J. OPERATOR THEORY
8(1982), 391–400

SPECTRAL PICTURES OF FUNCTIONS OF OPERATORS

C. BOSCH, C. HERNÁNDEZ, E. De OTEYZA and C. PEARCY

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. The concept of the spectral picture of an operator T in $\mathscr{L}(\mathscr{H})$ was originated by one of the authors in [9] and was subsequently studied by Chevreau [4]. The notion seems to be a useful one. In particular, it enables succinct statements to be given of two recent major theorems in operator theory. Thus the theorem of Brown-Douglas-Fillmore [3] characterizing essentially normal operators up to compalence becomes "Two essentially normal operators in $\mathscr{L}(\mathscr{H})$ are compalent if and only if they have the same spectral picture." Furthermore the Romanian characterization of quasitriangular operators [1] can be stated thus: "An operator in $\mathscr{L}(\mathscr{H})$ is quasitriangular if and only if its spectral picture contains no negative number." (See [9] for definitions.)

In this note we begin a program of calculating the spectral picture of various constructs of an operator T in $\mathscr{L}(\mathscr{H})$ in terms of the spectral picture of T. In particular, we completely determine the spectral pictures of all operators of the form (T) where f is a function analytic on an open set containing the spectrum of T. We then apply these results to give some very short proofs of various facts about quasitriangular operators. First some notation and terminology that will be needed later are introduced, and the relevant facts about spectral pictures are reviewed.

We denote by $\mathbf{K}$ the ideal of compact operators in $\mathscr{L}(\mathscr{H})$ and by π the quotient map of $\mathscr{L}(\mathscr{H})$ onto the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathbf{K}$. The *spectrum* of an operator T in $\mathscr{L}(\mathscr{H})$ will be denoted, as usual, by $\sigma(T)$, and the *essential spectrum* of T (i.e., the spectrum of $\pi(T)$ in the Calkin algebra) by $\sigma_e(T)$. Similarly, the *left* and *right essential spectra* of T (notation: $\sigma_{le}(T)$ and $\sigma_{re}(T)$) are the left and right spectra of $\pi(T)$, respectively. We shall also find it convenient to denote $\sigma_{le}(T)\cap\sigma_{re}(T)$ by $\sigma_{lre}(T)$. A *hole* in $\sigma_e(T)$ is a (nonempty) bounded component of $\mathbf{C}\setminus\sigma_e(T)$ (and thus is an open connected set in $\mathbf{C}$), and a *pseudohole* in $\sigma_e(T)$ is a (nonempty) component of $\sigma_e(T)\setminus\sigma_{le}(T)$ or of $\sigma_e(T)\setminus\sigma_{re}(T)$. Pseudoholes are also open connected sets in $\mathbf{C}$ by virtue of the fact that $\partial\sigma_e(T)\subset\sigma_{lre}(T)$. If Y is any compact set in $\mathbf{C}$, we denote by $\hat{Y}$ the complement of the unbounded component of $\mathbf{C}\setminus Y$. Thus $\hat{Y}$, which is called the *polynomial hull* of Y, consists of the union of Y and any holes in Y. In particular, it follows easily from the known relation between $\sigma(T)$ and $\sigma_e(T)$ (cf. [9], Proposition 1.27) that $\widehat{\sigma(T)}$ consists of the union of $\sigma_e(T)$, the holes in $\sigma_e(T)$, and the isolated points of $\sigma(T)$ not lying in any hole of $\sigma_e(T)$.

These concepts are closely related to the topic of *Fredholm theory*, the needed facts from which we now review. An operator T in $\mathcal{L}(\mathcal{H})$ is *semi-Fredholm* if T has closed range and either kernel T or kernel T^* is finite dimensional. The set of all semi-Fredholm operators in $\mathcal{L}(\mathcal{H})$ is an open set (in the norm topology) and will be denoted by $\mathcal{S}\mathcal{F}$. There is a map $i : \mathcal{S}\mathcal{F} \to \mathbf{Z} \cup \{+\infty, -\infty\}$, called the *Fredholm index*, defined by setting $i(T) = \dim(\text{kernel}T) - \dim(\text{kernel}T^*)$. If the discrete topology is given to $\mathbf{Z} \cup \{+\infty, -\infty\}$ (and the norm topology to $\mathcal{S}\mathcal{F}$), the map i is continuous. The (open) set of all T in $\mathcal{S}\mathcal{F}$ for which $i(T)$ is finite is the set $\mathcal{F}$ of *Fredholm operators* in $\mathcal{L}(\mathcal{H})$. The relationships between the various essential spectra of an operator T in $\mathcal{L}(\mathcal{H})$ and the classes of Fredholm and semi-Fredholm operators are given by the following equations:

$$\begin{aligned} &\sigma_e(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \mathcal{F}\}, \\ (1) \qquad &\sigma_e(T)\setminus\sigma_{le}(T) = \{\lambda \in \mathbf{C} : T - \lambda \in \mathcal{S}\mathcal{F} \text{ and } i(T - \lambda) = -\infty\}, \\ &\sigma_e(T)\setminus\sigma_{re}(T) = \{\lambda \in \mathbf{C} : T - \lambda \in \mathcal{S}\mathcal{F} \text{ and } i(T - \lambda) = +\infty\}. \end{aligned}$$

Since the mapping i is continuous, and for any T in $\mathcal{L}(\mathcal{H})$ holes and pseudoholes in $\sigma_e(T)$ are connected sets by definition, it follows easily that $i(T - \lambda)$, as a function of λ, is constant (and finite) on every hole in $\sigma_e(T)$ and constant (but infinite) on every pseudohole of $\sigma_e(T)$. This fact enables the following definition to be made (cf. [9, page 2]). The *spectral picture* of an operator T in $\mathcal{L}(\mathcal{H})$, denoted by $\mathrm{SP}(T)$, is the structure consisting of the set $\sigma_e(T)$, the collections $\{H_j\}$ and $\{P_k\}$ of holes and pseudoholes in $\sigma_e(T)$, and the respective collections of Fredholm indices $\{i(H_j)\}$ and $\{i(P_k)\}$. For example, if V is a unilateral shift operator of multiplicity one, then, as is easily seen, $\sigma_e(V)$ is the unit circle $\mathbf{T}$ and the open unit disc H is a hole in $\sigma_e(V)$ having index -1, so $\mathrm{SP}(V) = \{\mathbf{T}, H, -1\}$.

2. FUNCTIONS OF OPERATORS

Suppose now that an operator T in $\mathcal{L}(\mathcal{H})$ is given such that $\mathrm{SP}(T)$ is known. If f is any function defined and analytic on a domain containing $\sigma(T)$, then, of course, $f(T)$ is defined by the Riesz-Dunford functional calculus (cf. [2, page 390]). We take up the problem of computing $\mathrm{SP}(f(T))$. Obviously this problem has three components: to determine $\sigma_e(f(T))$, to determine how the holes and pseudoholes of $\sigma_e(f(T))$ are related to those of $\sigma_e(T)$, and to determine the Fredholm indices to be associated with each hole and pseudohole of $\sigma_e(f(T))$.

The first part of this problem has already been solved. Gramsch and Lay obtained in [6] a spectral mapping theorem for the essential spectrum of an operator on a Banach space. They showed, in particular, that if $T \in \mathcal{L}(\mathcal{H})$ and f is any function defined and analytic on an open set containing $\sigma(T)$, then

$$\sigma_{\mathrm{le}}(f(T)) = f(\sigma_{\mathrm{le}}(T)),$$

$$\sigma_{\mathrm{re}}(f(T)) = f(\sigma_{\mathrm{re}}(T)), \text{ and} \tag{2}$$

$$\sigma_{\mathrm{e}}(f(T)) = f(\sigma_{\mathrm{e}}(T)).$$

A very short argument is available to obtain these formulae, and for completeness we give it here. Since π is a continuous algebra homomorphism and $\pi(r(T)) = r(\pi(T))$ for all rational functions r with poles off $\sigma(T)$, it follows easily from Runge's theorem (cf. [2, page 421]) that $\pi(f(T)) = f(\pi(T))$. Furthermore, the following general spectral mapping theorem is well-known:

Let $\mathcal{B}$ be a unital Banach algebra and let $X \in \mathcal{B}$. If f is any function defined and analytic on a neighborhood of the spectrum of X, then the left (right) spectrum of $f(X)$ is equal to the image under f of the left (right) spectrum of X.

This principle applied to the Calkin algebra and the equation $f(\pi(T)) = \pi(f(T))$ immediately yields equations (2).

THEOREM 2.1. (B. Gramsch and D. Lay). *Let $T \in \mathcal{L}(\mathcal{H})$, and suppose f is a function defined and analytic on an open set containing $\sigma(T)$. Then the equations (2) are valid.*

3. HOLES AND PSEUDOHOLES OF $\sigma_e(f(T))$

In this section we take up the problem of determining the relation between the holes and pseudoholes of $\sigma_e(T)$ and those of $\sigma_e(f(T))$. Our analysis depends on the following rather general proposition. Recall that a *domain* in **C** is, by definition, a connected, open subset of **C**.

PROPOSITION 3.1. *Suppose that K is a nonempty compact set in* **C**, *that H is a hole in K, and that f is a function defined and analytic on a domain Ω containing $K \cup H$. Then $f(H) \setminus f(K)$ is an open set in* **C**, *and if this set is nonvoid, then its components are holes in $f(K)$.*

Proof. If f is constant, then $f(H) \setminus f(K) = \emptyset$. Otherwise, since analytic functions are open maps, $f(H)$ is open, and since K is compact, $f(K)$ is compact. Thus $f(H) \setminus f(K)$ is open. It is easy to construct examples in which $f(H) \setminus f(K)$ is void, but suppose now that $f(H) \setminus f(K) \neq \emptyset$. Since $f(H^{-})$ is compact, clearly $f(H) \setminus f(K)$ is a bounded subset of $\mathbf{C}\setminus f(K)$, and so each (nonvoid) component G of the bounded open set $f(H)\setminus f(K)$ is either contained in some unique hole $H' = H'(G)$ of $f(K)$ or is contained in the unbounded component U of $\mathbf{C}\setminus f(K)$.

We shall show that both $G \subsetneqq H'$ and $G \subset U$ are impossible, thus showing that $G = H'$ and completing the proof. In either of the former two cases, there exists a point ξ_0 belonging to $\partial G \cap (\mathbf{C}\setminus f(K))$. Choose a sequence $\{\xi_n\}$ of distinct points from G such that $\xi_n \to \xi_0$. Since $G \subset f(H)\setminus f(K)$, there exists a sequence $\{\lambda_n\}$ of distinct points in H such that $f(\lambda_n) = \xi_n$ for all n. Let $\{\lambda_{n_k}\}$ be a convergent subsequence of $\{\lambda_n\}$ with, say, $\lambda_{n_k} \to \lambda_0$. Then $\lambda_0 \in H^-$ and clearly $f(\lambda_0) = \xi_0$. But $\lambda_0 \in \partial H$ is impossible, since $\partial H \subset K$ and $f(\lambda_0) = \xi_0 \notin f(K)$. Furthermore $\xi_0 \in H$ is impossible, since this would imply that ξ_0 belongs to some component of $f(H)\setminus f(K)$, contrary to the fact that $\xi_0 \in \partial G$. Thus we have reached a contradiction, and we must have $G = H'$ as desired.

As an immediate corollary of Theorem 2.1 and Proposition 3.1 we have one relation between the holes of $\sigma_e(T)$ and those of $\sigma_e(f(T))$.

THEOREM 3.2. *Let $T \in \mathscr{L}(\mathscr{H})$, and suppose that H is a hole in $\sigma_e(T)$. If f is a function defined and analytic on a domain Ω containing $\sigma(T) \cup H$, then the set $f(H)\setminus f(\sigma_e(T))$ is open, and if this set is nonvoid, then the components of $f(H)\setminus f(\sigma_e(T))$ are holes in $\sigma_e(f(T))$.*

Proof. Set $K = \sigma_e(T)$ in Proposition 3.1.

The following result is the analog of Theorem 3.2 for pseudoholes.

THEOREM 3.3. *Let $T \in \mathscr{L}(\mathscr{H})$ and suppose f is a function defined and analytic on a domain Ω containing $\sigma(T)$. If J is a pseudohole in $\sigma_e(T)$ of index $-\infty$ ($+\infty$), then the set $f(J)\setminus f(\sigma_{le}(T))$ ($f(J)\setminus f(\sigma_{re}(T))$) is open, and if this set is nonvoid, then its components are pseudoholes in $\sigma_e(f(T))$ of index $-\infty$ ($+\infty$).*

Proof. We give the argument in case J is a pseudohole of index $-\infty$; the other case is dealt with similarily. Since $f(\sigma_{le}(T)) = \sigma_{le}(f(T))$ by Theorem 2.1, and the left spectrum of any element in a unital Banach algebra is compact, it is clear that $f(J)\setminus f(\sigma_{le}(T))$ is open. Since $f(J)$ is a subset of $\sigma_e(f(T))$ by Theorem 2.1, it is clear from the definitions that if $f(J)\setminus f(\sigma_{le}(T)) \neq \emptyset$, then each component G of $f(J)\setminus f(\sigma_{le}(T))$ is contained in a unique pseudohole $J' = J'(G)$ of $\sigma_e(f(T))$. Furthermore, an argument exactly like the one in the proof of Proposition 3.1 (using the fact that $\partial J \subset \sigma_{le}(T)$), shows that G must be all of J'. Thus the components of $f(J)\setminus f(\sigma_{le}(T))$ must be pseudoholes in $\sigma_e(f(T))$, and the fact the index of each such pseudohole must be $-\infty$ follows immediately from (1).

Suppose now that $T \in \mathscr{L}(\mathscr{H})$ and that f is a function defined and analytic on an open set Ω containing $\sigma(T)$. The preceding two theorems show how holes (on which f is defined) and pseudoholes in $\sigma_e(T)$ transform under f to contribute to the formation of holes and pseudoholes of $\sigma_e(f(T))$. The following discussion shows that, roughly speaking, this is the only way that holes and pseudoholes of $\sigma_e(f(T))$ can arise.

First let us note that no generality is lost by supposing that each component Ω_i of Ω intersects $\sigma(T)$, and thus, since $\sigma(T)$ is compact, we may suppose that Ω has only finitely many components. Moreover, in case Ω has $n > 1$ components Ω_i, then using the spectral idempotents for T associated with the sets Ω_i, one sees easily that T is a (not necessarily orthogonal) direct sum $T = T_1 \dotplus \ldots \dotplus T_n$, and the spectral properties of T are easily deducible from those of the T_i. (In particular, $f(T)$ is then the direct sum $f(T_1) \dotplus \ldots \dotplus f(T_n)$, and the spectral picture of a direct sum is, in a natural way, the "superposition" of the spectral pictures o the direct summands.) Thus we may and do assume in the ensuing discussion that Ω is connected and that the function f is not constant on Ω.

Let Λ denote the compact set consisting of $\sigma_e(T)$ and the union of all holes H in $\sigma_e(T)$ such that $H \subset \sigma(T)$, and suppose $u \in \widehat{\sigma_e(f(T))} = \widehat{f(\sigma_e(T))}$. Then there are two possibilities. The first is that $u \in f(\Lambda)$, and since Λ is compact and f is not constant on Ω, it follows easily that there exists a positive integer $n(u)$ such that the function $f(z) - u$ has exactly the zeros

$$\lambda_1(u), \ldots, \lambda_{n(u)}(u)$$

in Λ (counting multiplicities). Thus we may factor $f(z) - u$ as

$$f(z) - u = (z - \lambda_1(u)) \ldots (z - \lambda_{n(u)}(u))g_u(z), \quad z \in \Omega, \tag{3}$$

where $g_u(z)$ is a function defined and analytic on Ω that does not vanish on Λ, and some of the $\lambda_i(u)$ may coincide.

The second possibility is that $u \notin f(\Lambda)$, and therefore the function $f(z) - u$ has no zero in Λ. In this case, maintaining the notation of equation (3), we write $n(u) = 0$ and $g_u(z) = f(z) - u$, $z \in \Omega$, so $g_u(z)$ is invertible on Λ and (3) remains valid (where $\{\lambda_1(u), \ldots, \lambda_{n(u)}(u)\}$ is defined to be the empty set).

Thus, in either case, using the facts that π and the Riesz-Dunford functional calculus are homomorphisms, we obtain

$$\pi(f(T)) - u = [\pi(T) - \lambda_1(u)] \ldots [\pi(T) - \lambda_{n(u)}(u)][\pi(g_u(T))]. \tag{4}$$

Moreover, by Theorem 2.1, $\sigma_e(g_u(T)) = g_u(\sigma_e(T))$, and since $g_u(z)$ does not vanish on $\sigma_e(T)$, $\pi[g_u(T)]$ is invertible in the Calkin algebra.

We first use (4) to characterize $\sigma_{lre}(f(T))$. Note that

$$f(\sigma_{lre}(T)) = f(\sigma_{le}(T) \cap \sigma_{re}(T)) \subset f(\sigma_{le}(T)) \cap f(\sigma_{re}(T)) = \sigma_{lre}(f(T)),$$

and suppose next that $u \in \sigma_{lre}(f(T))$, so that $\pi(f(T)) - u$ has neither a left nor a right inverse in $\mathscr{L}(\mathscr{H})/\mathbf{K}$. It follows easily from (4) that this is equivalent to saying that there exist j, k satisfying $1 \leqslant j,\ k \leqslant n(u)$ such that $\pi(T) - \lambda_j(u)$ has no left inverse and $\pi(T) - \lambda_k(u)$ has no right inverse. If either $\pi(T) - \lambda_j(u)$ has no right inverse or $\pi(T) - \lambda_k(u)$ has no left inverse, then $u \in f(\sigma_{\mathrm{lre}}(T))$, but the other possibility is that $T - \lambda_j(u)$ and $T - \lambda_k(u)$ are semi-Fredholm operators such that $i(T - \lambda_j(u)) = +\infty$ and $i(T - \lambda_k(u)) = -\infty$; in other words, $\lambda_j(u)$ and $\lambda_k(u)$ belong to pseudoholes in $\sigma_e(T)$ of different indices. Thus let $P_{+\infty}$ and $P_{-\infty}$ denote the union of all pseudoholes in $\sigma_e(T)$ of index $+\infty$ and $-\infty$, respectively. The above discussion proves the following theorem.

THEOREM 3.4. *With f and T as in Theorem 3.3, we have*

$$\sigma_{\mathrm{lre}}(f(T)) = f(\sigma_{\mathrm{lre}}(T)) \cup \{f(P_{+\infty}) \cap f(P_{-\infty})\}.$$

Thus if SP(T) *does not contain two pseudoholes of different indices, then* $\sigma_{\mathrm{lre}}(f(T)) = f(\sigma_{\mathrm{lre}}(T))$.

We next use (3) to characterize the holes of $\sigma_e(f(T))$ in terms of the information we already have from Theorem 3.2. The following lemma from complex analysis (which could be formulated without reference to operators) will be central to our purposes.

LEMMA 3.5. *Suppose $T \in \mathscr{L}(\mathscr{H})$ and f is a nonconstant function defined and analytic on a domain Ω containing $\sigma(T)$. Let H be any hole of $\sigma_e(T)$ contained in $\sigma(T)$ and H' any hole of $\sigma_e(f(T))$. Then either $H' \subset f(H)$ or $H' \cap f(H) = \emptyset$. Moreover, if $H' \subset f(H)$, then there exists a positive integer $m = m_{H'}(H)$ such that for any fixed u in H', the equation $f(\lambda) = u$ has exactly m solutions λ in H.*

Proof. To prove the first assertion, it suffices to show that $H' \cap f(H) \neq \emptyset$ implies $H' \subset f(H)$. Thus, suppose that there exist $\lambda \in H$, $u \in H'$ with $f(\lambda) = u$. Then $u \in f(H) \setminus f(\sigma_e(T))$, so by Theorem 3.2, this set is open and its components are holes in $\sigma_e(f(T))$. In particular, if G is the component of $f(H) \setminus f(\sigma_e(T))$ containing u, then G must be a hole in $\sigma_e(f(T))$, and thus $G = H'$. But then $H' \subset f(H)$, as desired. To prove the second assertion, suppose that $H' \subset f(H)$, and let u be a fixed point of H'. Then there must be at least one $\lambda_i(u)$ satisfying (3) lying in H, and we denote by $m_{H'}(H)$ the number of such $\lambda_i(u)$, counting multiplicities, lying in H. Of course we must show that $m_{H'}(H)$ does not depend upon the point u chosen from H', but locally this is an easy consequence of Hurwitz's theorem (cf. [8, Theorem 2.5, p. 49]), and since H' is arcwise connected, that $m_{H'}(H)$ is independent of the point u follows from a standard argument.

The following theorem associates with each hole H' in $\sigma_e(f(T))$ a collection of holes of $\sigma_e(T)$ contained in $\sigma(T)$, and gives the sought after characterization.

THEOREM 3.6. *Suppose $T \in \mathscr{L}(\mathscr{H})$ and f is a non-constant function defined and analytic on a domain Ω containing $\sigma(T)$. If H' is any hole in $\sigma_e(f(T))$, we denote by $\mathscr{C}(H')$ the collection of all those holes H in $\sigma_e(T)$ such that $H \subset \sigma(T)$ and $H' \subset f(H)$. Then either $\mathscr{C}(H') \neq \emptyset$ (which says that H' is not contained in $f(H)$*

for any hole H of $\sigma_e(T)$ that is contained in $\sigma(T)$, and thus may arise from the nonunivalent character of f or from holes in $\sigma_e(T)$ not contained in the domain of f), or $\mathscr{C}(H') \neq \emptyset$, in which case H' is a component of each of the sets $f(H_i)\setminus f(\sigma_e(T))$, $H_i \in \mathscr{C}(H')$, and therefore of the set $\bigcap_{H_i \in \mathscr{C}(H')} f(H_i)\setminus f(\sigma_e(T))$.

Proof. If $\mathscr{C}(H') \neq \emptyset$, then for every $H_i \in \mathscr{C}(H')$, we have $H' \subset f(H_i)$ by definition, and thus $H' \subset f(H_i)\setminus f(\sigma_e(T))$. But then, by Theorem 3.2, we know that $f(H_i)\setminus f(\sigma_e(T))$ is open and that its components are holes of $\sigma_e(f(T))$. Since H' must be contained in one of these components, and is itself a hole in $\sigma_e(f(T))$, we conclude that H' is one of the components of $f(H_i)\setminus f(\sigma_e(T))$, and thus a component of $\bigcap_{H_i \in \mathscr{C}(H')} f(H_i)\setminus f(\sigma_e(T))$.

Using the multiplicity function $m_{H'}(H)$ defined in Lemma 3.5 we can now also calculate the Fredholm index $i_{f(T)}(H')$ of a hole associated with $\mathrm{SP}(f(T))$ in terms of the indices $i_T(H_j)$ of the corresponding holes H_j in $\mathscr{C}(H')$.

THEOREM 3.7. *Let $T \in \mathscr{L}(\mathscr{H})$, and suppose f is a nonconstant function defined and analytic on a domain Ω containing $\sigma(T)$. If H' is a hole in $\sigma_e(f(T))$, then either $\mathscr{C}(H')$ is empty, in which case $i_{f(T)}(H') = 0$, or $\mathscr{C}(H')$ is nonempty, in which case*

$$i_{f(T)}(H') = \sum_{H \in \mathscr{C}(H')} m_{H'}(H) i_T(H). \tag{5}$$

Proof. Choose a point u in H'. Then, by definition, $i_{f(T)}(H')$ is the Fredholm index $i(f(T) - u)$. If $\mathscr{C}(H') = \emptyset$, then we know from the discussion before Theorem 3.4 that $f(z) - u \equiv g_u(z)$ on Ω and $g_u(z)$ does not vanish on Λ. Thus $\pi(g_u(T))$ is invertible, $i(f(T) - u) = i(g_u(T))$, and there are two possibilities. If $g_u(z)$ does not vanish on $\sigma(T)$, then $0 \notin \sigma(g_u(\sigma(T))$, and obviously $i(g_u(T)) = 0$. On the other hand, if $g_u(T)$ does vanish at some point of $\sigma(T)$, then since $\sigma(T)\setminus\Lambda$ consists of at most a countable number of isolated points whose only possible points of accumulatiou belong to $\sigma_e(T)$ (cf. [9, Proposition 1.27]), one may factor $g_u(z)$ as $g_u(z) = (z - \beta_1)\ldots$ $\ldots(z - \beta_k)h(z)$ where each $\beta_i \in \sigma(T)\setminus\Lambda$ and $h(z)$ does not vanish on $\sigma(T)$. Since $\pi(g_u(T))$ is invertible, each of the operators $T - \beta_j$ and $h(T)$ must belong to $\mathscr{F}$, and by the well-known addition property of the Fredholm index,

$$i(g_u(T)) = i(h(T)) + \sum_{j=1}^{k} i(T - \beta_j). \tag{6}$$

Since each β_j is an isolated point in $\sigma(T)$ and h does not vanish on $\sigma(T)$, the index of each summand on the right hand side of (6) is zero. Thus $i_{f(T)}(H') =$ $= i(g_u(T)) = 0$.

Turning now to the case $\mathscr{C}(H') \neq \emptyset$, we have from (4) that

$$i_{f(T)}(H') = i(f(T) - u) = i(g_u(T)) + \sum_{j=1}^{n(m)} i(T - \lambda_j(u)), \tag{7}$$

and since an argument like that just given shows that $i(g_u(T)) = 0$, (5) results from (7) and the definitions.

We turn briefly now to consider the analogs of Theorems 3.6 and 3.7 for pseudoholes (with T and f as before). Given any pseudohole J' in $\sigma_e(f(T))$ and any u belonging to J', one may define the $\lambda_i(u)$ as in (3), and note that since $u \in f(\sigma_e(T))$, necessarily $n(u) > 0$. Furthermore it is clear from the definitions and (4) that each $\lambda_j(u)$, $1 \leqslant j \leqslant n(u)$, must belong to either a hole in $\sigma_e(T)$ or a pseudohole in $\sigma_e(T)$, but at least one $\lambda_j(u)$ must belong to some pseudohole J in $\sigma_e(T)$, and the index of J must be the same as that of J'. Proceeding in a fashion analogous to what was done above in the discussion of holes, one arrives easily at the following theorem.

THEOREM 3.8. *Let $T \in \mathscr{L}(\mathscr{H})$, and suppose f is a nonconstant function defined and analytic on a domain Ω containing $\sigma(T)$. If J' is a pseudohole in $\sigma_e(f(T))$ of index $-\infty$ $(+\infty)$, then there exists a unique nonempty collection $\mathscr{C}(J')$ of pseudoholes J in $\sigma_e(T)$ associated with J', and each J in $\mathscr{C}(J')$ has the same index as J'. Furthermore J' is a component of the set*

$$\bigcap_{J \in \mathscr{C}(J')} f(J) \setminus f(\sigma_{le}(T)) \quad \Big(\bigcap_{J \in \mathscr{C}(J')} f(J) \setminus f(\sigma_{re}(T))\Big).$$

4. SOME APPLICATIONS

The preceding theorems can be used to give short proofs of some nice theorems in the theories of quasitriangular and essentially normal operators. For the definitions of these classes of operators, cf., for example, [9]. The following theorem is contained in [10, Theorem 2.6], which was nontrivial when it was proved and remains nontrivial even in the presence of the Apostol-Foiaş-Voiculescu characterization of quasitriangular operators.

THEOREM 4.1. *Suppose T is a quasitriangular operator in $\mathscr{L}(\mathscr{H})$ and f is a function defined and analytic on an open set Ω containing $\sigma(T)$. Then $f(T)$ is quasitriangular.*

Proof. One knows that a direct sum of quasitriangular operators is quasitriangular [7] and that an operator similar to a quasitriangular operator is also quasitriangular [5]. Using these facts and the discussion following Theorem 3.3, one sees easily that without loss of generality we may suppose that the open set Ω is connected and that f is not a constant function. Furthermore, from the characterization of quasitriangular operators stated in the introduction, we see that it suffices to prove that $\mathrm{SP}(f(T))$ has no associated hole or pseudohole whose index is negative. But this is immediate from Theorems 3.5 and 3.6 and the fact that no hole or pseudohole associated with $\mathrm{SP}(T)$ has negative index.

The following immediate corollary of Theorem 4.1 is a nice example of the contribution of the present note.

COROLLARY 4.2. ([10]). *If T is an invertible quasitriangular operator, then T^{-1} is also quasitriangular.*

Similar results are available about other classes of operators that can be characterized in terms of their spectral pictures.

THEOREM 4.3. *Suppose T is an invertible operator in $\mathscr{L}(\mathscr{H})$ that is a norm limit of algebraic operators. Then T^{-1} (and, more generally, every analytic function of T) is also a norm limit of algebraic operators.*

Proof. The set of all norm limits of algebraic operators in $\mathscr{L}(\mathscr{H})$ has been characterized by Voiculescu [11] as the set of all S in $\mathscr{L}(\mathscr{H})$ such that $\mathrm{SP}(S)$ contains no pseudoholes and no holes associated with nonzero index. But then, by Theorems 3.5 and 3.6, the same must be true of all analytic functions of T.

THEOREM 4.4. *If $T \in \mathscr{L}(\mathscr{H})$ and T is the sum of a normal operator and a compact operator, then every rational function $r(T)$ (even every analytic function (T)) also has this property.*

Proof. The subset of $\mathscr{L}(\mathscr{H})$ consisting of all sums of the form $N + K$ where N is normal and K is compact has been characterized [3] as the set of all S in $\mathscr{L}(\mathscr{H})$ such that $\pi(S)$ is normal and $\mathrm{SP}(S)$ has no holes associated with nonzero numbers or pseudoholes. Since $\pi(f(T)) = f(\pi(T))$, as was shown earlier, and $\pi(T)$ is normal, it is clear that $\pi(f(T))$ is normal, and the result follows from Theorems 3.5 and 3.6 as before.

REFERENCES

1. APOSTOL, C.; FOIAŞ, C.; VOICULESCU, D., Some results on non-quasitriangular operators. IV, *Rev. Roumaine Math. Pures Appl.*, **18**(1973), 487–514.
2. BROWN, A.,; PEARCY, C., *Introduction to operator theory*. I. *Elements of functional analysis*, Springer-Verlag, New York, 1977.
3. BROWN, L.; DOUGLAS, R. G.; FILLMORE, P., Unitary equivalence modulo the compact operators and extension of C^*-algebras, Proc. Conf. Operator Theory, *Lectures Notes in Math.*, Springer-Verlag, Berlin, **345**(1973), 58–128.
4. CHEVREAU, B., On the spectral picture of an operator, *J. Operator Theory*, **4**(1980), 119–132.
5. DOUGLAS, R. G.; PEARCY, C., A note on quasitriangular operators, *Duke Math. J.*, **37**(1970), 177–188.

6. GRAMSH, B.; LAY, D., Spectral mapping theorem for essential spectra, *Math. Ann.*, **192**(1971), 17–32.
7. HALMOS, P. R., Quasitriangular operators, *Acta Sci. Math. (Szeged)*, **29**(1968), 283–293.
8. MARKUSHEVICH, A. I., *Theory of functions of a complex variable*. II, Prentice Hall, Englewood Cliffs, 1967.
9. PEARCY, C., *Some recent developments in operator theory*, C.B.M.S. Regional Conference Series in Mathematics, No. 36, Amer. Math. Soc., Providence, 1978.
10. PEARCY, C.; SALINAS, N., An invariant-subspace theorem, *Michigan Math. J.*, **20**(1973), 21–31.
11. VOICULESCU, D., Norm limits of algebraic operators, *Rev. Roumaine Math. Pures Appl.*, **19**(1974), 371–378.

C. BOSCH, C. HERNÁNDEZ and E. De OTEYZA
Institute of Mathematics,
National Autonomous University of Mexico,
Mexico. D. F., Mexico.

C. PEARCY
Department of Mathematics,
University of Michigan,
Ann Arbor, Michigan 48109,
U.S.A.

Received October 7, 1981.