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ON DOMAINS OF POWERS OF CLOSED SYMMETRIC OPERATORS

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INTRODUCTION

Let T be a densely defined closed symmetric operator in a Hilbert space $\mathscr{H}$. As M. A. Naimark has observed, the domain $\mathscr{D}(T^2)$ of its square need not to be dense in $\mathscr{H}$. Naimark [5] even proved (using a result of J. von Neumann) the existence of a closed symmetric operator T with $\mathscr{D}(T^2) = \{0\}$.

In this paper we investigate some properties of the domains $\mathscr{D}(T^n)$ of the powers of T. To be more precise, we are mainly (but not only) concerned with the following two questions:

When $\mathscr{D}(T^n)$ is dense in $\mathscr{H}$? When $\mathscr{D}(T^{n+r})$ is a core for T^r?

Except from Naimark's Doklady notes [5] published in 1940, this problems appearently have not yet been further studied (as far as the author is aware).

Let us briefly describe the content of this paper and some of the results.

In § 1 we investigate some general properties of powers of a closed symmetric operator T. Among other things we prove that if at least one of the deficiency indices of T is finite, then $\mathscr{D}_\infty(T) := \bigcap_{n\in\mathbf{N}} \mathscr{D}(T^n)$ is a core for each operator T^k, $k \in \mathbf{N}$, and hence $\mathscr{D}_\infty(T)$ is dense in $\mathscr{H}$. Consequently, in our further study of the problems mentioned above we may restrict ourselves to the case where both deficiency indices are infinite. If the underlying Hilbert space $\mathscr{H}$ is separable, then T has a self-adjoint extension in $\mathscr{H}$. This enables us (at least in principle) to investigate both problems for restrictions T of a (given) unbounded self-adjoint operator A in $\mathscr{H}$. Throughout the remaining parts of the paper, we will adopt this view point.

In § 2 we give general answers to the questions from above. The criterions depend on the self-adjoint operator A and the size of the deficiency space $\mathscr{H}_+$ of T.

In § 3 we prove some technical results which are needed in § 4.

§ 4 contains the main result (Theorem 4.5) concerning the two questions. Let A be an arbitrary unbounded self-adjoint operator in $\mathscr{H}$ and $\mathfrak{N}$ a set of positive integers. Then there exists a closed symmetric operator T with $T \subseteq A$ such that

$\mathscr{D}(T^{n+r})$, $n, r \in \mathbf{N}$, is a core for T^r if and only if $r \notin \mathfrak{N}$. Moreover, if $\mathfrak{N}$ is bounded and k is an upper bound for $\mathfrak{N}$, then we can choose T such that this is true for $n+r \leqslant k$ and in addition $\mathscr{D}(T^{k+1})$ is not dense in $\mathscr{H}$, while $\mathscr{D}(T^k)$ is dense. This theorem illustrates from a new view point the difficulties and pathologies which can occur in working with unbounded operators.

The second part of Theorem 4.5 can be strengthened in the following manner. There exists a closed restriction T of A such that $\mathscr{D}(T^k)$ is dense in $\mathscr{H}$, but $\mathscr{D}(T^{k+1})$ reduces to the zero vector. The proof is lengthy and will not be included in this paper. In § 5 we give a short proof in the case $k = 1$. It is based on an argument which the author has seen in [2] and [3].

The main result of § 5 (Theorem 5.1) is the following. For each unbounded self-adjoint operator A in a separable Hilbert space $\mathscr{H}$, there exists a projection P in $\mathscr{H}$ such that

$$(I-P)\mathscr{H} \cap \mathscr{D}(A) = P\mathscr{H} \cap \mathscr{D}(A) = \{0\}.$$

From this theorem some other appearently different (known and new) results describing "pathological" phenomenona of unbounded operators follow easily. Setting $U = I - 2P$, we have the existence of a unitary operator U in $\mathscr{H}$ with $U\mathscr{D} \cap \mathscr{D} = \{0\}$, a result due to J. von Neumann [6]. Taking $P\mathscr{H}$ and $(I-P)\mathscr{H}$ as deficiency spaces, we obtain densely defined closed restrictions T_1 and T_2 of A with $\mathscr{D}(T_1^2) \cap \mathscr{D}(T_2^2) = \{0\}$ and $\mathscr{D}(T_1) \cap \mathscr{D}(T_2) = \{0\}$. The second part of this assertion is a recent result of van Daele [2]. Finally, for each $k \in \mathbf{N}$, there is a linear subspace $\mathscr{D}$ of $\mathscr{D}(A^{k+1})$ with $\mathscr{D} \cap \mathscr{D}(A^{k+2}) = \{0\}$ such that A^k is essentially self-adjoint on $\mathscr{D}$ and $A^{k+1} \upharpoonright \mathscr{D}$ is closed and has infinite deficiency indices. In some sense, Theorem 5.1 helps to see somewhat more clearly in the matter of these phenomenona.

NOTATION

We fix the notation and recall some basic definitions which will be used in this paper.

$\mathbf{N}_0$, $\mathbf{N}$ and $\mathbf{Z}$ denote the non-negative integers, positive integer and integers, respectively. Sequences are usually denoted by $\gamma = \{\gamma_n, n \in \mathbf{N}\}$, $\delta = \{\delta_n, n \in \mathbf{N}\}$, and so on. We let $\gamma\delta := \{\gamma_n\delta_n, n \in \mathbf{N}\}$ and $\gamma/\delta := \{\gamma_n/\delta_n, n \in \mathbf{N}\}$. $\mathscr{d}$ is the space of all complex sequences having only a finite number of non-zero terms. F_k, $k \in \mathbf{N}$, denotes the projection in ℓ_2 which is defined by $F_k x := (x_1, \ldots, x_k, 0, \ldots)$ for $x = \{x_n, n \in \mathbf{N}\} \in \ell_2$. We set $F_0 = 0$.

$\mathscr{H}$, $\mathscr{K}$ etc. are infinite dimensional complex Hilbert spaces. We write $\langle \cdot , \cdot \rangle$ and $\|\cdot\|$ for the scalar product and the norm of these spaces. To avoid ambiguities, the scalar product and the norm of the sequence space ℓ_2 will be denoted by $(\cdot,\cdot)$ and $|||\cdot|||$, respectively. If $\mathscr{L}_j$, $j = 1, \ldots, n$, are subsets of $\mathscr{H}$, then $\mathscr{L}_1 + \ldots + \mathscr{L}_n$

stands for the set $\{\varphi_1 + \ldots + \varphi_n;\ \varphi_j \in \mathscr{L}_j\}$. $\mathscr{L}^\perp$ and $\overline{\mathscr{L}}$ are the orthogonal complement resp. the closure of a linear subspace $\mathscr{L}$ in $\mathscr{H}$.

Because the domains of powers of densely defined closed symmetric operators are not dense in general, we have to deal with operators whose domains are not dense in $\mathscr{H}$. Let T be a linear operator in $\mathscr{H}$. $\mathscr{D}(T)$, $\mathscr{R}(T)$, $\mathscr{N}(T)$ denote the domain, the range, the nullspace of T, respectively. Let us define $T^0 := I$, I the identity mapping in $\mathscr{H}$, $p(T) := \sum a_j T^j$ for each complex polynomial $p(z) = \sum a_j z^j$ and $\mathscr{D}_\infty(T) := \bigcap_{n\in\mathbf{N}} \mathscr{D}(T^n)$. $T_1 \subseteq T_2$ means that $\mathscr{D}(T_1) \subseteq \mathscr{D}(T_2)$ and $T_1\varphi = T_2\varphi$ for $\varphi \in \mathscr{D}(T)$. T is closed if and only if $\mathscr{D}(T)$ is complete with respect to the graph norm $\|\varphi\|_T := (\|T\varphi\|^2 + \|\varphi\|^2)^{1/2}$. A linear subspace $\mathscr{D}$ of $\mathscr{D}(T)$ is a core for T if and only if $\mathscr{D}(T)$ is the closure of $\mathscr{D}$ in the graph norm. Let T_1, T_2 be linear operators in $\mathscr{H}$, $\mathscr{K}$. The orthogonal direct sum $T_1 \oplus T_2$ is the linear operator in the orthogonal sum $\mathscr{H} \oplus \mathscr{K}$ which is defined by $(T_1 \oplus T_2)(\varphi_1 + \varphi_2) := T_1\varphi_1 + T_2\varphi_2$, $\varphi_1 \in \mathscr{D}(T_1)$, $\varphi_2 \in \mathscr{D}(T_2)$. A similar meaning is attached to finite and countable orthogonal direct sums.

Now let T be a closed symmetric operator in $\mathscr{H}$. Then, $(T + \mathrm{i})\mathscr{D}(T)$ and $(T - \mathrm{i})\mathscr{D}(T)$ are closed subspaces of $\mathscr{H}$. Their orthogonal complements $\mathscr{H}_+$ and $\mathscr{H}_-$ are the deficiency spaces of T. The dimensions $\dim \mathscr{H}_+$ and $\dim \mathscr{H}_-$ are called the deficiency indices of T. The Cayley transform W of T is a partial isometry from $\mathscr{H} \ominus \mathscr{H}_+ = (T + \mathrm{i})\mathscr{D}(T)$ on $\mathscr{H} \ominus \mathscr{H}_- = (T - \mathrm{i})\mathscr{D}(T)$ which is defined by $W((T + \mathrm{i})\varphi) := (T - \mathrm{i})\varphi$, $\varphi \in \mathscr{D}(T)$.

Throughout the whole paper, let A denote an arbitrary densely defined unbounded self-adjoint operator in $\mathscr{H}$. Let $E(\lambda)$, $\lambda \in \mathbf{R}_1$, be the spectral projections, $V = (A - \mathrm{i})(A + \mathrm{i})^{-1}$ the Cayley transform, and $\sigma(A)$ the spectrum of A.

Finally, let us say that two vectors $\varphi, \psi \in \mathscr{H}$ are A-orthogonal if $f(A)\varphi \perp g(A)\psi$ for all bounded continuous functions f, g.

§ 1

1.1. We begin with some well-known facts. We include the (short) proofs because this is easier than localizing an explicit reference. Suppose that T is a densely defined closed symmetric operator in the Hilbert space $\mathscr{H}$.

LEMMA 1.1. *Let $p_j(t)$, $j = 1,2,3$, be complex polynomials in t with complex coefficients. Suppose* degree $p_j \leqslant n$ *where $n \in \mathbf{N}$ and $j = 1, 2, 3$.*

(i) *If $p_1(t) \geqslant 0$ on $\mathbf{R}_1$, then $\langle p(T)\varphi, \varphi\rangle \geqslant 0$ for $\varphi \in \mathscr{D}(T^n)$.*

(ii) *If $|p_1(t)|^2 \leqslant |p_2(t)|^2 + |p_3(t)|^2$ on $\mathbf{R}_1$, then*

$$\|p_1(T)\varphi\|^2 \leqslant \|p_2(T)\varphi\|^2 + \|p_3(T)\varphi\|^2$$

for all $\varphi \in \mathscr{D}(T^n)$.

Proof. Extend T to a self-adjoint operator, say A, in a larger Hilbert space and apply the functional calculus for A.

REMARK. (i) follows also immediately from the fundamental theorem of algebra. The latter implies the existence of a polynomial $q(t)$ such that $p_1(t) = \overline{q(t)}\, q(t)$. Since T is symmetric, $\langle p_1(T)\varphi, \varphi\rangle = \|q(T)\varphi\|^2 \geqslant 0$ for $\varphi \in \mathscr{D}(T^n)$.

Let $\|\cdot\|_m$, $m \in \mathbf{N}$, be the graph norm defined by $\|\varphi\|_m^2 := \|T^m\varphi\|^2 + \|\varphi\|^2$ on $\mathscr{D}(T^m)$. Applying Lemma 1.1, (ii), with $p_1(t) = t^k$, $p_2(t) = t^n$, $p_3(t) = 1$, we obtain

COROLLARY 1.2. *If* $k \leqslant n$, $k, n \in \mathbf{N}$, *then* $\|\varphi\|_k^2 \leqslant \|\varphi\|_n^2 + \|\varphi\|^2$ *for* $\varphi \in \mathscr{D}(T^n)$.

Recall that a linear subspace $\mathscr{D} \subseteq \mathscr{D}(T^m)$, $m \in \mathbf{N}$, is a core for T^m iff $\mathscr{D}$ is $\|\cdot\|_m$-dense in $\mathscr{D}(T^m)$. Therefore, Corollary 1.2 immediately implies

COROLLARY 1.3. *Suppose* $k \leqslant n$, $k, n \in \mathbf{N}$. *Let* $p(z)$ *be a complex polynomia of degree* k.

If $\mathscr{D} \subseteq \mathscr{D}(T^n)$ *is a core for* T^n *and* $\mathscr{D}(T^n)$ *is a core for* T^k, *then* $\mathscr{D}$ *is also a core for* $p(T)$.

COROLLARY 1.4. *For any complex polynomial* $p(z)$, $p(T)$ *is a closed linear operator.*

Proof. Suppose that $n := \text{degree}\, p(z) \geqslant 1$. Since, by Corollary 2, the graph norms $\|\cdot\|_{p(T)}$ and $\|\cdot\|_n$ are equivalent, it suffices to show that T^n is closed, i.e., $\mathscr{D}(T^n)$ is $\|\cdot\|_n$-complete. We argue by induction on n. Assume that T^{n-1} is closed. Let $\{\varphi_m\}$ be a $\|\cdot\|_n$-Cauchy sequence. Applying Corollary 2 (with $k = 1$), we see that $\{T\varphi_m\}$ is a Cauchy sequence in $\mathscr{H}$. Let $\varphi_m \to \varphi$, $\psi_m := T\varphi_m \to \psi$, $T^{n-1}\psi_m = T^n\varphi_m \to \xi$ as $m \to +\infty$ in $\mathscr{H}$. Since T is closed, $\varphi \in \mathscr{D}(T)$ and $\psi = T\varphi$. Because T^{n-1} is closed, $\xi \in \mathscr{D}(T^{n-1})$ and $\xi = T^{n-1}\psi = T^n\varphi$.

REMARKS. 1) By Corollary 2 and Corollary 3, $\mathscr{D}$ is a core for $p(T)$ if and only if T is a core for T^n, $n := \text{degree}\, p(z)$. Therefore, we can devote ourselves in what follows to the study of cores for powers of T.

2) Corollary 4 is no longer true without the assumption that T is symmetric (see Remark 3 in § 2 for counter-examples).

1.2. Let $\mathscr{H}_+, \mathscr{H}_-$ be the deficiency spaces and W the Cayley transform of T Let $\mathscr{G}_n := \mathscr{H}_+ + W^*\mathscr{H}_+ + \ldots + (W^*)^{n-1}\mathscr{H}_+$ for $n \in \mathbf{N}$. The closed linear subspaces $\mathscr{H}_n := \overline{\mathscr{G}_n}$, $n \in \mathbf{N}$, of $\mathscr{H}$ are called the *iterated deficiency spaces* of T (with respect to i). Let P_n be the projection on $\mathscr{H}_n$.

LEMMA 1.5. $\mathscr{D}(T^n) = (W - I)^n (I - P_n)\mathscr{H}$ *for each* $n \in \mathbf{N}$.

Proof. We proceed by induction on n. For $n = 1$ the assertion is clear. Assume now that $\mathscr{D}(T^n) = (W - I)^n (I - P_n)\mathscr{H}$. Let $\psi \in \mathscr{D}(T^{n+1})$. Since $\psi \in \mathscr{D}(T^n)$, $\psi = (W - I)^n\varphi$ with $\varphi \perp \mathscr{H}_n$. $T\psi = \mathrm{i}(W + I)(W - I)^{n-1}\varphi \in \mathscr{D}(T^n)$ implies $T\psi - \mathrm{i}\psi =$

$= 2\mathrm{i}(W - I)^{n-1}\varphi \in \mathcal{D}(T^n)$, that is, $(W - I)^{n-1}\varphi = (W - I)^n\xi$ with $\xi \perp \mathcal{H}_n$. Since $(W - I)^{k-1}\varphi \perp \mathcal{H}_+$ and $(W - I)^k\xi \perp \mathcal{H}_+$ for $k = 1, \ldots, n - 1$, the injectivity of $W - I$ on $\mathcal{H} \ominus \mathcal{H}_+$ gives $\varphi = (W - I)\xi$. From $\varphi \perp \mathcal{H}_n$ and $\xi \perp \mathcal{H}_n$ it follows that $\xi \perp W^*\mathcal{H}_n$ and thus $\xi \perp \mathcal{H}_{n+1}$. Therefore, $\psi = (W - I)^{n+1}(I - P_{n+1})\xi$. Conversely, it is easy to check that these vectors are in $\mathcal{D}(T^{n+1})$.

LEMMA. 1.6. *Let k, $n \in \mathbf{N}$.*

(i) *$\mathcal{D}(T^n)$ is dense in $\mathcal{H}$ if and only if $(W^* - I)^n \psi \in \mathcal{H}_n$, $\psi \in \mathcal{H}$, always implies* $\psi = 0$.

(ii) *Suppose that $k \leqslant n$. $\mathcal{D}(T^n)$ is a core for T^k if and only if for $z = \mathrm{i}$ and $z = -\mathrm{i}$, $(W^*-I)^n(z-\mathrm{i}^k(W^* + I)^k)\psi \in \mathcal{H}_n$, $\psi \in \mathcal{H}$, implies $(W^* - I)^k(z - \mathrm{i}^k(W^* + I)^k)\psi \in \mathcal{H}_k$.*

Proof. (i) By Lemma 5, $\mathcal{D}(T^n)$ is dense in $\mathcal{H}$ if and only if the range of the operator $S: = (W - I)^n(I - P_n)$ is dense, that is, $\mathcal{N}(S^*) = \{0\}$. The above condition is only a reformulation of $\mathcal{N}(S^*) = \{0\}$.

(ii) $\mathcal{D}(T^n)$ is a core for T^k if and only if $\overline{(T^k + z)\mathcal{D}(T^k)} \subseteq \overline{(T^k + z)\mathcal{D}(T^n)}$ for $z = \mathrm{i}$ and $z = -\mathrm{i}$. Letting $S_{m,k,z}: = (\mathrm{i}^k(W + I)^k + z)(W - I)^m(I - P_m)$, $m \in \mathbf{N}$, $m \geqslant k$, and using again Lemma 5, this is equivalent to $\overline{S_{k,k,z}\mathcal{H}} \subseteq \overline{S_{n,k,z}\mathcal{H}}$, $z = \pm\mathrm{i}$. Since the condition mentioned in the lemma is a reformulation of $\mathcal{N}(S^*_{n,k,z}) \subseteq \mathcal{N}(S^*_{k,k,z})$, the assertion follows.

The next lemma has been shown in the proof of Proposition 2.1 in [8].

LEMMA 1.7. *If $\mathcal{D}(T^n)$ is a core for T^{n-1} for each $n \in \mathbf{N}$, then $\mathcal{D}_\infty(T)$ is a core for each T^k, $k \in \mathbf{N}$.*

PROPOSITION 1.8. *Let $k, n \in \mathbf{N}$, $k < n$. Suppose that the linear subspace $\mathcal{G}_n$ is closed in $\mathcal{H}$.*

If $\mathcal{D} \subseteq \mathcal{D}(T^n)$ is a core for T^n, then $\mathcal{D}$ is also a core for T^k.

Proof. By Corollary 3, we have to show that $\mathcal{D}(T^n)$ is a core for T^k. By Lemma 4, (ii), it suffices to prove that $(W^*-I)^{n-k}\varphi \in \mathcal{H}_n$ for $\varphi \in \mathcal{H}$ always implies $\varphi \in \mathcal{H}_k$. Fix a vector $\varphi \in \mathcal{H}$ with $(W^*-I)^{n-k}\varphi \in \mathcal{H}_n$. Since $\mathcal{G}_n = \mathcal{H}_n$ by assumption, we can find a vector $\psi \in \mathcal{G}_k$ such that $(W^* - I)^{n-k}(\varphi - \psi) \in \mathcal{G}_{n-k-1}$. Here we will set $\mathcal{G}_{-1} = \{0\}$. Our proof is complete if we have shown that $\varphi = \psi$. We proceed by induction. Let $r \in \{0, \ldots, n - k - 1\}$. Assume that $(W^* - I)^{r+1}(\varphi - \psi) \in \mathcal{G}_r$. For $r = n - k - 1$ this is true. We want to prove $(W^* - I)^r(\varphi - \psi) \in \mathcal{G}_{r-1}$. Let us write $(W^* - I)^{r+1}(\varphi - \psi)$ as $\xi + (W^* - I)\eta$ where $\xi \in \mathcal{G}_1$, $\eta \in \mathcal{G}_{r-1}$. Since $W^*P_- = 0$, this gives

$$W^* - I)(I - P_-)[(W^* - I)^r(\varphi - \psi) - \eta] - P_-[(W^* - I)^r(\varphi - \psi) - \eta] + \xi = 0.$$

Clearly, W^* is the Cayley transform of the closed symmetric operator $-T$ and $\mathcal{D}(-T) = (W^* - I)(I - P_-)\mathcal{H}$. Since $\mathcal{D}((-T)^*)$ is the *direct* sum $\mathcal{D}(-T) + \mathcal{H}_+ + \mathcal{H}_-$, the above equation can only be true if all components are vanishing, that is, $P_-\zeta = 0$ and $(W^* - I)(I - P_-)\zeta = 0$ where $\zeta := (W^* - I)^r(\varphi - \psi) - \eta$. Because $W^* - I$ is injective on $\mathscr{H} \ominus \mathscr{H}_-$, we obtain $(I - P_-)\zeta = 0$. Hence $\zeta = 0$ and $(W^* - I)^r(\varphi - \psi) = \eta \in \mathscr{G}_{r-1}$. For $r = 0$ this gives $\varphi = \psi$ which completes the proof.

REMARK. It follows from the preceding proof that $\mathscr{G}_n$ is the (algebraical) direct sum of its subspaces $\mathscr{H}_+$, $(W^* - I)\mathscr{H}_+, \ldots, (W^* - I)^{n-1}\mathscr{H}_+$ for each $n \in \mathbf{N}$. Hence $\mathscr{G}_n$ is also a direct sum of $\mathscr{H}_+, W^*\mathscr{H}_+, \ldots, (W^*)^{n-1}\mathscr{H}_+$.

1.3. The main result in this section is

THEOREM 1.9. *Let T be a (densely defined) closed symmetric linear operator in $\mathscr{H}$ with deficiency spaces $\mathscr{H}_+, \mathscr{H}_-$. Suppose that one of the following conditions is satisfied:*

(a) *$\mathscr{G}_n := \mathscr{H}_+ + W^*\mathscr{H}_+ + \ldots + (W^*)^{n-1}\mathscr{H}_+$ is closed in $\mathscr{H}$ for each $n \in \mathbf{N}$.*

(b) *At least one of the spaces $\mathscr{H}_+$ or $\mathscr{H}_-$ is finite dimensional.*

Then, $\mathscr{D}_\infty(T)$ is a core for each operator T^k, $k \in \mathbf{N}$. In particular, $\mathscr{D}_\infty(T)$ is dense in $\mathscr{H}$.

Proof. (a) The assertion follows from Lemma 7 and Proposition 8.

(b) Without loss of generality we may assume that $\dim \mathscr{H}_+ < \infty$ (otherwise we replace T by $-T$). Then, $\mathscr{G}_n$ is finite dimensional for $n \in \mathbf{N}$ and (a) applies.

REMARKS. 1) If the operator T has equal finite deficiency indices, then Theorem 9 reduces to Proposition 2.1, (ii), in [8]. Our proof has used some arguments from [8].

2) The questions formulated in the introduction could be answered in terms of the adjoint operator T^*. We mention a simple example: $\mathscr{D}(T^n)$ is dense in $\mathscr{H}$ if and only if the operator $(T^*)^n$ is closable in $\mathscr{H}$. But we do not go further along this line. As we have explained in the introduction, Theorem 9 enables us to take a different point of view throughout the remaining sections of this paper: The closed symmetric operator T will be considered as a restriction of a self-adjoint operator A acting in the same Hilbert space.

1.4. We conclude this section with a result concerning the deficiency indices of T^n. It is probably known.

Following [1], p. 305, a complex number λ is called a *point of regular type* for T if there is a constant $c = c(\lambda) > 0$ such that $\|(T - \lambda)\varphi\| \geqslant c\|\varphi\|$ for $\varphi \in \mathscr{D}(T)$. Let $\rho_r(T)$ denote the set of these points. $\rho_r(T)$ is an open subset of $\mathbf{C}_1$[1]. By $\mathscr{P}_n(\rho_r(T))$ we denote the set of all complex polynomials of degree n which have no zeros in $\mathbf{C}_1 \setminus \rho_r(T)$.

For the next lemma we merely assume that T is a closed operator on $\mathscr{H}$.

LEMMA 1.10. *Let $p_1, p_2 \in \mathscr{P}_n(\rho_r(T))$ for some $n \in \mathbf{N}$. Suppose, for each connected component Q of $\rho_r(T)$, the number of zeros of p_1 and p_2 in Q (counted with multiplicity) coincide. Then,* $\dim \mathscr{R}(p_1(T))^\perp = \dim \mathscr{R}(p_2(T))^\perp$.

For $p_1(z) = z - \lambda_1, p_2(z) = z - \lambda_2$, Lemma 10 is a classical result due to Kreĭn and Krasnoselski (see [1], § 100). Using the argument of the proof from this case (as given, for example in [1], p. 306), Lemma 10 can be shown by induction on the number of common zeros of p_1 and p_2. We omit the details.

Now we set $k_n = n/2$ if $n \in \mathbf{N}$ is even and $k_n = (n+1)/2$ if $n \in \mathbf{N}$ is odd. Let $\mathscr{P}_n^+$ resp. $\mathscr{P}_n^-$ be the set of all $p \in \mathscr{P}_n(\rho_r(T))$ which have k_n resp. $n-k_n$ zeros in the upper half-plane and $n-k_n$ resp. k_n zeros in the lower half-plane. Again the zeros are counted up to multiplicity.

COROLLARY 1.11. *Let T be a closed symmetric operator in $\mathscr{H}$.*

(a) *Let $p_\pm$ be arbitrary polynomials from $\mathscr{P}_n^\pm$. Then,* $\dim \mathscr{R}(p_+(T))^\perp$ *and* $\dim \mathscr{R}(p_-(T))^\perp$ *are the deficiency indices of T^n for* i *resp.* $-$i.

(b) *If there is a real number in $\rho_r(T)$ (in particular, if T is semibounded), then both deficiency indices of T are equal to* $\dim \mathscr{R}(p(T))^\perp$ *for each $p \in \mathscr{P}_n(\rho_r(T))$.*

Proof. Since T is symmetric, the complement of the real line is in $\rho_r(T)$ and $\rho_r(T)$ has at most two connected components. To prove (a), we apply Lemma 10 with $p_1(z) = z^n \pm \mathrm{i}$, $p_2(z) = p_\pm(z)$ and Q the connected component of the upper resp. the lower half-plane in $\rho_r(T)$. By the definition of $\mathscr{P}_n^\pm$, the assumption concerning the zeros of p_1 and p_2 is satisfied. The second assertion (b) follows again from Lemma 10 if we take into account that $\rho_r(T)$ is connected in that case.

§ 2

In this section we begin to study restrictions of the unbounded self-adjoint operator A. Let $\mathscr{H}_+$ be a closed subspace in $\mathscr{H}$ and P_+ the projection on $\mathscr{H}_+$. Obviously, $W := V(I - P_+)$ is a partial isometry in $\mathscr{H}$ with initial space $\mathscr{H} \ominus \mathscr{H}_+$ and final space $\mathscr{H} \ominus V\mathscr{H}_+$. It is easy to check that W is the Cayley transform of a closed symmetric (in general not densely defined) restriction T of A with deficiency spaces $\mathscr{H}_+$ and $\mathscr{H}_- = V\mathscr{H}_+$. First we need a more convenient form of the iterated deficiency spaces $\mathscr{H}_n$ which are, by definition, the closures of $\mathscr{G}_n = \mathscr{H}_+ + W^*\mathscr{H}_+ + \ldots + (W^*)^{n-1}\mathscr{H}_+$.

LEMMA 2.1.

$$\mathscr{G}_n = \mathscr{H}_+ + V^{-1}\mathscr{H}_+ + \ldots + V^{-(n-1)}\mathscr{H}_+ = \mathscr{H}_+ + (A-\mathrm{i})^{-1}\mathscr{H}_+ + \ldots + (A-\mathrm{i})^{-(n-1)}\mathscr{H}_+$$

for each $n \in \mathbf{N}$.

Using $W^* = V^{-1} - P_+V^{-1}$, the first equality follows by induction. The second equality is an immediate consequence of $V^{-1} - I = 2\mathrm{i}(A - \mathrm{i})^{-1}$.

From [8] we recall

Proposition 2.2. *$\mathscr{D}(T^n)$, $n \in \mathbf{N}$, is dense in $\mathscr{H}$ if and only if $\mathscr{H}_n \cap \mathscr{D}(A^n) = \{0\}$.*

For $n = 1$, Proposition 2.2 has been independently obtained in [4].

We now give a general answer to the second question.

Proposition 2.3. *Let $k, n \in \mathbf{N}$, $k < n$. $\mathscr{D}(T^n)$ is a core for T^k if and only if*

$$\mathscr{H}_n \cap \mathscr{D}(A^{n-k}) \subseteq (A - \mathrm{i})^{-(n-k)}\mathscr{H}_k .$$

Proof. The proof is similar to the proof of Lemma 1.6, (ii). As in Lemma 1.5, $\mathscr{D}(T^m) = (V - I)^m(I - P_m)\mathscr{H}$ for $m \in \mathbf{N}$. Set $S_{m,k,z} := (T^k + z)(V - I)^m(I - P_m)$ where $m \in \mathbf{N}$, $m \geqslant k$, $z \in \mathbf{C}_1$. $\mathscr{D}(T^n)$ is a core for T^k if and only if

$$\overline{(T^k + z)\,\mathscr{D}(T^k)} \equiv \overline{S_{k,k,z}\mathscr{H}} \subseteq \overline{S_{n,k,z}\mathscr{H}} \equiv \overline{(T^k + z)\mathscr{D}(T^n)}$$

for $z = \mathrm{i}$ and $z = -\mathrm{i}$. Clearly, this is equivalent to $\mathscr{N}(S^*_{n,k,z}) \subseteq \mathscr{N}(S^*_{k,k,z})$, $z = \pm\mathrm{i}$. The operator $(T^k + z)(V - I)^m = (A^k + z)(A + \mathrm{i})^m(2\mathrm{i})^m$ is bounded if $m \geqslant k$. Hence we have

$$\mathscr{N}(S^*_{n,k,z}) = \{\varphi \in \mathscr{H} : (A^k + \bar{z})(A - \mathrm{i})^{-n}\varphi \in \mathscr{H}_n\}$$

and

$$\mathscr{N}(S^*_{k,k,z}) = \{\varphi \in \mathscr{H} : (A^k + \bar{z})(A - \mathrm{i})^{-k}\varphi \in \mathscr{H}_k\}.$$

Since $\varphi \to (A^k + \bar{z})(A - \mathrm{i})^k\varphi$ is a one-to-one map of $\mathscr{H}$ onto $\mathscr{H}$ for $\operatorname{Im} z \neq 0$, $\mathscr{N}(S^*_{n,k,z}) \subseteq \mathscr{N}(S^*_{k,k,z})$, $z = \pm\mathrm{i}$, is equivalent to $\{\psi \in \mathscr{H} : (A - \mathrm{i})^{k-n}\psi \in \mathscr{H}_n\} \subseteq \mathscr{H}_k$. Substituting $\xi = (A - \mathrm{i})^{k-n}\psi$, the latter leads to $\mathscr{H}_n \cap \mathscr{D}(A^{n-k}) \subseteq (A - \mathrm{i})^{-(n-k)}\mathscr{H}_k$.

Remarks. 1) Obviously, the opposite inclusion $(A - \mathrm{i})^{-(n-k)}\mathscr{H}_k \subseteq \mathscr{H}_n \cap \mathscr{D}(A^{n-k})$ of the condition occurring in Proposition 2.3 is always valid.

2) We take this opportunity to correct an (essential) error in the Zentralblatt review [11] of Naimark's paper [5]. In this review (but not in Naimark's paper!), it has been asserted that $\mathscr{D}(T^2)$ is dense if and only if $\mathscr{H}_+ + \mathscr{H}_-$ is closed in $\mathscr{H}$. This condition is clearly sufficient because $\mathscr{D}(T^2)$ is even a core for T under this assumption (note that $\mathscr{H}_+ + \mathscr{H}_- = V\mathscr{G}_2$) by Proposition 1.8. We show by an easy example that it is not necessary.

Let $\mathscr{H} = \sum_{n=1}^{\infty} \oplus L_2(a_n, b_n)$, where $a_n, b_n \in \mathbf{R}_1$, $a_n < b_n$, for $n \in \mathbf{N}$. Let T be the (closed symmetric) differential operator $-\mathrm{i}\dfrac{\mathrm{d}}{\mathrm{d}x}$ in $\mathscr{H}$ acting in each component with boundary conditions zero at a_n and b_n for each $n \in \mathbf{N}$. The deficiency spaces $\mathscr{H}_+$ and $\mathscr{H}_-$ are spanned by the vectors $\varphi_k^+ = (f_{kn}(x) = \mathrm{e}^{-x}\delta_{kn})$, $k \in \mathbf{N}$, resp. $\varphi_k^- = (g_{kn}(x) = \mathrm{e}^{x}\delta_{kn})$, $k \in \mathbf{N}$. Here δ_{kn} stands for the Kronecker symbol. It is well-known that $\mathscr{H}_+ + \mathscr{H}_-$ is closed if and only if there is a constant q, $0 < q < 1$, such that

$$|\langle\varphi^+, \varphi^-\rangle| \leqslant q\|\varphi^+\|\,\|\varphi^-\| \quad \text{for } \varphi^+ \in \mathscr{H}_+,\ \varphi^- \in \mathscr{H}_-.$$

Suppose that $\lim_{n\to\infty} b_n - a_n = 0$. Then,

$$\lim_{k\to\infty} |\langle \varphi_k^+, \varphi_k^- \rangle| \, \|\varphi_k^+\|^{-1} \, \|\varphi_k^-\|^{-1} = 1$$

and hence $\mathscr{H}_+ + \mathscr{H}_-$ is not closed in $\mathscr{H}$. On the other hand, $\mathscr{D}_\infty(T)$ is a core for each operator T^k, $k \in \mathbf{N}$. Of course, $\mathscr{D}(T^2)$ is dense in $\mathscr{H}$.

3) Let T be the closed symmetric operator from the preceding example. Because $\mathscr{D}(T)$ is dense, T^* is a well-defined closed operator in $\mathscr{H}$. Since $\mathscr{H}_+ + \mathscr{H}_-$ is not closed in $\mathscr{H}$, it can be easily seen that $(T^*)^2$ is not a closed operator (but closable because $\mathscr{D}(T^2)$ is dense). If we choose T such that $\mathscr{D}(T^2)$ is not dense in $\mathscr{H}$ (by Theorem 4.5), then $(T^*)^2$ is not even closable. Hence Corollary 1.4 becomes false if we do not assume that T is symmetric.

§ 3

The main aim of this section is to prove Corollary 3 below which will be applied in § 4. Nevertheless, some results (in particular, Proposition 1) are essentially stronger than needed, because they seem to be of some interest in itself and they could have further applications.

3.1. **Proposition** 3.1. *Let $\{\varepsilon_n, n \in \mathbf{N}\}$ and $\{\delta_n, n \in \mathbf{N}\}$ be positive sequences. There exists a vector $\varphi \in \mathscr{D}_\infty(A)$ such that*

(1) $$|\langle (A - \mathrm{i})^k \varphi, (A - \mathrm{i})^m \varphi \rangle| \leqslant \varepsilon_m \|(A - \mathrm{i})^k \varphi\| \, \|(A - \mathrm{i})^m \varphi\|$$

and

(2) $$\|(A - \mathrm{i})^m \varphi\| \leqslant \delta_{m+1} \|(A - \mathrm{i})^{m+1} \varphi\|$$

for all $k, m \in \mathbf{N}_0$ with $k \neq m$.

The proof is based on a technique which the author has developpped in studying unbounded operator algebras (see, for example, [9]).

Proof. First we prove the following assertion. For any $\varepsilon > 0$, $\alpha > 0$ and $m \in \mathbf{N}$, there is a vector $\xi \in \mathscr{D}_\infty(A)$ such that

(3) $$|\langle (A - \mathrm{i})^k \xi, (A - \mathrm{i})^m \xi \rangle| \leqslant \varepsilon \|(A - \mathrm{i})^k \xi\| \, \|(A - \mathrm{i})^m \xi\|$$

for $k = 0, \ldots, m - 1$,

(4′) $$\|(A - \mathrm{i})^m \xi\| \geqslant \alpha,$$

and

(4″) $$\|(A - \mathrm{i})^{m-1} \xi\| \leqslant \varepsilon.$$

Without loss of generality we can assume that $\sup\{\lambda\,;\, \lambda \in \sigma(A)\} = +\infty$ and $\alpha = 1$. Let $n \in \mathbf{N}$ and $0 < q < 1$. Since A is an unbounded self-adjoint operator, we can find real numbers λ_s, $s = 0, \ldots, n-1$, and vectors $\xi_s \in \mathscr{H}$, $s = 0, \ldots, n-1$, such that the intervals $[\lambda_s, \lambda_s + 1]$ are mutually disjoint, $2 < \lambda_s < q\lambda_{s+1}$, $\|\xi_s\| = 1$ and $\xi_s = E(\lambda_s, \lambda_s + 1)\xi_s$ for $s = 0, \ldots, n-1$. Let $\xi := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \|(A-\mathrm{i})^m \xi_j\|^{-1} \xi_j$.
Obviously,

$$\|(A-\mathrm{i})^m \xi\| = 1. \tag{5}$$

Suppose that $k \in \{0, \ldots, m-1\}$. Then

$$|\langle (A-\mathrm{i})^k \xi, (A-\mathrm{i})^m \xi\rangle| = \frac{1}{n} \sum_{j=0}^{n-1} \frac{|\langle (A-\mathrm{i})^k \xi_j, (A-\mathrm{i})^m \xi_j\rangle|}{\|(A-\mathrm{i})^m \xi_j\|^2} \leqslant$$

$$\leqslant \frac{1}{n} \sum_{j=0}^{n-1} \frac{|\lambda_j + 1 + \mathrm{i}|^{m+k}}{|\lambda_j - \mathrm{i}|^{2m}} \leqslant \frac{1}{n} \sum_{j=0}^{n-1} \frac{2^{m+k}}{\lambda_j^{m-k}} \leqslant \frac{2^{m+k}}{n} \sum_{j=0}^{n-1} \frac{q^{(m-k)j}}{\lambda_0^{m-k}} \leqslant \tag{6}$$

$$\leqslant \frac{2^{m+k}}{n} \lambda_0^{-m+k} (1 - q^{m-k})^{-1}.$$

On the other hand,

$$\|(A-\mathrm{i})^k \xi\| \geqslant \frac{\|(A-\mathrm{i})^k \xi_0\|}{\sqrt{n}\, \|(A-\mathrm{i})^m \xi_0\|} \geqslant \frac{|\lambda_0 - \mathrm{i}|^k}{|\lambda_0 + 1 - \mathrm{i}|^m \sqrt{n}} \geqslant \frac{1}{2^m \lambda_0^{m-k} \sqrt{n}}. \tag{7}$$

If we choose n and q so that $2^{2m+k} \leqslant \sqrt{n}\, \varepsilon (1 - q^{m-k})$ for all $k = 0, \ldots, m-1$, then (5), (6) and (7) yield (3) and (4′). It remains to verify (4″). A similar estimation as (7) gives

$$\|(A-\mathrm{i})^{m-1} \xi\|^2 \leqslant \frac{1}{n} 2^{2m-2} \frac{1}{\lambda_0^2 (1 - q^2)}.$$

Taking λ_0 sufficiently large, (4)″ is satisfied.

Now we complete the proof of the proposition. Without loss of generality we assume that $\varepsilon_{n+1} \leqslant \varepsilon_n$ and $\delta_n \leqslant 1$ for all $n \in \mathbf{N}$. By the preceding assertion, there is a sequence $\{\varphi_n, n \in \mathbf{N}_0\}$ of vectors in $\mathscr{D}_\infty(A)$ such that for all $k, m \in \mathbf{N}_0$:

$$\|(A-\mathrm{i})^m \varphi_m\| \geqslant \frac{3}{\delta_m} \left(\sum_{j=0}^{m-1} \|(A-\mathrm{i})^m \varphi_j\| + 1 \right), \tag{8}$$

$$\|(A-\mathrm{i})^m \varphi_m\| \geqslant \frac{1}{\varepsilon_m} \left(\sum_{j=0}^{m-1} |\langle (A-\mathrm{i})^k \varphi_j, (A-\mathrm{i})^m \varphi_j\rangle| + 1 \right) \quad \text{if } k < m, \tag{9}$$

$$\|(A-\mathrm{i})^{m-1} \varphi_m\| \leqslant 2^{-m}, \tag{10}$$

$$|\langle (A-\mathrm{i})^k \varphi_m, (A-\mathrm{i})^m \varphi_m\rangle| \leqslant \varepsilon_m \|(A-\mathrm{i})^k \varphi_m\| \, \|(A-\mathrm{i})^m \varphi_m\| \quad \text{if } k < m. \tag{11}$$

Moreover, by decomposing A as an orthogonal direct sum of countably many unbounded self-adjoint operators, we may assume in addition that

$$A^r\varphi_m \perp A^s\varphi_k \quad \text{for } k \neq m, \quad k, m, r, s \in \mathbf{N}_0. \tag{12}$$

Let $\varphi = \sum_{j=0}^{\infty} \varphi_j$. (10) ensures the convergence of the series in all graph norms $\|\cdot\|_{A^n}$, $n \in \mathbf{N}$. Therefore, $\varphi \in \mathscr{D}_\infty(A)$. Using (8), (10) and $\delta_n \leqslant 1$, we get

$$\begin{aligned} \|(A-\mathrm{i})^m\varphi\| \geqslant \|(A-\mathrm{i})^m\varphi_m\| - \sum_{j=0}^{m-1} \|(A-\mathrm{i})^m\varphi_j\| - \\ - \sum_{j=m+1}^{\infty} \|(A-\mathrm{i})^m\varphi_j\| \geqslant \frac{2}{3}\|(A-\mathrm{i})^m\varphi_m\| \end{aligned} \tag{13}$$

and similarly

$$\|(A-\mathrm{i})^m\varphi\| \leqslant \frac{4}{3}\|(A-\mathrm{i})^m\varphi_m\|. \tag{14}$$

Hence (8), (13), (14) give us

$$\begin{aligned} \|(A-\mathrm{i})^m\varphi\| \geqslant \frac{2}{3}\cdot\frac{3}{\delta_m}\|(A-\mathrm{i})^m\varphi_{m-1}\| \geqslant \frac{2}{\delta_m}\|(A-\mathrm{i})^{m-1}\varphi_{m-1}\| \geqslant \\ \geqslant \frac{2}{\delta_m}\cdot\frac{3}{4}\|(A-\mathrm{i})^{m-1}\varphi\|, \end{aligned}$$

i.e. (2) is satisfied.

Suppose now that $k < m$, $k, m \in \mathbf{N}_0$. By (13) and (8),

$$\|(A-\mathrm{i})^k\varphi\| \geqslant \frac{2}{3}\|(A-\mathrm{i})^k\varphi_k\| \geqslant 2.$$

Applying this and (8)—(13), we obtain

$$\begin{aligned} |\langle(A-\mathrm{i})^k\varphi, (A-\mathrm{i})^m\varphi\rangle| \leqslant |\langle(A-\mathrm{i})^k\varphi_m, (A-\mathrm{i})^m\varphi_m\rangle| + \\ + \sum_{j=0}^{m-1} |\langle(A-\mathrm{i})^k\varphi_j, (A-\mathrm{i})^m\varphi_j\rangle| + \sum_{j=m+1}^{\infty} |\langle(A-\mathrm{i})^k\varphi_j, (A-\mathrm{i})^m\varphi_j\rangle| \leqslant \\ \leqslant \varepsilon_m\|(A-\mathrm{i})^k\varphi_m\|\,\|(A-\mathrm{i})^m\varphi_m\| + \varepsilon_m\|(A-\mathrm{i})^m\varphi_m\| \leqslant 2\varepsilon_m\|(A-\mathrm{i})^m\varphi_m\| \leqslant \\ \leqslant \varepsilon_m\|(A-\mathrm{i})^k\varphi\|\,\|(A-\mathrm{i})^m\varphi\|. \end{aligned}$$

Since the left hand side of (1) is symmetric in k and m and $\{\varepsilon_n\}$ is monotonic, this implies (1). Now the proof is complete.

REMARK. As we can see from the proof, $A - \mathrm{i}$ could be replaced by $A - \lambda$ where λ is an arbitrary complex number. In particular, Proposition 1 is valid for A instead of $A - \mathrm{i}$. The technique works in fact for each unbounded normal operator.

3.2. COROLLARY 3.2. *Let* $1 > \varepsilon > 0$, $\delta > 0$ *and* $n \in \mathbf{N}$. *There exist a vector* $\psi \in \mathscr{H}$, $\psi \notin \mathscr{D}(A)$, *such that*

$$|\langle (A - \mathrm{i})^{-r}\psi, (A - \mathrm{i})^{-s}\psi\rangle| \leqslant \varepsilon \|(A - \mathrm{i})^{-r}\psi\| \, \|(A - \mathrm{i})^{-s}\psi\| \tag{15}$$

$$\Big\| \sum_{j=0}^{n} x_j (A - \mathrm{i})^{-j}\psi \Big\|^2 \geqslant (1 - n\varepsilon) \sum_{j=0}^{n} |x_j|^2 \, \|(A - \mathrm{i})^{-j}\psi\|^2 \tag{16}$$

$$\|(A - \mathrm{i})^{-r}\psi\| \leqslant \delta \|(A - \mathrm{i})^{-(r-1)}\psi\| \tag{17}$$

for all $r, s = 0, \ldots, n$ *with* $r \neq s$ *and* $x_j \in \mathbf{C}_1$, $j = 0, \ldots, n$.

Proof. We apply Proposition 1 with $\varepsilon_m = \varepsilon/2$, $\delta_m = \delta/2$ for $m \in \mathbf{N}$. Let φ be the corresponding vector from $\mathscr{D}_\infty(A)$ satisfying (1) and (2). Set $\tilde{\psi} := (A - \mathrm{i})^n \varphi$. Then

$$|\langle (A - \mathrm{i})^{-r}\tilde{\psi}, (A - \mathrm{i})^{-s}\tilde{\psi}\rangle| \leqslant \frac{\varepsilon}{2} \|(A - \mathrm{i})^{-r}\tilde{\psi}\| \, \|(A - \mathrm{i})^{-s}\tilde{\psi}\|$$

and

$$\|(A - \mathrm{i})^{-r}\tilde{\psi}\| \leqslant \frac{\delta}{2} \|(A - \mathrm{i})^{-(r-1)}\tilde{\psi}\| \quad \text{for } r, s = 0, \ldots, n, r \neq s.$$

Now let ξ be an arbitrary vector in $\mathscr{H}$ with $\xi \notin \mathscr{D}(A)$. Taking $\psi = \tilde{\psi} + \alpha\xi$ and choosing the positive number α sufficiently small, (15) and (17) are satisfied. (16) follows easily from (15).

3.3. For the construction in § 4 we only need

COROLLARY 3.3. *Let* $\{\delta_n, n \in \mathbf{N}\}$ *be an arbitrary positive sequence. There exist a sequence* $\{\psi_n, n \in \mathbf{N}\}$ *of mutually A-orthogonal vectors* ψ_n, $\psi_n \notin \mathscr{D}(A)$ *for all* $n \in \mathbf{N}$, *and a positive sequence* $\{q_n, n \in \mathbf{N}_0\}$ *such that*

$$\Big\| \sum_{j=0}^{n} \sum_{l=1}^{\infty} x_{jl} (A - \mathrm{i})^{-j}\psi_l \Big\|^2 \geqslant q_n \sum_{j=0}^{n} \sum_{l=1}^{\infty} |x_{jl}|^2 \, \|(A - \mathrm{i})^{-j}\psi_l\|^2 \tag{18}$$

for all $n \in \mathbf{N}_0$ *and* $x_j := \{x_{jl}, l \in \mathbf{N}\} \in d$, $j = 0, \ldots, n$,

$$\|(A - \mathrm{i})^{-k}\psi_n\| \leqslant \delta_n \|(A - \mathrm{i})^{-(k-1)}\psi_n\| \tag{19}$$

for all $n \in \mathbf{N}, k \in \mathbf{N}_0, k \leqslant n$.

Proof. By the spectral theorem it is possible to write A as an orthogonal direct sum $\sum_n \oplus A_n$ of unbounded self-adjoint operators A_n in Hilbert spaces $\mathscr{H}_n$, $n \in \mathbf{N}$. We apply Corollary 2 to A_n with $\varepsilon = 1/n$ and $\delta = \delta_n$. Let ψ_n be the corresponding vector. (19) follows immediately from (17). We prove (18). Let $n \in \mathbf{N}$. Since the vectors ψ_l, $l \in \mathbf{N}$, are mutually A-orthogonal,

$$(20)\qquad \left\| \sum_{j=0}^{n} \sum_{l=1}^{\infty} x_{jl}(A-\mathrm{i})^{-j}\psi_l \right\|^2 = \sum_{l=1}^{\infty} \left\| \sum_{j=0}^{n} x_{jl}(A-\mathrm{i})^{-j}\psi_l \right\|^2 .$$

If $l \geqslant n$, then Corollary 2, (16), and $\varepsilon = 1/2\,n$ yield

$$\left\| \sum_{j=0}^{n} x_{jl}(A-\mathrm{i})^{-j}\psi_l \right\|^2 \geqslant \frac{1}{2} \sum_{j=0}^{n} |x_{jl}|^2 \, \|(A-\mathrm{i})^{-j}\psi_l\|^2 .$$

Now suppose that $l < n$, $l \in \mathbf{N}$. Clearly, the vectors $(A-\mathrm{i})^{-j}\psi_l$, $j=0, \dots, n$, are linearly independent because $\psi_l \notin \mathscr{D}(A)$. Hence $p_l(y) := \left\| \sum_{j=0}^{n} y_j(A-\mathrm{i})^{-j}\psi_l \right\|$ is a norm on the $(n+1)$-dimensional vector space $\mathscr{H}_{l,n} := \mathrm{Lin}\{(A-\mathrm{i})^{-j}\psi_l,\ j=0, \dots, n\}$. Therefore, we can find a positive constant β_l with

$$p_l(y)^2 \geqslant \beta_l \sum_{j=0}^{n} |y_j|^2 \, \|(A-\mathrm{i})^{-j}\psi_l\|^2$$

for all vectors y. Setting $q_n = \min\{1/2, \beta_1, \dots, \beta_{n-1}\}$ and putting both cases $l \geqslant n$ and $l < n$ into (20), we obtain the desired inequality (18). Now the proof is complete.

Let $\mathscr{H}_+$ be the closed subspace of $\mathscr{H}$ generated by the sequence $\{\psi_n, n \in \mathbf{N}\}$ from Corollary 3 and let T be the corresponding closed restriction of A. Since $\mathscr{H}_+ \cap \mathscr{D}(A) = \{0\}$, T is densely defined. Moreover, from (18) we see that the spaces

$$\mathscr{G}_n = \mathscr{H}_+ + (A-\mathrm{i})^{-1}\mathscr{H}_+ + \dots + (A-\mathrm{i})^{-(n-1)}\mathscr{H}_+, \quad n \in \mathbf{N},$$

are closed in $\mathscr{H}$ and hence coincide with the iterated deficiency spaces $\mathscr{H}_n$. This fact will be stated separately as

COROLLARY 3.4. *There exists a (densely defined) closed symmetric restriction T of A with infinite dimensional deficiency spaces $\mathscr{H}_+$, $\mathscr{H}_-$ such that, for each $n \in \mathbf{N}$, the space*

$$\mathscr{G}_n = \mathscr{H}_+ + (A-\mathrm{i})^{-1}\mathscr{H}_+ + \dots + (A-\mathrm{i})^{-(n-1)}\mathscr{H}_+ \ (= \mathscr{H}_n)$$

is closed in $\mathscr{H}$.

§ 4

Throughout this section, let us adopt the following convention concerning the notation: *For small greek letters such as* $\gamma^s_{rl}, \eta^1_n, \eta^2_n$ *etc. and capital script letters such as* $\mathscr{T}^s_n, \mathscr{D}^s_n, \mathscr{L}^s_n$ *etc. the upper symbols* s, 1, 2 *are always indices. They do not refer to powers.*

4.1. Let k be a fixed natural number. Recall that A is an arbitrary unbounded self-adjoint operator in $\mathscr{H}$. By the spectral theorem, we can find unbounded self-adjoint operators A_1, A_2, B acting in Hilbert spaces $\mathscr{K}_1$, $\mathscr{K}_2$, $\mathscr{K}$ such that $\mathscr{H} = \mathscr{K}_1 \oplus \mathscr{K}_2 \oplus \mathscr{K}$ and $A = A_1 \oplus A_2 \oplus B$. The elements of $\mathscr{H}$ will be written as sums $\varphi^1 + \varphi^2 + \varphi$, $\varphi^1 \in \mathscr{K}_1$, $\varphi^2 \in \mathscr{K}_2$, $\varphi \in \mathscr{K}$. Let $\{\eta^1_n, n \in \mathrm{N}\}$ be a sequence of B-orthogonal vectors in $\mathscr{K}$ such that the series $\eta := \sum_{n=1}^{\infty} (A - \mathrm{i})^{-k} \eta^1_n$ is converging in $\mathscr{K}$, that is,

$$\{\|(A - \mathrm{i})^{-k} \eta^1_n\|, n \in \mathrm{N}\} \in \ell_2. \tag{1}$$

Let $\{m_n, n \in \mathrm{N}\}$ be a sequence with $m_n \in \mathrm{N}_0$ for $n \in \mathrm{N}$. Suppose that s is either 1 or 2. Let $z_s = \{z_{sn}, n \in \mathrm{N}\}$ be a complex sequence. For A_s we choose a sequence $\{\psi^s_n, n \in \mathrm{N}\}$ in $\mathscr{K}_s$ satisfying the conditions of Corollary 3.3. Let $\mathscr{L}^s_+$ be the closed subspace of $\mathscr{K}_s$ generated by $\{\psi^s_n, n \in \mathrm{N}\}$ and let $\mathscr{L}^s_n = \mathscr{L}^s_+ + (A - \mathrm{i})^{-1} \mathscr{L}^s_+ + \ldots + (A - \mathrm{i})^{-(n-1)} \mathscr{L}^s_+$, $n \in \mathrm{N}$. Recall that $\mathscr{L}^s_n$ is a closed subspace of $\mathscr{K}_s$ for all $n \in \mathrm{N}$ because of the properties of the sequence $\{\psi^s_n\}$. Clearly,

$$\mathscr{D}^s_n := \left\{\psi = \sum_{j=0}^{n-1} \sum_{l=1}^{\infty} x_{jl} (A - \mathrm{i})^{-j} \psi^s_l \,;\, x_j := \{x_{jl}, l \in \mathrm{N}\} \in d \text{ for } j = 0, \ldots, n-1\right\}$$

is a dense linear subspace of $\mathscr{L}^s_n$. For simplicity, we identify $\psi \in \mathscr{D}^s_n$ with the n-tuple $(x_0, \ldots, x_{n-1})$ of sequences from d. (This is possible because the vectors $(A-\mathrm{i})^{-j}\psi^s_l$, $l \in \mathrm{N}$, $j = 0, \ldots, n-1$, are, of course, linearly independent.) Set $\eta^2_n = (A - \mathrm{i})^{-1} \eta^1_n$ for $n \in \mathrm{N}$.

Now we define linear operators $\mathscr{T}^s_n$ from $\mathscr{D}^s_n$ into $\mathscr{K}$ by

$$\mathscr{T}^s_1 \psi = \mathscr{T}^s_1(x_0) := \sum_{l=1}^{\infty} (x_0, (I - F_{m_l}) z_s) \eta^s_l,$$

$$\mathscr{T}^s_n \psi = \mathscr{T}^s_n(x_0, \ldots, x_{n-1}) := \sum_{j=0}^{n-1} (A - \mathrm{i})^{-j} \mathscr{T}^s_1(x_j) \quad \text{if } n \leqslant k$$

and

$$\mathscr{T}^s_n \psi = \mathscr{T}^s_n(x_0, \ldots, x_{n-1}) :=$$

$$:= \mathscr{T}^s_k(x_0, \ldots, x_{k-1}) - \sum_{j=k}^{n-1} (A - \mathrm{i})^{-j} \sum_{l=1}^{\infty} (x_j, F_{m_l} z_s) \eta^s_l \quad \text{if } n \geqslant k + 1.$$

Note that all sums are finite because $x_j \in d$. Recall that the scalar product and the norm of ℓ_2 will be denoted by $(\cdot\,,\,\cdot)$ resp. $|||\cdot|||$.

Let $\mathscr{H}_+$ denote the closure of

$$\operatorname{graph}\mathscr{T}_1^1 + \operatorname{graph}\mathscr{T}_1^2 \equiv \{\varphi^1 + \mathscr{T}_1^1\varphi^1 + \varphi^2 + \mathscr{T}_1^2\varphi^2;\ \varphi^1 \in \mathscr{D}_1^1,\ \varphi^2 \in \mathscr{D}_1^2\}.$$

To describe the corresponding iterated spaces $\mathscr{H}_n$ (defined as the closure of $\mathscr{H}_+ + (A-\mathrm{i})^{-1}\mathscr{H}_+ + \ldots + (A-\mathrm{i})^{-(n-1)}\mathscr{H}_+$), we use the notations $\gamma_{rl}^s := \|(A-\mathrm{i})^{-r}\psi_l^s\|$, $\gamma_r^s := \{\gamma_{rl}^s,\ l \in \mathrm{N}\}$ for $r, l \in \mathrm{N}$ and $\mathscr{E}_n := \operatorname{Lin}\{(A-\mathrm{i})^{-j}\eta;\ j = 0, 1, \ldots, n-k\}$ for $n \in \mathrm{N}$, $n \geqslant k+1$.

LEMMA 4.1. *Suppose that*

$$\varkappa_s/\gamma_{k-1}^s \in \ell_2, \quad \varkappa_s/\gamma_k^s \notin \ell_2, \tag{2}$$

$$\{|||(I - F_{m_n})\varkappa_s/\gamma_r^s|||\ \|(A-\mathrm{i})^{-r}\eta_n^s\|,\ n \in \mathrm{N}\} \in \ell_2 \quad \text{for } r = 0, \ldots, k-1,\ s = 1, 2, \tag{3}$$

and

$$\{|||F_{m_n}\varkappa_s/\gamma_r^s|||\ \|(A-\mathrm{i})^{-r}\eta_n^s\|,\ n \in \mathrm{N}\} \in \ell_2 \quad \text{for } r \in \mathrm{N},\ r \geqslant k,\ s = 1, 2. \tag{4}$$

Then the operators $\mathscr{T}_n^s$, $n \in \mathrm{N}$, $s = 1,2$, extend continuously to bounded operators $\mathscr{T}_n^s$ from $\mathscr{L}_n^s$ into $\mathscr{H}$. We have

$$\mathscr{H}_n = \operatorname{graph}\mathscr{T}_n^1 + \operatorname{graph}\mathscr{T}_n^2 \quad \text{for } n = 0, \ldots, k \tag{5}$$

and

$$\mathscr{E}_n \subseteq \mathscr{H}_n \subseteq \operatorname{graph}\mathscr{T}_n^1 + \operatorname{graph}\mathscr{T}_n^2 + \mathscr{E}_n \quad \text{for } n \in \mathrm{N},\ n \geqslant k+1. \tag{6}$$

Proof. We prove that $\mathscr{T}_n^s$ is bounded from $\mathscr{D}_n^s$ into $\mathscr{H}$ for $n = 0, \ldots, k$; $s = 1, 2$. Let $j \in \{0, 1, \ldots, k-1\}$. By (3), we have

$$\|(A-\mathrm{i})^{-j}\mathscr{T}_1^s(x_j)\|^2 = \|\sum_l (x_j, (I - F_{m_l})\varkappa_s)(A-\mathrm{i})^{-j}\eta_l^s\|^2 =$$

$$= \sum_l |(x_j\gamma_j^s, (I - F_{m_l})\varkappa_s/\gamma_j^s)|^2\ \|(A-\mathrm{i})^{-j}\eta_l^s\|^2 \leqslant$$

$$\leqslant \left(\sum_l |||(I - F_{m_l})\varkappa_s/\gamma_j^s|||^2\ \|(A-\mathrm{i})^{-j}\eta_l^s\|^2\right)|||x_j\gamma_j^s|||^2 \leqslant$$

$$\leqslant \text{const.}\ q_{k-1}^{-1}\|\psi\|^2.$$

The latter follows from formula (18) in Corollary 3.3. Therefore, $\mathscr{T}_n^s$ is bounded. A similar estimation applies to $\mathscr{T}_n^s$, $n \geqslant k$.

Now we prove the second assertion of the lemma, that is, we verify (5) and (6). First let $n \leqslant k$. By the definition of $\mathscr{T}_n^s$, $\{\varphi^1 + \mathscr{T}_n^1\varphi^1 + \varphi^2 + \mathscr{T}_1^2\varphi^2; \varphi^s \in \mathscr{D}_n^s\}$

is a dense linear subspace of $\mathcal{H}_n$. Since $\mathcal{T}_n^s$ is continuous, (5) follows. Now suppose that $n \geqslant k+1$. If $x_j \in d$ for $j = 0, \ldots, n-1$ and $s = 1,2$, then by definition of $\mathcal{T}_n^s$ and

$$\sum_{j=0}^{n-1} (A-\mathrm{i})^{-j} \mathcal{T}_1^s(x_j) = \mathcal{T}_n^s(x_0, \ldots, x_{n-1}) + \sum_{j=k}^{n-1} (x_j, z_s) \sum_l (A-\mathrm{i})^{-j} \eta_l^s \tag{7}$$

$$= \mathcal{T}_n^s(x_0, \ldots, x_{n-1}) + \sum_{j=k}^{n-1} (x_j \gamma_j^s, z_s/\gamma_j^s)(A-\mathrm{i})^{-j+k+1-s} \eta.$$

The second sum in (7) is contained in $\mathcal{E}_n$. Since $\mathcal{T}_n^s$ is continuous and the finite dimensional subspace $\mathcal{E}_n$ is closed in $\mathcal{H}$, (7) implies $\mathcal{H}_n \subseteq \operatorname{graph}\mathcal{T}_n^1 + \operatorname{graph}\mathcal{T}_n^2 + \mathcal{E}_n$. It remains to show that $\mathcal{E}_n \subseteq \mathcal{H}_n$. We have to prove that $(A-\mathrm{i})^{-r}\eta \in \mathcal{H}_n$ for $r = 0, \ldots, n-k$. First assume that $r \leqslant n-k-1$. Since obviously $|\gamma_{k,l}^1| \geqslant |\gamma_{k+r,l}^1|$ for $l \in \mathbf{N}$, (2) implies $z_1/\gamma_{k+r}^1 \notin \ell_2$. Hence $f(x) := (x, z_1/\gamma_{k+r}^1)$ is a discontinuous linear functional on d with respect to the ℓ_2-norm. Therefore, there are vectors $y_m = \{y_{ml}, l \in \mathbf{N}\}$, $m \in \mathbf{N}$, in d such that $y_m \to 0$ in the ℓ_2-norm and $f(y_m) \to 1$ as $m \to +\infty$. Letting

$$\varphi_m := \sum_{l=1}^{\infty} \frac{y_{ml}}{\gamma_{k+r,l}^1}(A-\mathrm{i})^{-r-k}\psi_l^1 \quad \text{for } m \in \mathbf{N},$$

we have $\lim_m \varphi_m = 0$ in $\mathcal{H}_1$. Because $\mathcal{T}_n^1$ is bounded, $\lim_m \mathcal{T}_n^1\varphi_m = 0$. By (7),

$$(y_m, z_1/\gamma_{r+k}^1)(A-\mathrm{i})^{-r-k+k}\eta \to (A-\mathrm{i})^{-r}\eta \in \mathcal{H}_n.$$

Replacing $\mathcal{T}_n^1$ by $\mathcal{T}_n^2$, the same argument works for $r = n-k$ and yields $(A-\mathrm{i})^{-(n-k)}\eta \in \mathcal{H}_n$. This completes the proof of Lemma 1.

4.2. LEMMA 4.2. *The sequences* $\{\eta_n^1\}$, $\{m_n\}$ *and* z_s, $\{\psi_n^s\}$ *for* $s = 1,2$ *can be chosen such that* $\eta \notin \mathcal{D}(A)$ *and the conditions* (1)—(4) *are satisfied.*

Proof. Here we essentially make use of the growth conditions contained in Corollary 3.3, (19). Let us assume that $\{\psi_n^s\}$, $s = 1,2$, and $\{\eta_n^1\}$ satisfy formula (19) in Corollary 3.3 with positive sequences $\delta^s = \{\delta_n^s\}$ resp. $c = \{c_n\}$ which will be chosen now. (Note that for $\{\eta_n^1\}$ the other conditions of Corollary 3.3 need not be true.)

Then we have $\gamma_{k,n}^s = \|(A-\mathrm{i})^{-k}\psi_n^s\| \leqslant \delta_n^s \|(A-\mathrm{i})^{-(k-1)}\psi_n^s\| = \delta_n^s\gamma_{k-1,n}^s$ for $s = 1,2$, $n \in \mathbf{N}$, $n \geqslant k$. Choosing $\delta_n^s = 1/n$ for $n \in \mathbf{N}$, the sequence $\gamma_{k-1}^s/\gamma_k^s$ will be unbounded. Hence there is a $y_s \in \ell_2$ so that $y_s\gamma_{k-1}^s/\gamma_k^s \notin \ell_2$. Setting $z_s := y_s\gamma_{k-1}^s$, (2) follows.

Again by Corollary 3.3, (19), we have

$$\|(A-\mathrm{i})^{-r}\eta_n^1\| \leqslant c_n\|(A-\mathrm{i})^{-(r-1)}\eta_n^1\| \quad \text{for } n \in \mathbf{N}, r \in \mathbf{N}_0, n \geqslant r. \tag{8}$$

Multiplying with appropriate positive numbers we can assume that $\|(A - \mathrm{i})^{-k+1}\eta_n^1\| = 1$ for $n \in \mathbf{N}$. Moreover, we choose c such that $c_n \leqslant 2^{-n}$ for $n \in \mathbf{N}$. Then (1) is satisfied and $\eta \notin \mathscr{D}(A)$.

Now we turn to (3). Since $\varkappa_s/\gamma_k^s \in \ell_2$, we can choose m_n so large that

$$|||(I - F_{m_n})\varkappa_s/\gamma_k^s|||\, \|\eta_n^1\| \leqslant 2^{-n} \quad \text{for } n \in \mathbf{N},\ s = 1, 2.$$

Then $\varkappa_s/\gamma_r^s \in \ell_2$ and

$$|||(I - F_{m_n})\varkappa_s/\gamma_r^s|||\, \|(A - \mathrm{i})^{-r}\eta_n^1\| \leqslant |||(I - F_{m_n})\varkappa_s/\gamma_k^s|||\, \|\eta_n^1\|$$

for $r = 0, \ldots, k - 1$ imply (3).

Finally, we prove (4). Let $r \in \mathbf{N}$, $r \geqslant k$. Since $\psi_l^s \neq 0$ by construction, there are real numbers λ_{sl} such that $E(\lambda_{sl}, \lambda_{sl} + 1)\psi_l^s =: \varphi_l^s \neq 0$ for all $n \in \mathbf{N}$ and $s = 1, 2$. Hence $\gamma_{rl}^s = \|(A - \mathrm{i})^{-r}\psi_l^s\| \geqslant \|\varphi_l^s\|\, a_{sl}^r$ where $a_{sl} := \max(|\lambda_{sl} - \mathrm{i}|, |\lambda_{sl} + 1 - \mathrm{i}|)$. Letting

$$a_n := \min\{\|\varphi_l^s\|, a_{sl} : s = 1, 2;\ l = 1, \ldots, m_n\} \quad \text{for } n \in \mathbf{N},$$

we get $\gamma_{rl}^s \geqslant a_n^{r+1}$ for $l = 1, \ldots, m_n$ and therefore

$$|||F_{m_n}\varkappa_s/\gamma_r^s||| \leqslant a_n^{-r-1}\, |||\varkappa_s||| \quad \text{for } n \in \mathbf{N} \text{ and } s = 1, 2. \tag{9}$$

By a repeated application of (8), we obtain for $n \geqslant r$

$$\|(A - \mathrm{i})^{-r}\eta_n^2\| \leqslant \|(A - \mathrm{i})^{-r}\eta_n^1\| \leqslant c_n^{r-k+1}\|(A - \mathrm{i})^{-k+1}\eta_n^1\| = c_n^{r-k+1} \leqslant c_n. \tag{10}$$

Now we choose c_n such that $c_n a_n^{-r-1}\, |||\varkappa_s||| \leqslant 2^{-n}$ for $n \in \mathbf{N}$ and $s = 1, 2$. From (9) and (10) it follows that for $n \geqslant r$

$$|||F_{m_n}\varkappa_s/\gamma_r^s|||\, \|(A - \mathrm{i})^{-r}\eta_n\| \leqslant 2^{-n}.$$

Hence (4) is fulfilled, thus completing the proof of Lemma 2.

4.3. The following is the key to our main result.

COROLLARY 4.3. *There exists a closed symmetric restriction T of A such that*

(a) $\mathscr{D}(T^n)$ *is dense in* $\mathscr{H}$ *for all* $n \in \mathbf{N}$.
(b) $\mathscr{D}(T^{n+r})$ *is a core for* T^r *where* $n, r \in \mathbf{N}$ *if and only if* $r \neq k$.

Proof. Suppose the sequences $\{\eta_n^1\}$, $\{m_n\}$, $\varkappa_s$, $\{\psi_n^s\}$ are chosen as in Lemma 2. From § 3 we know that the spaces $\mathscr{L}_n^s := \mathscr{L}_+^s + (A - \mathrm{i})^{-1}\mathscr{L}_+^s + \ldots + (A - \mathrm{i})^{-(n+1)}\mathscr{L}_+^s$, $n \in \mathbf{N}$, $s = 1, 2$, are closed subspaces of $\mathscr{K}_s$. Since $\psi_l^s \in \mathscr{D}(A)$ for all $l \in \mathbf{N}$, $\mathscr{L}_+^s \cap \mathscr{D}(A) = \{0\}$. From Lemma 1 and $\eta \notin \mathscr{D}(A)$ we infer that $\mathscr{H}_n \cap \mathscr{D}(A^n) =$

$=\{0\}$, $n\in\mathbf{N}$. Let T be the closed symmetric restriction of A defined by means of $\mathscr{H}_+$, that is, the Cayley transform of T is $W=V(I-P_+)$ where P_+ is the projection on $\mathscr{H}_+$. Then, by Proposition 2.2, $\mathscr{D}(T^n)$ is dense in $\mathscr{H}$ for all $n\in\mathbf{N}$.

To prove (b), first let $r=k$. By Lemma 1,

$$(A-\mathrm{i})^{-n}\eta\in\mathscr{D}(A^n)\cap\mathscr{E}_{n+k}\subseteq\mathscr{D}(A^n)\cap\mathscr{H}_{n+k}\quad\text{and}\quad\eta\notin\mathscr{H}_k.$$

According to Proposition 2.3, $\mathscr{D}(T^{n+k})$ is not a core for T^k. Now we consider the case $r\neq k$. Let $r\in\mathbf{N}$. By Proposition 2.3, it is sufficient to prove that

$$\mathscr{H}_{n+r}\cap\mathscr{D}(A^n)\subseteq(A-\mathrm{i})^{-n}\mathscr{H}_r. \tag{11}$$

From Lemma 1 we see that for $s=1,2$

$$\operatorname{graph}\mathscr{T}_{n+r}\cap\mathscr{D}(A^n)\subseteq\operatorname{graph}\mathscr{T}^s_{n+r}\upharpoonright(A-\mathrm{i})^{-n}\mathscr{L}^s_r=(A-\mathrm{i})^{-n}\operatorname{graph}\mathscr{T}^s_{n+r}\upharpoonright\mathscr{L}^s_r=$$

$$=(A-\mathrm{i})^{-n}\operatorname{graph}\mathscr{T}^s_r.$$

Therefore,

$$(\operatorname{graph}\mathscr{T}^1_{n+r}\dotplus\operatorname{graph}\mathscr{T}^2_{n+r})\cap\mathscr{H}_{n+r}\cap\mathscr{D}(A^n)\subseteq(A-\mathrm{i})^{-n}\mathscr{H}_r.$$

If $n+r\leqslant k$, then $\mathscr{H}_{n+r}=\operatorname{graph}\mathscr{T}^1_{n+r}+\operatorname{graph}\mathscr{T}^2_{n+r}$ and (11) is valid. Now suppose that $n+r\geqslant k+1$. By the preceding, it suffices to check (11) for the subset $\mathscr{E}_{n+r}$ of $\mathscr{H}_{n+r}$. First let $r<k$. Because $\eta\notin\mathscr{D}(A)$, $\mathscr{D}(A^n)\cap\mathscr{E}_{n+r}=\{0\}$. Suppose now that $r>k$. Again by $\eta\notin\mathscr{D}(A)$,

$$\mathscr{D}(A^n)\cap\mathscr{E}_{n+r}=\operatorname{Lin}\{(A-\mathrm{i})^{-n}\eta,\ (A-\mathrm{i})^{-n-1}\eta,\ \ldots,(A-\mathrm{i})^{-n-r+k}\eta\}\subseteq(A-\mathrm{i})^{-n}\mathscr{E}_r.$$

Since $\mathscr{E}_{n+r}\subseteq\mathscr{H}_{n+r}$ and $\mathscr{E}_r\subseteq\mathscr{H}_r$ by Lemma 1, (11) is fulfilled in this case which completes the proof of Corollary 3.

4.4. The next corollary only needs Lemma 1 and condition (2).

COROLLARY 4.4. *There is a closed symmetric restriction T of A such that*

(a) *$\mathscr{D}(T^k)$ is dense in $\mathscr{H}$. $\mathscr{D}(T^{k+1})$ is not dense in $\mathscr{H}$.*

(b) *If $n+r\leqslant k$, $n,r\in\mathbf{N}$, then $\mathscr{D}(T^{n+r})$ is a core for T^r.*

Proof. We choose the sequences $\{\psi^s_n\}$, $\varkappa_s$ so that (2) is satisfies. Let $m_n=0$ for $n\in\mathbf{N}$, $\eta^1_1\neq0$, $\eta^1_1\in\mathscr{D}(A)$ and $\eta^1_n=0$ for $n\geqslant2$. Then the assumptions of Lemma 1 are fulfilled. (Recall that $F_{m_n}=F_0=0$ by definition.) Since $\eta^1_1\in\mathscr{D}(A)$, $\eta=(A-\mathrm{i})^{-k}\eta^1_1\in\mathscr{D}(A^{k+1})\cap\mathscr{H}_{k+1}$. Therefore, by Proposition 2.2, $\mathscr{D}(T^{k+1})$ is not dense in $\mathscr{H}$. The remaining assertions follow similarly as in the preceding proof. Note that $\mathscr{H}_{n+r}=\operatorname{graph}\mathscr{T}^1_{n+r}+\operatorname{graph}\mathscr{T}^2_{n+r}$ for $n+r\leqslant k$ by Lemma 1. This completes the proof.

REMARK. Corollary 4.4 alone could be obtained by an essentially simpler construction. We decompose A as $A_1 \oplus B$, where A_1 and B are unbounded self-adjoint operators. We only use the operators $\mathcal{T}_n^1$. As in the proof of Corollary 4.4, we assume that (2) is satisfied and $\mathcal{T}_n^1$ has one-dimensional range, i.e., $\mathcal{T}_1^1(x_0) = (x_0, x_1)\eta_1^1$. We only mention the results in that case because the proofs are similar as above.

Then, for $r \leqslant k, n, r \in \mathbf{N}$, $\mathcal{D}(T^{n+r})$ is a core for T^r if and only if $n + r \leqslant k$. To study the density of $\mathcal{D}(T^m)$, it will be convenient to discuss the two cases $\eta_1^1 \in \mathcal{D}(A)$ and $\eta_1^1 \notin \mathcal{D}(A)$ separately. First let $\eta_1^1 \in \mathcal{D}(A)$. Then, $\mathcal{D}(T^{k+1})$ is not dense in $\mathcal{H}$, while $\mathcal{D}(T^k)$ is dense. Now suppose that $\eta_1^1 \notin \mathcal{D}(A)$. Then, all domains $\mathcal{D}(T^m)$, $m \in \mathbf{N}$, are dense in $\mathcal{H}$. The latter case is also interesting from another point of view which we will shortly discuss. Let $\tilde{\mathcal{H}}_+$ be the closed subspace of $\mathcal{H}$ generated by η_1^1 and $\{\psi_n^1, n \in \mathbf{N}\}$ and let $\tilde{T}$ be the corresponding closed restriction of A. Obviously, $\tilde{\mathcal{H}}_+ \supseteq \mathcal{H}_+$ and thus $T \supseteq \tilde{T}$. Since $\tilde{T}$ is an orthogonal direct sum of restrictions with deficiency indices (1,1), $\mathcal{D}_\infty(\tilde{T})$ is a core for all $\tilde{T}^m$, $m \in \mathbf{N}$. As we have seen, $\mathcal{D}(T^{n+r})$ is not a core for T^r if $n + r > k \geqslant r$. In other words, by restricting T to a closed operator T on a smaller (but still dense) domain it may happen that all domains $\mathcal{D}(\tilde{T}^{n+r})$ are cores for $\tilde{T}^r$, $n, r \in \mathbf{N}$, while T has not this property. Moreover, we see that $\mathcal{D}_\infty(T)$ is still dense in $\mathcal{H}$ (because $\mathcal{D}_\infty(\tilde{T}) \subseteq \mathcal{D}_\infty(T)$ is dense by Theorem 1.9). But the assumptions of Lemma 1.7 are not fulfilled. Hence, "$\mathcal{D}(T^{n+1})$ is a core for T^n, $n \in \mathbf{N}$" is sufficient (by Lemma 1.7), but not necessary for the density of $\mathcal{D}_\infty(T)$.

4.5. THEOREM 4.5. *Let A be an arbitrary unbounded self-adjoint operator in the Hilbert space $\mathcal{H}$. Let $\mathfrak{N}$ be a (possibly empty) subset of* $\mathbf{N}$.

I. *There exists a closed symmetric operator T with $T \subseteq A$ such that:*

(a) *$\mathcal{D}(T^m)$ is dense in $\mathcal{H}$ for all $m \in \mathbf{N}$.*

(b) *$\mathcal{D}(T^{n+r})$ is a core for T^r, $n, r \in \mathbf{N}$, if and only if $r \notin \mathfrak{N}$.*

II. *Suppose that $\mathfrak{N}$ is bounded. Let $k \in \mathbf{N}$ such that $k \geqslant n$ for all $n \in \mathfrak{N}$. There is a closed symmetric operator T with $T \subseteq A$ such that:*

(a) *$\mathcal{D}(T^k)$ is dense in $\mathcal{H}$. $\mathcal{D}(T^{k+1})$ is not dense in $\mathcal{H}$.*

(b) *For any $n, r \in \mathbf{N}$ with $n + r \leqslant k$, $\mathcal{D}(T^{n+r})$ is a core for T^r if and only if $r \notin \mathfrak{N}$.*

Proof. First we consider the case where $\mathfrak{N}$ is empty. Then, it suffices to take $T = A$ in part I and to apply Corollary 4 in part II.

Now suppose that $\mathfrak{N} \neq \emptyset$. To prove I, we decompose A as an orthogonal sum $\sum_{n \in \mathfrak{N}} \oplus A_n$ of unbounded self-adjoint operators A_n, $n \in \mathfrak{N}$, and apply Corollary 3 to each operator A_n. Part II will be treated similarly. Let $A = A_0 \oplus \sum_{n \in \mathfrak{N}} \oplus A_n$ where A_0 and A_n, $n \in \mathfrak{N}$, are unbounded self-adjoint operators. We now apply Corollary 4 to A_0 and Corollary 3 to A_n, $n \in \mathfrak{N}$. This ends the proof.

4.6. REMARKS. 1. Our main aim was to show that the phenomenon stated in Theorem 5 can occur for restrictions of an *arbitrary* unbounded self-adjoint operator A. If we only want to have *examples* of this kind, then the construction could be made quite simpler. One possibility has been discussed in the remark in 4.4. Further, for special operators A (for example, $A = A_1 \otimes A_2$, A_1, A_2 unbounded self-adjoint operators), the construction in § 3 could be essentially simplified.

2. The preceding construction has many degrees of freedom. They can be used to obtain restrictions T having some additional properties. We give a sample of this: There is an essentially self-adjoint operator Q on a dense domain $\mathscr{D}$ contained in $\mathscr{D}(T)$ such that T and Q satisfy the Heisenberg commutation relation $TQ\varphi - QT\varphi = i\varphi$ for all $\varphi \in \mathscr{D}$. Examples of this kind (based on a quite different construction) can be found in [10], Subsection 6.4.

3. Using the technique established above, we can prove further results in the spirit of Theorem 5. Thus part II can be essentially generalized. For example, on the subspace $\mathscr{H}^1 := \overline{\mathscr{D}(T^{k+1})}$ the same phenomenon as described in I, (b), can occur etc. (of course, provided that T is still unbounded on $\mathscr{D}(T^{k+1})$). With a more refined technique it can be shown in part II, (a), that there is a restriction T such that $\mathscr{D}(T^k)$ is still dense in $\mathscr{H}$, but $\mathscr{D}(T^{k+1}) = \{0\}$. For $k = 1$ this assertion is proved in the next section.

§ 5

Throughout this section, closed operators and self-adjoint operators are always meant to be densely defined. Moreover, we assume that the Hilbert space $\mathscr{H}$ is separable.

5.1. The main results of this section are the following.

THEOREM 5.1. *Let B be a closed unbounded operator in $\mathscr{H}$. There exists an orthogonal projection P in $\mathscr{H}$ such that*

$$P\mathscr{H} \cap \mathscr{D}(B) = (I - P)\mathscr{H} \cap \mathscr{D}(B) = \{0\}.$$

THEOREM 5.2. *For each unbounded self-adjoint operator A in $\mathscr{H}$, there exist closed symmetric operators T_1, T_2 with $T_1 \subseteq A$ and $T_2 \subseteq A$ such that $\mathscr{D}(T_1) \cap \mathscr{D}(T_2) = \{0\}$ and $\mathscr{D}(T_1^2) = \mathscr{D}(T_2^2) = \{0\}$.*

The main idea of the proof of Theorem 1 given below I have first seen in a recent preprint of van Daele [2]. Later I learned that it already occurs in [3]. This argument essentially simplified my more involved original proof of the theorem.

The existence of restrictions T_1, T_2 with $\mathscr{D}(T_1) \cap \mathscr{D}(T_2) = \{0\}$ is a result of van Daele [2] (stated in [2] under the additional assumption that A is non-singular and positive). Examples of this kind has been first constructed by J. von Neumann [6].

5.2. First we derive Theorem 2 from Theorem 1. Suppose that Theorem 1 is already shown. Let P be the projection from Theorem 1 for the operator A. $W_1 := V(I-P)$ and $W_2 := VP$ are the Cayley transforms of closed symmetric restrictions T_1 resp. T_2 of A. Recall that V is the Cayley transform of A. Since $P\mathscr{H} \cap \mathscr{D}(A) = (I-P)\mathscr{H} \cap \mathscr{D}(A) = \{0\}$ by Theorem 1, T_1 and T_2 are densely defined by Proposition 2.2.

Now let $\varphi \in \mathscr{D}(T_1) \cap \mathscr{D}(T_2)$. Then, $\varphi \in (I-V)P\mathscr{H} \cap (I-V)(I-P)\mathscr{H}$. Since $I-V$ is injective, we obtain $\varphi = 0$, thus proving the first part of Theorem 2.

Next we check that $\mathscr{D}(T_1^2) = \mathscr{D}(T_2^2) = \{0\}$. Clearly, $P\mathscr{H}$ is the deficiency space of T_1 for i. Since $\mathscr{D}(T_2) = (I-V)P\mathscr{H}$ is dense in $\mathscr{H}$, $P\mathscr{H} + V^{-1}P\mathscr{H} \supseteq V^{-1}(I-V)P\mathscr{H}$ is dense in $\mathscr{H}$. Because $\mathscr{D}(T_1^2) = (I-V)^2(\mathscr{H} \ominus (\overline{P\mathscr{H} + V^{-1}P\mathscr{H}}))$, we have $\mathscr{D}(T_1^2) = \{0\}$ and similarly $\mathscr{D}(T_2^2) = \{0\}$. This completes the proof of Theorem 2.

5.3. *Proof of Theorem 1.* First we note that $A := (B^*B)^{1/2}$ is an unbounded self-adjoint operator in $\mathscr{H}$ with $\mathscr{D}(A) = \mathscr{D}(B)$. Using the spectral resolution of A, we can find an orthonormal basis $\{\varphi_{kn}, k \in \mathbf{N}, n \in \mathbf{Z}\}$ of $\mathscr{H}$ such that

$$\|(A+\mathrm{i})^{-1}\varphi_{kn}\| \leqslant 1/|n|! \quad \text{for all } k \in \mathbf{N} \text{ and } n \in \mathbf{Z}. \tag{1}$$

Let $\mathscr{K}$ be the set of all vectors $\psi \in \mathscr{H}$ with

$$\sum_{n=-\infty}^{+\infty} \langle \psi, \varphi_{kn}\rangle e^{int} = 0 \text{ a.e. on } [0, \pi] \quad \text{for all } k \in \mathbf{N}. \tag{2}$$

Obviously, $\mathscr{K}$ is a closed linear subspace of $\mathscr{H}$. Let P be the orthogonal projection on $\mathscr{K}$. Assume now that $\psi_0 \in \mathscr{D}(A) \cap \mathscr{K}$. Since

$$|\langle \psi_0, \varphi_{kn}\rangle| = |\langle \psi_0, (A+\mathrm{i})(A+\mathrm{i})^{-1}\varphi_{kn}\rangle| = |\langle (A-\mathrm{i})\psi_0, (A+\mathrm{i})^{-1}\varphi_{kn}\rangle| \leqslant$$
$$\leqslant \|(A-\mathrm{i})\psi_0\|/|n|! \qquad \text{by (1)},$$

$$f_k(z) := \sum_{n=-\infty}^{+\infty} \langle \psi_0, \varphi_{kn}\rangle z^n$$

defines a holomorphic function on $\mathbf{C}_1 \setminus \{0\}$. By (2), $f_k(z)$ is vanishing on a set of non-zero Lebesgue measure. Hence $f_k(z) \equiv 0$. By the uniqueness theorem for Laurent series, this implies $\langle \psi_0, \varphi_{kn}\rangle = 0$ for all $n \in \mathbf{Z}$ and $k \in \mathbf{N}$. Consequently, $\psi_0 = 0$.

From the theory of Fourier series it is clear that $\mathscr{K}^\perp = (I-P)\mathscr{H}$ is the set of all $\xi \in \mathscr{H}$ such that

$$\sum_{n=-\infty}^{+\infty} \langle \xi, \varphi_{kn}\rangle e^{int} = 0 \text{ a.e. on } [\pi, 2\pi] \quad \text{for all } k \in \mathbf{N}.$$

Therefore, the same argument applies to $\mathscr{K}^\perp$. This completes the proof of Theorem 1.

5.4. Let A again an unbounded self-adjoint operator in $\mathscr{H}$. If $\mathscr{D} \subseteq \mathscr{D}(A^n)$ is a core for A^n, then $\mathscr{D}$ is a core for A^k where $k \leqslant n$, $k, n \in \mathbf{N}$ (see Proposition 1.8). Conversely, a core $\mathscr{D} \subseteq \mathscr{D}(A^{k+1})$ of A^k need not to be a core for A^{k+1}. As a by-product of the preceding proof we obtain a much more striking result along this line.

THEOREM 5.3. *Let A be an arbitrary unbounded self-adjoint operator in $\mathscr{H}$. Let $k \in \mathbf{N}$. There exists a linear subspace $\mathscr{D}$ of $\mathscr{D}(A^{k+1})$ such that:*

(a) *A^k is essentially self-adjoint on $\mathscr{D}$, i.e. $\mathscr{D}$ is a core for A^k.*

(b) *$A^{k+1} \restriction \mathscr{D}$ is a closed symmetric operator with infinite deficiency indices.*

(c) *$\mathscr{D} \cap \mathscr{D}(A^{k+2}) = \{0\}$.*

Proof. Put $\mathscr{D} := (I - V)^{k+1}\mathscr{K}$, where $\mathscr{K}$ is the closed subspace of $\mathscr{H}$, as defined in 4.3.

Let us begin by proving (a). By definition, $(A^k+z)\mathscr{D}=(A^k+z)(I-V)^k(I-V)\mathscr{K}$ for $z = \mathrm{i}$ and $z = -\mathrm{i}$. Since $\mathscr{K}^{\perp} \cap \mathscr{D}(A) = \{0\}$, $(I-V)\mathscr{K}$ is dense by Proposition 2.2 (applied in the case $\mathscr{H}_+ = \mathscr{K}^{\perp}$, $n = 1$). Because

$$S_{k,z} := (A^k + z)(I - V)^k = (A^k + z)(A + \mathrm{i})^{-k}(2\mathrm{i})^k$$

is a bounded isomorphism of $\mathscr{H}$ onto $\mathscr{H}$ for $z = \pm\mathrm{i}$, we obtain $\overline{(A^k + z)\mathscr{D}} = \mathscr{H}$ for $z = \mathrm{i}$ and $z = -\mathrm{i}$. This means that A^k is essentially self-adjoint on $\mathscr{D}$. In particular, $\mathscr{D}$ is dense in $\mathscr{H}$.

Next we prove (b). We have

$$(A^{k+1} + z)\mathscr{D} = (A^{k+1} + z)(I - V)^{k+1}\mathscr{K} = S_{k+1,z}\mathscr{K}$$

for $z = \mathrm{i}$ and $z = -\mathrm{i}$. Since $\mathscr{K}$ is closed, $(A^{k+1} + z)\mathscr{D}$ is closed. Therefore, $A^{k+1} \restriction \mathscr{D}$ is a closed operator. It is easy to check that $A^{k+1} \restriction \mathscr{D}$ has the deficiency spaces $(A - \mathrm{i})^{k+1}(A^{k+1} + \bar{z})\mathscr{K}^{\perp}$. Of course, these spaces are infinite dimensional. (The latter also follows from Theorem 1.9 and condition (c) which will be shown now.)

Assume that $\varphi \in \mathscr{D} \cap \mathscr{D}(A^{k+2})$. Let $\varphi = (I - V)^{k+1}\varphi_1$, where $\varphi_1 \in \mathscr{K}$. Then $(A + \mathrm{i})^{k+1}\varphi = (2\mathrm{i})^{k+1}\varphi_1 \in \mathscr{D}(A) \cap \mathscr{K}$. By 4.3, $\varphi_1 = 0$ and hence $\varphi = 0$. This proves (c), and completes the proof of Theorem 3.

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