LEMMA 5.2. *For any G-invariant measurable domain D, $c \in D(H)$ if and only if $\tilde{c}|H_c \in D(H_c)$. Moreover in this case*

$$F(\tilde{c}|H_c)\tilde{h} = (F(c)h)^{\sim} \tag{5.3}$$

for any G-invariant measurable operator function F defined on D and any $h \in H$.

Proof. Assume that $c \in D(H)$. Then $gc \in D(H)$ for all $g \in G$, because D is G-invariant. Using the condition 1) of Definition 1.1 we get $\tilde{c} = \int_G^{\oplus} (gc) d\mu(g) \in D(\tilde{H})$ and (cf. (5.2)) $\tilde{c}|H_c \in D(H_c)$.

Conversely assume now that $\tilde{c}|H_c \in D(H_c)$. A $\tilde{c}$-invariant subspace $K \subset \tilde{H}$ will be called admissible if and only if $\tilde{c}|K \in D(K)$. Our assumption means that H_c is admissible.

If K_1 and K_2 are subspaces of $\tilde{H}$ then the smallest subspace of $\tilde{H}$ containing K_1 and K_2 will be denoted by $K_1 \vee K_2$.

Let K_1 and K_2 be admissible subspaces. If $K_1 \perp K_2$ then clearly $K_1 \oplus K_2$ is admissible. In fact it follows directly from the condition 2) of Definition 1.1. We shall prove that in general:

$$(K_1, K_2 \text{ admissible}) \Rightarrow (K_1 \vee K_2 \text{ admissible}). \tag{5.4}$$

To this end we consider the subspaces

$$K_3 = K_2 \ominus (K_1 \cap K_2)$$

$$K_4 = (K_1 \vee K_2) \ominus K_1.$$

The subspaces K_3 and K_4 are $\tilde{c}$-invariant and K_3 is admissible (because it is a $\tilde{c}$--invariant subspace of admissible K_2). Let P_3 be the orthogonal projection onto K_3. We consider this projection restricted to K_4:

$$P_3|K_4 : K_4 \to K_3. \tag{5.5}$$

One can easily check that the kernel of this operator is trivial and the image is dense in K_3. Let S be the unitary factor in the polar decomposition of (5.5). Remembering that all subspaces considered so far are $\tilde{c}$-invariant one can prove that

$$\tilde{c}|K_4 = S^*(\tilde{c}|K_3)\, S.$$

We noticed that K_3 is admissible. It means that $\tilde{c}|K_3 \in D(K_3)$. Using the above formula we get: $\tilde{c}|K_4 \in D(K_4)$, i.e. K_4 is admissible. Therefore $K_1 \vee K_2 = K_1 \oplus K_4$ is admissible.

Now let $u_1, u_2, \ldots$ be functions on G with values in $\mathbf{S}^1 = \{z \in \mathbf{C}^1 : |z| = 1\}$ such that their linear combinations form a dense subset in $L^2(G)$. They give rise to a sequence of unitary operators $U_1, U_2, \ldots$ acting on $L^2(G, H)$: U_k is the multiplication by u_k $(k = 1, 2, \ldots)$. Since the operators $\tilde{c}$ are decomposable, they commute with all U_k. It means that

$$U_k \tilde{c}\, U_k^* = \tilde{c}$$

for $k = 1, 2, \ldots$. Restricting both sides to $U_k H_c$ we get

$$U_k(\tilde{c}|H_c)U_k^* = \tilde{c}|U_k H_c$$

and using the condition 1) of Definition 1.1 we obtain

$$\tilde{c}|U_k H_c \in D(U_k H_c),$$

i.e. $U_k H_c$ is admissible.

Let $L_k = H_c \vee U_1 H_c \vee \ldots \vee U_k H_c$. According to (5.4) $L_1 \subset L_2 \subset \ldots$ is an increasing sequence of admissible subspaces of $\tilde{H}$. Then the subspaces $L_{k+1} \ominus L_k$

are admissible and using once more the condition 2) of Definition 1.1 we see that

$$L_\infty = L_1 \oplus (L_2 \ominus L_1) \oplus (L_3 \ominus L_2) \oplus \dots$$

is admissible. We shall prove that $L_\infty = \tilde{H}$. Indeed $L_\infty \supset L_k$ for $k = 1, 2, \dots$. Therefore $L_\infty \supset U_k H_c$ for $k = 1, 2, \dots$. Remembering that H_c contains all constant functions we see that L_∞ contains all functions of the form $h(g) = u_k(g)h$, where $h \in H$ and $k = 1, 2, \dots$. Due to the assumed property of u_k, the functions of this form span a dense subset in $\tilde{H}$. Therefore $L_\infty \supset \tilde{H}$. This way we proved that $\tilde{H}$ is admissible. It means that

$$\tilde{c} = \tilde{c}|\tilde{H} \in D(\tilde{H}).$$

Now using the condition 2) of Definition 1.1 we see that $gc \in D(H)$ for μ-a.a. $g \in G$. Let $g \in G$ be such that $gc \in D(H)$. Then $c = g^{-1}gc \in D(H)$ because D is G-invariant. This ends the proof of the first part of Lemma 5.2.

Now assume that $c \in D(H)$ and that F is a G-invariant measurable operator function defined on D. Using the condition 2) of Definition 1.3, we obtain

$$F(\tilde{c}) = F\left(\int_G^{\oplus} (gc)\mathrm{d}\mu(g)\right) = \int_G^{\oplus} F(gc)\mathrm{d}\mu(g) = \int_G^{\oplus} F(c)\mathrm{d}\mu(g).$$

It means that $F(\tilde{c})$ is the direct integral of a constant field of operators. Therefore the subspace of constant functions in H is invariant under $F(\tilde{c})$. More precisely we have

$$F(\tilde{c})\,\tilde{h} = (F(c)h)^{\sim}$$

for all $h \in H$. Remembering that $\tilde{h} \in H_c$ and using the condition 2) of Definition 1.3 one easily obtain

$$F(\tilde{c})\tilde{h} = (F(\tilde{c})|\tilde{H}_c)h = F(\tilde{c}|H_c)\tilde{h}$$

and (5.3) follows. Q.E.D.

Proof of Theorem 5.1. Let $c \in B(H)^N$ and $w \in P_N$; then

$$w(\tilde{c}) = \int_G^{\oplus} w(gc)\mathrm{d}\mu(g)$$

and for any $f, h \in H$ we have:

$$(\tilde{f}\,|\,w(\tilde{c})\tilde{h}) = \int_G^{\oplus} (\tilde{f}(g)|\,w(gc)\tilde{h}(g))\mathrm{d}\mu(g) =$$

$$= \int_G (f|\,(g^* w)(c)h)\mathrm{d}\mu(g) = (f|\,\tilde{w}(c)h)$$

where $\tilde{w} = \int_G (g^* w) d\mu(g) \in P_N^G$. We assumed that $u(a) = u(b)$ for all $u \in P_N^G$. Therefore using the above computation we get

$$(\tilde{f} \mid w(\tilde{a})\tilde{h}) = (\tilde{f} \mid w(\tilde{b})\tilde{h}) \tag{5.6}$$

for all $f, h \in H$ and $w \in P_N$.

Let $w_1, w_2, \ldots, w_k \in P_N$ and $h_1, h_2, \ldots, k_k \in H$. Inserting in (5.6) h_i, h_j and $w_i^* w_j$ instead of f, h and w and summing over i and j we obtain

$$\| \sum_i w_i(\tilde{a})\tilde{h}_i \|^2 = \| \sum_i w_i(\tilde{b})\tilde{h}_i \|^2.$$

Therefore there exists a unitary operator

$$V: H_a \to H_b$$

such that

$$Vw(\tilde{a})\tilde{h} = w(\tilde{b})\tilde{h} \tag{5.7}$$

for any $w \in P_N$ and $h \in H$. Inserting here $X^i w$ instead of w (X^i is a generator of P_N) and using once more (5.7) we obtain

$$V \tilde{a}^i w(\tilde{a})\tilde{h} = \tilde{b}^i w(\tilde{b})\tilde{h} = \tilde{b}^i V w(\tilde{a})\tilde{h}.$$

It shows that

$$\tilde{b} \mid H_b = V(\tilde{a} \mid H_a)V^*. \tag{5.8}$$

Assume now that a belongs to a G-invariant domain D. Then according to Lemma 5.2 $\tilde{a}|H_a \in D(H_a)$. Taking into account the relation (5.8) we obtain $\tilde{b}|H_b \in D(H_b)$ and using Lemma 5.2 once more we get $b \in D(H)$. This proves the first part of Theorem 5.1.

Now let F be a G-invariant measurable operator function defined on D. Using (5.3), (5.8) and the condition 1) of Definition 1.3 we have

$$(F(b)h)^\sim = F(\tilde{b}|H_b)\tilde{h} = F(V(\tilde{a}\ |H_a)V^*)\tilde{h} =$$

$$= VF(\tilde{a}|H_a)V^*\tilde{h}.$$

Inserting in (5.7) the unit of P_N instead of w we see that constant functions are invariant under V: $V\tilde{h} = \tilde{h}$ and $V^*\tilde{h} = \tilde{h}$ for all $h \in H$. Therefore using once more (5.3) we obtain

$$(F(b)h)^\sim = VF(\tilde{a}|H_a)\tilde{h} = V(F(a)h)^\sim = (F(a)h)^\sim$$

for all $h \in H$. This shows that $F(b) = F(a)$. Q.E.D.

Let us indicate certain possible applications of the notions introduced above.

Assume that G is a compact group of $N \times N$ matrices with complex entries and that D is a compact G-invariant N-dimensional domain. Then for any $g \in G$ the map $a \mapsto ga$ is a homeomorphism of D (in the sense considered in Section 3). Let g^* denote the corresponding automorphism of the C^*-algebra $C(D)$:

$$(g^* F)(a) = F(ga)$$

for any $F \in C(D)$ and any element a of D.

This way we found an action of G on $C(D)$. This action is continuous; more precisely for each $F \in C(D)$ the mapping $G \ni g \mapsto g^* F \in C(D)$ is norm continuous. This fact is obvious if F is a coordinate function (cf. (3.5)). The general case follows easily from the Theorem 3.4.

Let us recall [3] that an action of a group on a C^*-algebra is said to be *ergodic* if the multiples of the unit are the only elements of A invariant under the action of the group.

Using Theorem 5.1, we obtain the following nice criterion:

THEOREM 5.3. *Let G be a compact group of $N \times N$ matrices and D be a G-invariant compact domain.*

Then the action of G on $C(D)$ is ergodic if and only if there exists a function

$$\chi : P_N^G \to \mathbf{C}^1 \tag{5.9}$$

such that for any Hilbert space H:

$$D(H) = \{a \in B(H)^N : \text{for any } w \in P_N^G,\ w(a) = \chi(w) \cdot I\}. \tag{5.10}$$

REMARK. For an arbitrary function (5.9) the formula (5.10) defines usually the empty set. In order to obtain a nontrivial domain, the function χ must satisfy certain compatibility conditions. They are listed in the following theorem.

THEOREM 5.4. *The equation* (5.10) *defines a compact non-empty domain if χ satisfies the following conditions:*

1) *χ is a linear multiplicative functional on P_N^G such that $\chi(w^*) = \overline{\chi(w)}$ for all $w \in P_N^G$;*

2) *For any finite sequence $w_1, w_2, \ldots, w_k \in P_N$ such that $\sum_{i=1}^{k} w_i^* w_i \in P_N^G$ we have*

$$\chi\left(\sum_{i=1}^{k} w_i^* w_i\right) \geqslant 0;$$

3) *For any finite sequences $w_1, w_2, \ldots, w_k \in P_N$ and $u_1, u_2, \ldots, u_k \in P_N$ such that*

$$\sum_{i=1}^{k} w_i u_i \in P_N^G \quad \text{and} \quad \sum_{i=1}^{k} u_i w_i \in P_N^G$$

we have

$$\chi\left(\sum_{i=1}^{k} w_i u_i\right) = \chi\left(\sum_{i=1}^{k} u_i w_i\right).$$

REMARK. A domain D is called non-empty if there exists a Hilbert space H such that $D(H) \neq \emptyset$. For example the domain D introduced by (3.12) is non-empty although $D(H) = \emptyset$ for every finite dimensional Hilbert space H.

Theorem 5.3 follows easily from the Theorem 5.1. To prove Theorem 5.4 one has to combine GNS construction with the main result of [3]. We left the details of the proof to the reader.

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PAWEŁ KRUSZYŃSKI and STANISŁAW L. WORONOWICZ
Department of Mathematical Methods in Physics,
University of Warsaw,
Hoża 74, 00-682 Warsaw,
Poland.

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