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COMPOSITION OPERATORS ON H^2

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1. INTRODUCTION

For φ analytic in the unit disk $\mathbf{D}$ such that $\varphi(\mathbf{D}) \subset \mathbf{D}$, the composition operator C_φ is the operator on the Hardy space H^2 of the unit disk given by $C_\varphi f = f \circ \varphi$ for all f in H^2. Ryff [24] showed that C_φ is always a bounded operator.

Several authors have found that the properties of C_φ depend, to a great extent, on the behaviour of φ near its fixed points. We will say *a point b in $\overline{\mathbf{D}}$ is a fixed point of φ* if $\lim\limits_{r\to 1^-} \varphi(rb) = b$. We will write $\varphi'(b)$ for $\lim\limits_{r\to 1^-} \varphi'(rb)$: the limit obviously exists if $|b| < 1$, and if $|b| = 1$, the theorem of Julia, Carathéodory, and Wolff [20, page 57] shows that this limit exists and $0 < \varphi'(b) \leqslant \infty$. Although it is not a priori evident that φ has fixed points, it has at least one.

DENJOY-WOLFF THEOREM ([11], [29], [1]). *If φ, not an elliptic Möbius transformation of $\mathbf{D}$ onto $\mathbf{D}$, is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$, then there is a unique fixed point a of φ (with $|a| \leqslant 1$) such that $|\varphi'(a)| \leqslant 1$.*

We will call the distinguished fixed point a the *Denjoy-Wolff point of φ*, and we reiterate that if $|a| = 1$ then $0 < \varphi'(a) \leqslant 1$ and if $|a| < 1$ then $0 \leqslant |\varphi'(a)| < 1$.

The results of this paper, the most important of which are noted below, strengthen the observation that properties of C_φ depend on the behaviour of φ near its fixed points. The hypotheses of all the following theorems include the assumption that φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ and that a denotes the Denjoy-Wolff point of φ, but we omit this statement for brevity.

THEOREM 2.1. *If $|a| < 1$, the spectral radius of C_φ is 1. If $|a| = 1$, the spectral radius of C_φ is $\varphi'(a)^{-1/2}$.*

If T is any operator, the *essential norm of T* is $\|T\|_e = \inf\{\|T + A\| : A$ is a compact operator$\}$ and the *essential spectrum of T* is $\sigma_e(T) = \{\mu : \mu - T$ is not a

Fredholm operator}. The following result is based on a new estimate of the radial maximal function due to B. J. Davis (Theorem 2.2).

THEOREM 2.4. *If φ' is continuous on $\bar{\mathbf{D}}$, then*

$$M^{1/2} \leqslant \|C_\varphi\|_e \leqslant 2M^{1/2}$$

where $M := \max\{\sum_{e^{i\theta}\in\varphi^{-1}(\{\lambda\})} |\varphi'(e^{i\theta})|^{-1} : |\lambda| = 1\}$.

Under more restrictive hypotheses, this is used to compute the essential spectral radius of C_φ.

In [21] Nordgren found $\sigma(C_\varphi)$ when φ is an inner function with $|a| < 1$ and when φ is an inner Möbius transformation. The following result covers the remaining cases.

COROLLARY 5.2. *If φ is an inner function, not a Möbius transformation, and* $|a| = 1$, *then* $\sigma(C_\varphi) = \sigma_e(C_\varphi) = \{\lambda : |\lambda| \leqslant \varphi'(a)^{-1/2}\}$.

For general φ, our information is most specific in case $|a| = 1$ and $\varphi'(a) < 1$.

THEOREM. *If $|a| = 1$ and $\varphi'(a) < 1$, then:*
(4.3) *C_φ is similar to $e^{i\theta}C_\varphi$ for θ real,*
(4.5) *if $\varphi'(a)^{1/2} < |\lambda| < \varphi'(a)^{-1/2}$ then λ is an eigenvalue of infinite multiplicity for C_φ.*

We obtain further results when φ is smooth enough, for example we have the following.

COROLLARY 4.8. *If φ, not a finite Blaschke product, is analytic in a neighborhood of $\bar{\mathbf{D}}$ and has $|a| = 1$ with $\varphi'(a) < 1$, then* $\sigma(C_\varphi) = \{\lambda : |\lambda| \leqslant \varphi'(a)^{-1/2}\}$.

In [17, page 142] Kamowitz asks if $|a| = 1$ and $\varphi'(a) = 1$ implies $\sigma(C_\varphi) = \bar{\mathbf{D}}$. In Section 6, we show that if $\varphi(z) = (2 - z)^{-1}$ then $\sigma(C_\varphi) = \{\lambda : 0 \leqslant \lambda \leqslant 1\}$, and give other examples for which the spectrum is a heart-shaped subset of $\bar{\mathbf{D}}$.

This paper has two major themes. The first is the analogy between C_φ^* and weighted shifts, which is developed in Section 3. The results of Section 3 give information about the point spectrum and approximate point spectrum of C_φ^*. These results are based on the trivial observation that $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$ where $K_\alpha(z) := (1 - \bar{\alpha}z)^{-1}$ is the kernel for evaluation of H^2 functions at $\alpha \in \mathbf{D}$, and the deeper fact that if $\{z_k\}$ is an interpolating sequence in $\mathbf{D}$ then $\{(1 - |z_k|^2)^{1/2}K_{z_k}\}$ is a basic sequence in H^2 equivalent in the Banach space sense to an orthonormal set.

The second theme of the paper (developed in Sections 4 and 6) is the exploitation of the solution of Schroeder's functional equation $f \circ \varphi = \lambda f$ given in [8]. Solutions of this equation *that are in* H^2 are eigenfunctions for C_φ. The crucial (and usually difficult) issue is to determine which of the solutions of Schroeder's equation (an infinite dimensional manifold for all $\lambda \neq 0$ if $|a| = 1$) are actually in H^2.

Composition operators have been studied in this context for at least 15 years [21], but implicitly for much longer. Koenigs [18] gave a solution to Schroeder's functional equation for the case $|a| < 1$ in 1884 and much has been written on this equation since then (see the bibliography of [22)]. Littlewood [19] and Ryff [24] established inequalities that, in our context, give estimates for the norm of a composition operator.

Nordgren [21] studied C_φ when φ is a Möbius transformation mapping $\mathbf{D}$ onto $\mathbf{D}$ and when φ is inner and $|a| < 1$. In particular, he found the spectrum of C_φ in these cases. In his thesis Schwartz [25], gave a characterization (A is a composition operator if and only if $A(z^n) = (Az)^n$ for $n = 0,1,2, \ldots$), gave norm estimates, studied convergence of sequences of composition operators, and showed that C_φ is a normal operator if and only if $\varphi(z) = \alpha z$ for $|\alpha| \leqslant 1$. Deddens [10] studied the composition operators given by $\varphi(z) = \alpha z + \beta$ where $|\alpha| + |\beta| \leqslant 1$ and found spectra in these cases.

Several authors have studied compact composition operators [25], [27], [4] developing interesting criteria (but not sharp) for determining compactness and finding the spectrum. This paper contributes little to this aspect of the subject, although Section 2 and 3 use the kernel function point of view developed in [3] and [4]. It is helpful to observe that the F-sequence and B-sequence arguments of Sections 2 and 3 can easily be used to give a more elementary proof of Theorem 2.1 of [27, page 478]: "if φ has a finite angular derivative at any point of $\partial\mathbf{D}$ then C_φ is not compact". This, together with the Denjoy-Wolff theorem, implies Theorem 2 of [4, page 128]: "if C_φ is compact, $|a| < 1$".

The most extensive study of spectra of composition operators was undertaken by Kamowitz [17]. This paper adds little to his treatment of the case $|a| < 1$, [17, Theorem 3.8, page 149]. Unfortunately however, the proof of his Theorem 3.1 [17, page 139], which covers the case $|a| = 1$, contains an error. The error occurs on page 140, line 9 which reads: "thus $[(\lambda - C_\varphi)^{-1}g](z) = \lambda^{-1} \sum_{k=0}^{\infty} g(\varphi_k(z))\lambda^{-k}$ whenever the right-hand side converges". To be correct, it should read: "... whenever the right-hand side converges *and is in* H^p". Since there are infinitely many analytic functions f that satisfy $\lambda f - f \circ \varphi = g$ if there are any [8, Theorem 4.7, page 89], we have no reason to suppose, a priori, that the given series is the solution in H^p. Fortunately, the theorem is true: (it is Corollary 4.8 of this paper), although the proof given here does not fill the gap in Kamowitz' proof.

In [5], Cima, Thomson, and Wogen give necessary and sufficient conditions for C_φ to have closed range. This result is especially interesting because it comes to grips with non-inner functions whose boundary values include $\partial\mathbf{D}$, a class about which most results say little.

The last section of this paper points out some of the many problems concerning composition operators that remain open and gives conjectures concerning some of them.

This paper makes no attempt to consider composition operators on H^p for $p \neq 2$, although it is clear that most of the results have easy generalizations for $1 < p < \infty$. For $p \leqslant 1$ or $p = \infty$, many of the techniques break down and the situation is less clear.

I am especially grateful to Burgess Davis for considering the problem of estimating the norm of the radial maximal function and for permitting me to include his result (Theorem 2.2) which has not appeared elsewhere. I would also like to thank R. P. Kaufman for suggesting the proof of Theorem 6.1. However, my greatest debt is to Eric Nordgren whose excellent lectures at Long Beach in 1977 [22] inspired this work.

2. NORM INEQUALITIES

In this section, we give estimates for the norm of a composition operator, determine its spectral radius, and in some cases, estimate the essential norm and the essential spectral radius. We begin with an easy estimate of the norm which leads to the spectral radius computation. The upper bound for the norm is due to Ryff [24], the lower bound to Schwartz [25].

THEOREM 2.1. *Suppose φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ and suppose φ has Denjoy-Wolff point a. Then C_φ is a bounded operator on H^2 with*

$$(1 - |\varphi(0)|^2)^{-1/2} \leqslant \|C_\varphi\| \leqslant (1 + |\varphi(0)|)(1 - |\varphi(0)|^2)^{-1/2}.$$

The spectral radius of C_φ is 1 when $|a| < 1$ and $\varphi'(a)^{-1/2}$ when $|a| = 1$.

Proof. For a proof of boundedness and the upper bound, see [22, page 48].

Since $1 = K_0$, the kernel for evaluation of H^2 functions at zero, $C_\varphi^* 1 = K_{\varphi(0)}$. It follows that $\|C_\varphi\| = \|C_\varphi^*\| \geqslant \|C_\varphi^* 1\| = (1 - |\varphi(0)|^2)^{-1/2}$.

The spectral radius of C_φ is $\lim_{n\to\infty} \|C_\varphi^n\|^{1/n} = \lim_{n\to\infty} \|C_{\varphi_n}\|^{1/n}$. Using the norm estimates, we see

$$\limsup_{n\to\infty} (1 - |\varphi_n(0)|^2)^{-1/2n} \leqslant \lim_{n\to\infty} \|C_{\varphi_n}\|^{1/n} \leqslant$$

$$\leqslant \liminf_{n\to\infty} (1 + |\varphi_n(0)|)^{1/n} (1 - |\varphi_n(0)|^2)^{-1/2n} = \liminf_{n\to\infty} (1 - |\varphi_n(0)|^2)^{-1/2n}.$$

That is, the spectral radius of C_φ is

$$\lim_{n\to\infty}(1-|\varphi_n(0)|^2)^{-1/2n} = \lim_{n\to\infty}(1-|\varphi_n(0)|)^{-1/2n}.$$

Since $a = \lim_{n\to\infty}\varphi_n(0)$, we have $\lim_{n\to\infty}(1-|\varphi_n(0)|)^{-1/2n} = 1$ when $|a| < 1$.

When $|a| = 1$ and $\varphi'(a) < 1$, the sequence $\{\varphi_n(0)\}$ converges to a in a Stolz angle [8, page 73] which means that $\lim_{n\to\infty}\dfrac{1-|\varphi_n(0)|}{1-|\varphi_{n-1}(0)|} = \varphi'(a)$ [1, page 32]. Thus,

$$\lim_{n\to\infty}(1-|\varphi_n(0)|)^{-1/2n} = \lim_{n\to\infty}\left(\prod_{k=0}^{n-1}\frac{1-|\varphi_k(0)|}{1-|\varphi_{k+1}(0)|}\right)^{-1/2n} =$$

$$= \lim_{n\to\infty}\left(\frac{(1-|\varphi_{n-1}(0)|}{1-|\varphi_n(0)|}\right)^{1/2} = \varphi'(a)^{-1/2}.$$

Suppose now that $|a| = 1$ and $\varphi'(a) = 1$. If z_n is a sequence in $\mathbf{D}$ converging to a such that $\varphi(z_n) \to a$ and $\alpha = \lim_{n\to\infty}\dfrac{1-|\varphi(z_n)|}{1-|z_n|}$ exists then $\alpha \geqslant \varphi'(a) = 1$ [2, pages 25—32]. It follows that $\liminf_{n\to\infty}\left(\dfrac{1-|\varphi_n(0)|}{1-|\varphi_{n-1}(0)|}\right) \geqslant 1$. Thus

$$\lim_{n\to\infty}(1-|\varphi_n(0)|)^{-1/2n} = \lim_{n\to\infty}\left(\prod_{k=0}^{n-1}\frac{1-|\varphi_k(0)|}{1-|\varphi_{k+1}(0)|}\right)^{1/2n} \leqslant$$

$$\leqslant \limsup_{n\to\infty}\left(\frac{(1-|\varphi_{n-1}(0)|}{1-|\varphi_n(0)|}\right)^{1/2} \leqslant 1.$$

On the other hand, since $1-|\varphi_n(0)| \leqslant 1$ for each n, $\lim_{n\to\infty}(1-|\varphi_n(0)|)^{-1/2n} \geqslant 1$. Therefore, $\lim_{n\to\infty}(1-|\varphi_n(0)|)^{-1/2n} = 1 = \varphi'(a)^{-1/2}$ as was asserted. ▨

Although better estimates of the norm exist for functions in special classes (see [25, Theorem 3.10] or the results below), the given estimates are the best possible based only on the value of φ at zero. Indeed, if $\varphi(z) \equiv \varphi(0)$, an easy computation shows $\|C_\varphi\| = (1-|\varphi(0)|^2)^{-1/2}$ and if φ is an inner function, $\|C_\varphi\| = (1+|\varphi(0)|)(1-|\varphi(0)|^2)^{-1/2}$ [21, page 443].

COROLLARY. *If φ satisfies the hypotheses of Theorem* 2.1 *and* $|a| = 1$ *then* $\|C_\varphi\| \geqslant \varphi'(a)^{-1/2}$.

Proof. For any bounded operator the spectral radius is less than or equal to the norm. ▨

The corollary is sometimes an improvement over the theorem. If $\varphi(z) = sz + (1 - s)$ for $0 < s < 1$, then the Denjoy-Wolff point is 1 and $\varphi'(1) = s$. Since C_φ^* is subnormal [10, page 801], the norm of C_φ is its spectral radius, that is,

$$\|C_\varphi\| = s^{-1/2} > s^{-1/2}(2 - s)^{-1/2} = (1 - |\varphi(0)|^2)^{-1/2}.$$

We need two preliminary results before proceeding to the theorems on essential norm and essential spectral radius. The first of these is an estimate for the radial maximal function due to Burgess Davies.

DEFINITION. If f is in H^2, the *radial maximal function of* f is the function $\mathrm{R}_f(e^{i\theta}) = \sup\{|f(re^{i\theta})| : 0 \leqslant r < 1\}$.

THEOREM 2.2. (B. J. Davis). *Let f be in H^2 and let R_f be its radial maximal function. Then*

$$\int_0^{2\pi} \mathrm{R}_f(e^{i\theta})^2 \, d\theta \leqslant 4 \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta.$$

Proof. Let m be normalized Lebesgue measure on $\partial\mathbf{D}$, let $Z_t = X_t + iY_t$ be standard two dimensional Brownian motion (started at 0), let Z_t^θ be Brownian motion conditioned to first hit $\partial\mathbf{D}$ at $e^{i\theta}$, also started at 0. A reference for Z_t^θ and its properties is [14]. Let τ be the first time Z_t (or Z_t^θ) hits $\partial\mathbf{D}$, and define

$$f^* = \sup_{0 \leqslant t \leqslant \tau} |f(Z_t)|$$

and

$$f_\theta^* = \sup_{0 \leqslant t \leqslant \tau} |f(Z_t^\theta)|.$$

Now since $|f|$ is subharmonic, $|f(Z_{t \wedge \tau})|$, $0 \leqslant t < \infty$, is a submartingale ([13]) so that, using the continuous time version of Theorem 3.4 on page 317 of [12], and the fact that Z_τ is uniformly distributed (i.e. m) on $\partial\mathbf{D}$, we have

$$Ef^{*2} \leqslant 2E|f(Z_\tau)|^2 = 2\int_{\partial\mathbf{D}} |f(e^{i\theta})|^2 dm.$$

Next we show

$$\int \mathrm{R}_f(e^{i\theta})^2 dm \leqslant 2Ef^{*2},$$

which will complete the proof. We show, in fact, the stronger fact that for each positive λ,

$$m\{\theta : \mathrm{R}_f(e^{i\theta}) > \lambda\} \leqslant 2P(f^* > \lambda),$$

and this in turn, will follow upon showing

$$(*) \qquad P(f_\theta^* > \lambda) > \frac{1}{2} \quad \text{if } \mathrm{R}_f(\theta) > \lambda,$$

and integrating over θ.

To prove (*) let $\mathrm{R}_f(\theta) > \lambda$ and let $re^{i\theta}$ be a point on the ray from 0 to $e^{i\theta}$ such that $|f(re^{i\theta})| > \lambda$. Let γ be a curve from $re^{i\theta}$ to $\partial\mathbf{D}$ on which $|f| > \lambda$ and let γ' be the reflection of γ about the line through 0 and $e^{i\theta}$. We note that since,

$$P(Z_t^0 \text{ hits } \{se^{i\theta},\ r \leqslant s < 1\} \text{ for some } t < \tau) = 1,$$

we have

$$P(Z_t^0 \text{ hits } \gamma \cup \gamma' \text{ for some } t < \tau) = 1.$$

Since

$$P(Z_t^0 \text{ hits } \gamma \text{ for some } t < \tau) = \\ = P(Z_t^0 \text{ hits } \gamma' \text{ for some } t < \tau),$$

it follows that $P(Z_t^0$ hits γ for some $t < \tau) \geqslant 1/2$, and this implies (*). ▧

LEMMA 2.3. *Suppose φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ and φ' is continuous on $\bar{\mathbf{D}}$. Given $\varepsilon > 0$ and $0 < r_0 < 1$, there is a finite collection of disjoint open intervals $I_1, \ldots, I_m$ such that*

(1) *for some r, with $r_0 < r < 1$, we have*

$$\{\theta : |\varphi(e^{i\theta})| > r\} \subset \bigcup_{j=1}^{m} \bar{I}_j,$$

(2) *for θ in $\bigcup_{j=1}^{m} I_j$, we have* $\dfrac{d}{d\theta}(\arg\varphi(e^{i\theta})) \geqslant (1+\varepsilon)^{-1}|\varphi'(e^{i\theta})|$, *and*

(3) *for each $j = 1, \ldots, m$, the function* $\exp(i\arg\varphi(e^{i\theta}))$ *is univalent on I_j.*

Proof. It follows from the lemma of Julia, Carathéodory, and Wolff [20, page 57] that whenever $|\varphi(e^{i\theta})| = 1$, we have

$$\frac{d}{d\theta}(\arg\varphi(e^{i\theta})) = i^{-1}\frac{d}{d\theta}(\log\varphi(e^{i\theta})) = \varphi(e^{i\theta})^{-1}\varphi'(e^{i\theta})e^{i\theta} = |\varphi'(e^{i\theta})| > 0.$$

Thus there is a number r_1, with $r_0 < r_1 < 1$, so that if $|\varphi(e^{i\theta})| > r_1$ then

$$\frac{d}{d\theta}(\arg\varphi(e^{i\theta})) \geqslant (1+\varepsilon)^{-1}|\varphi'(e^{i\theta})|.$$

Let J_k, $k = 1,2, \ldots$ be the collection of disjoint open arcs of $\partial\mathbf{D}$ comprising $\{e^{i\theta} : |\varphi(e^{i\theta})| > r_1\}$. Let J_{k_1}, $J_{k_2}, \ldots$ be the set of these arcs that do not intersect $\varphi^{-1}(\partial\mathbf{D})$. If there are none, take $r = r_1$, otherwise let $r = \sup\{|\varphi(e^{i\theta})| : e^{i\theta} \in \bigcup_j J_{k_j}\}$. We claim $r < 1$. Indeed, if $e^{i\theta_n}$ is a sequence in $\bigcup_j J_{k_j}$ with $\lim_{n\to\infty}|\varphi(e^{i\theta_n})| = 1$, we choose $e^{i\theta^*}$ a limit point of $\{e^{i\theta_n}\}$. By continuity, $|\varphi(e^{i\theta^*})| = 1$ so $e^{i\theta^*}$ isin J_{k^*} for some, k^*. Since J_{k^*} is open, $e^{i\theta_n}$ is in J_{k^*} for some n, but this is impossible because $e^{i\theta_n}$ belongs to $\bigcup_j J_{k_j}$ which is disjoint from J_{k^*}.

Let $\tilde{I}_k$, $k = 1,2, \ldots$ be the collection of disjoint open arcs of $\partial\mathbf{D}$ comprising $\{e^{i\theta} : |\varphi(e^{i\theta})| > r\}$. We claim that there are only finitely many intervals $\tilde{I}_k$. Indeed, if there are infinitely many, choose $e^{i\theta_k}$ in $\tilde{I}_k$ with $|\varphi(e^{i\theta_k})| = 1$ and let $e^{i\theta^*}$ be a limit point of the sequence $\{e^{i\theta_k}\}$. Again $|\varphi(e^{i\theta^*})| = 1$ and $e^{i\theta^*}$ is an element of $\tilde{I}_{k^*}$ for some k^*. Since $\tilde{I}_{k^*}$ is open, for all k large, $e^{i\theta_k}$ is in $\tilde{I}_{k^*}$, which contradicts the disjointness of the $\tilde{I}_k$.

Since φ' is continuous, $\varphi(\partial\mathbf{D})$ has finite length which means, since $r > 0$, that for each k, the set $\tilde{I}_k \cap \varphi^{-1}(\{1\})$ is finite. Let I_j, $j = 1,2, \ldots, m$, be the collection of disjoint open intervals that comprise the set

$$\{\theta : 0 < \theta < 2\pi \text{ such that } e^{i\theta} \in \bigcup \tilde{I}_k \text{ and } \varphi(e^{i\theta}) \neq 1\}.$$

This collection is finite because there are only finitely many $\tilde{I}_k$ and because $\varphi^{-1}(\{1\})$ is finite. Properties (1) and (2) are direct consequences of the construction and (3) is evident since $\arg\varphi(e^{i\theta})$ is strictly increasing on each interval I_j and $\arg\varphi(e^{i\theta}) \neq 2\pi n$ for any n. ▨

COROLLARY. *If φ satisfies the hypotheses of the lemma, there is an integer m such that for every λ in $\partial\mathbf{D}$, $\varphi^{-1}(\{\lambda\})$ has at most m elements.*

Proof. There can be at most one element of $\varphi^{-1}(\{\lambda\})$ in each of the intervals $I_1, \ldots, I_m$ of the lemma. ▨

If T is a bounded operator, the essential norm of T is the number $\|T\|_e = \inf\{\|T + A\| : A \text{ is a compact operator}\}$.

THEOREM. 2.4. *Suppose φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ and φ' is continuous on $\bar{\mathbf{D}}$. Then*

$$M^{1/2} \leqslant \|C_\varphi\|_e \leqslant 2M^{1/2}$$

where

$$M = \max\{\sum_{e^{i\theta}\in\varphi^{-1}(\{\lambda\})} |\varphi'(e^{i\theta})|^{-1} : |\lambda| = 1\}.$$

Proof. We have

$$\|C_\varphi\|_e = \limsup_{n\to\infty} \sup_{\|f\|=1} \|C_\varphi(z^n f)\| = \limsup_{n\to\infty} \sup_{\|f\|=1} \|\varphi^n f\circ\varphi\|.$$

Given $\varepsilon > 0$ and $0 < r_0 < 1$, let $I_1, \ldots, I_m$ and r be as in Lemma 2.3, and find N large enough that $n > N$ implies $r^{2n} < \varepsilon$.

For $n > N$ and $\|f\| = 1$ we have

$$\|C_\varphi z^n f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |\varphi(e^{i\theta})|^{2n} |f(\varphi(e^{i\theta}))|^2 \,d\theta \leqslant$$

$$\leqslant \sum_{k=1}^{m} \frac{1}{2\pi}\int_{I_k} |f(\varphi(e^{i\theta}))|^2 \,d\theta + \frac{r^{2n}}{2\pi}\int_{[0,2\pi]\setminus\cup I_k} |f(\varphi(e^{i\theta}))|^2 d\theta \leqslant$$

$$\leqslant \sum_{k=1}^{m} \frac{1}{2\pi}\int_{I_k} |f(\varphi(e^{i\theta}))|^2 \,d\theta + \varepsilon\|C_\varphi\|^2.$$

Let $I_k = (\theta_k, \theta_k')$ and let $t_k = \arg\varphi(e^{i\theta_k})$, let $t_k' = \arg\varphi(e^{i\theta_k'})$ and define $t(\theta)$ on I_k by $t(\theta) = \arg\varphi(e^{i\theta})$. Now

$$\int_{\theta_k}^{\theta_k'} |f(\varphi(e^{i\theta}))|^2 d\theta \leqslant \int_{\theta_k}^{\theta_k'} R_f(e^{it(\theta)})^2 \,d\theta =$$

$$= \int_{t_k}^{t_k'} R_f(e^{it})^2 \left(\frac{dt}{d\theta}\right)^{-1} dt \leqslant (1+\varepsilon)\int_{t_k}^{t_k'} R_f(e^{it})^2 |\varphi'(e^{i\theta_k(t)})|^{-1} dt$$

where $\theta_k(t)$ is the inverse of $t(\theta)$ defined on (t_k, t_k').

Thus

$$\|C_\varphi(z^n f)\|^2 \leqslant \sum_{k=1}^{m} \frac{1+\varepsilon}{2\pi}\int_{t_k}^{t_k'} R_f(e^{it})^2 |\varphi'(e^{i\theta_k(t)})|^{-1} \,dt + \varepsilon\|C_\varphi\|^2 \leqslant$$

$$\leqslant (1+\varepsilon)\left(\int_0^{2\pi} R_f(e^{it})^2 \frac{dt}{2\pi}\right) \max\left\{ \sum_{(\arg\varphi)^{-1}(\{t\})\cap I_k \neq \varnothing} |\varphi'(e^{i\theta_k(t)})|^{-1} : e^{it} \in \partial\mathbf{D} \right\} + \varepsilon\|C_\varphi\|^2 \leqslant$$

$$\leqslant (1+\varepsilon)\, 4 \max\{ \sum_{(\arg\varphi)^{-1}(\{t\})\cap I_k \neq \varnothing} |\varphi'(e^{i\theta_k(t)})|^{-1} : e^{it} \in \partial\mathbf{D}\} + \varepsilon\|C_\varphi\|^2.$$

Since the right hand side of this inequality is independent of f and $n > N$, this expression dominates $\|C_\varphi\|_e^2$. As $r_0 \to 1^-$, by the continuity of φ', the right hand side tends to

$$4(1+\varepsilon)\max\{\sum_{\varphi(e^{i\theta})=e^{it}} |\varphi'(e^{i\theta})|^{-1} : e^{it} \in \partial\mathbf{D}\} + \varepsilon\|C_\varphi\|^2.$$

Since ε was an arbitrary positive number, we obtain $\|C_\varphi\|_e^2 \leqslant 4M$.

On the other hand, if $|\lambda| = 1$ and $\varphi(e^{i\theta_j}) = \lambda$ for $j = 1,2, \ldots, m$, then for r_0 close to 1, the set $\varphi^{-1}(\{r\lambda : r_0 \leqslant r \leqslant 1\})$ consists of m disjoint arcs terminating at the points $e^{i\theta_j}$: let $\gamma_j(r)$ denote the branch of $\varphi^{-1}(r\lambda)$ whose image includes $e^{i\theta_j}$.

For $\alpha_1, \ldots, \alpha_m$ complex numbers,

$$\lim_{r\to 1^-} \left\| \sum_{j=1}^{m} \alpha_j \sqrt{1-|\gamma_j(r)|^2} K_{\gamma_j(r)} \right\|^2 =$$

$$= \lim_{r\to 1^-} \sum_{j=1}^{m} \sum_{k=1}^{m} \alpha_j \bar{\alpha}_k \sqrt{1-|\gamma_j(r)|^2} \sqrt{1-|\gamma_k(r)|^2} (1-\overline{\gamma_j(r)}\gamma_k(r))^{-1} = \sum_{j=1}^{m} |\alpha_j|^2.$$

Since $\sqrt{1-|\beta|^2}K_\beta$ tends to 0 weakly as $|\beta| \to 1$, we have

$$\lim_{r\to 1^-} \left\| A\left(\sum_{j=1}^{m} \alpha_j \sqrt{1-|\gamma_j(r)|^2} K_{\gamma_j(r)} \right) \right\| = 0$$

for all compact operators A. This means that

$$\|C_\varphi^*\|_e^2 \geqslant \left(\sum_{j=1}^{m} |\alpha_j|^2 \right)^{-1} \lim_{r\to 1^-} \left\| C_\varphi^* \left(\sum_{j=1}^{m} \alpha_j \sqrt{1-|\gamma_j(r)|^2} K_{\gamma_j(r)} \right) \right\|^2.$$

Now

$$\left\| C_\varphi^* \left(\sum_{j=1}^{m} \alpha_j \sqrt{1-|\gamma_j(r)|^2} K_{\gamma_j(r)} \right) \right\|^2 =$$

$$= \left\| \sum_{j=1}^{m} \alpha_j \sqrt{1-|\gamma_j(r)|^2} K_{\varphi(\gamma_j(r))} \right\|^2 = \left\| \sum_{j=1}^{m} \alpha_j \sqrt{1-|\gamma_j(r)|^2}\, K_r \right\|^2 =$$

$$= \left| \sum_{j=1}^{m} \alpha_j \frac{\sqrt{1-|\gamma_j(r)|^2}}{1-r^2} \right|^2.$$

Since φ is conformal at $e^{i\theta_j}$, we find [2, page 30] that $\lim_{r\to 1-} \frac{1-|\gamma_j(r)|^2}{1-r^2} = |\varphi'(e^{i\theta_j})|^{-1}$ which implies that

$$\|C_\varphi^*\|_e^2 \geqslant \left| \sum_{j=1}^{m} \alpha_j |\varphi'(e^{i\theta_j})|^{-1/2} \right|^2 \left(\sum_{j=1}^{m} |\alpha_j|^2 \right)^{-1}.$$

Choosing $\alpha_j = |\varphi'(e^{i\theta_j})|^{-1/2}$ yields $\|C_\varphi^*\|_e^2 \geqslant \sum_{j=1}^{m} |\varphi'(e^{i\theta_j})|^{-1}$. Since λ was an arbitrary point of $\partial\mathbf{D}$, $\|C_\varphi\|_e^2 \geqslant M$. ∎

As in the corollary of Theorem 2.1, since $\|C_\varphi\| \geqslant \|C_\varphi\|_e$, this sometimes leads to an improved estimate of $\|C_\varphi\|$. Indeed, if $\varphi(z) = sz^2 + (1 - s)$ where $0 < s < 1$, we find

$$\|C_\varphi\| \geqslant \|C_\varphi\|_e \geqslant (|\varphi'(-1)|^{-1} + |\varphi'(1)|^{-1})^{1/2} = s^{-1/2}.$$

Comparing this with the earlier example, and noting that C_{z^2} is an isometry, we see that this indeed equals $\|C_\varphi\|$.

By the essential spectrum of an operator T, denoted $\sigma_e(T)$, we mean the set of complex numbers μ for which $\mu - T$ is not Fredholm. We can now compute the essential spectral radius in certain cases.

COROLLARY 2.5. *If φ satisfies the hypotheses of Theorem 2.4 and for some integer N the set $\{e^{i\theta} : |\varphi_N(e^{i\theta})| = 1\}$ consists only of fixed points of φ_N then*

$$\sup\{|\mu| : \mu \in \sigma_e(C_\varphi)\} = \sup\{|\varphi_N'(e^{i\theta})|^{-1/2N} : \varphi_N(e^{i\theta}) = e^{i\theta}\}.$$

Proof. Applying the spectral mapping theorem to the Calkin algebra we obtain

$$\sup\{|\mu| : \mu \in \sigma_e(C_\varphi)\} = \sup\{|\mu|^{1/N} : \mu \in \sigma_e(C_\varphi^N)\} = \lim_{n\to\infty}\| C_\varphi^{nN}\|_e^{1/nN} = \lim_{n\to\infty} \|C_{\varphi_{nN}}\|_e^{1/nN}.$$

Now by hypothesis, for any λ in $\partial\mathbb{D}$ and any integer n, the set $\{e^{i\theta} : \varphi_{nN}(e^{i\theta}) = \lambda\}$ is $\{\lambda\}$ if λ is a fixed point of φ_N and empty if it is not.

The theorem then implies

$$\max\{|\varphi_N'(\lambda)^n|^{-1/2nN} : \varphi_N(\lambda) = \lambda\} \leqslant \|C_{\varphi_{r,N}}\|_e^{1/nN} \leqslant 2^{1/n}\max\{|\varphi_N'(\lambda)^n|^{-1/2nN} : \varphi_N(\lambda) = \lambda\}.$$

Taking the limit as $n \to \infty$, we obtain the desired result. ▨

3. A WEIGHTED SHIFT ANALOGY FOR C_φ^*

As we shall see (Proposition 4.2), for some φ, C_φ^* has an invariant subspace on which it is similar to a weighted shift. In this section, we exploit the fact that this is almost true for all φ. We begin by formalizing the notions of forward and backward iteration sequences.

DEFINITION. A non-constant sequence $\{z_k\}_{k=0}^\infty$ is a *B-sequence* for φ if $\varphi(z_k) = z_{k-1}$, $k = 1, 2, \ldots$. A point b of $\partial\mathbb{D}$ is a *B-point* of φ *if* $\lim_{r\to1^-} \varphi(rb) = b$ and there is a B-sequence for φ converging to b.

A non-constant sequence $\{z_k\}_{k=0}^\infty$ or $\{z_k\}_{k=-\infty}^\infty$ is an *F-sequence* for φ if $\varphi(z_k) = z_{k+1}$ for all k.

We note that for a B-sequence $\{z_k\}$, we have $\lim_{k\to\infty}|z_k| = 1$, and for an F-sequence $\lim_{k\to\infty} z_k = a$, the Denjoy-Wolff point. If b is a B-point of φ and $\{z_k\}$ is a B-sequence converging to b, then $\lim_{k\to\infty}\frac{1-|z_{k-1}|}{1-|z_k|} \geqslant \varphi'(b)$ and if $|a| = 1$ and $\{z_k\}$ is an F-sequence then $\lim_{k\to\infty}\frac{1-|z_{k+1}|}{1-|z_k|} \geqslant \varphi'(a)$. (These remarks follow from the results of [2, pages 25—32].)

Determining whether a given φ has a B-sequence is usually difficult. If $\{z_k\}$ is a B-sequence then $z_k \in \bigcap_{n=1}^{\infty}\varphi_n(\mathbf{D})$ for each k, so if $\bigcap_{n=1}^{\infty}\varphi_n(\mathbf{D})$ is empty or is the singleton $\{a\}$ then φ has no B-sequences. On the other hand, it can be shown (although we will not) that if φ is univalent and has a fixed point b with $1 < \varphi'(b) < \infty$, then b is a B-point. We will be content to prove the following weak result.

LEMMA 3.1. *Let φ be analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$. If b in $\partial\mathbf{D}$ is a fixed point of φ with $\varphi'(b) > 1$ and φ is analytic in a neighborhood of b, then there are uncountably many B-sequences for φ converging to b with the property that* $\lim_{k\to\infty}\frac{1-|z_{k-1}|}{1-|z_k|} = \varphi'(b)$.

Proof. Since $\varphi'(b) > 1$, there is $\varepsilon > 0$ so that φ^{-1} is single-valued and analytic on the disk $U = \{z: |b-z| < \varepsilon\}$ and $\varphi^{-1}(U) \subset U$.

Theorems 3.2 and 3.3 of [8, pages 78—80] produce a univalent analytic function $\sigma : U \to \mathbf{C}$ so that $\varphi^{-1}(z) = \sigma^{-1}(\varphi'(b)^{-1}\sigma(z))$. Since σ is conformal at b, there is point z^* in $U \cap \mathbf{D}$ so that the line-segment $\{r\sigma(z^*) : 0 \leqslant r \leqslant 1\}$ is contained in $\sigma(U \cap \mathbf{D})$ and is not tangent to $\sigma(U \cap \partial\mathbf{D})$. The sequences $z_k = \sigma^{-1}(\varphi'(b)^{-k}r\sigma(z^*))$, $k = 1,2,\ldots$ for $\varphi'(b)^{-1} < r \leqslant 1$ are distinct B-sequences converging to b. Moreover, since $z_k \to b$ in a Stolz angle

$$\lim_{k\to\infty}\frac{1-|z_{k-1}|}{1-|z_k|} = \lim_{k\to\infty}\left|\frac{b-z_{k-1}}{b-z_k}\right| = \lim_{k\to\infty}\left|\frac{\varphi(b)-\varphi(z_k)}{b-z_k}\right| = \varphi'(b). \qquad ▨$$

Our use of B-sequences in studying composition operators is based on the fact that $C_\varphi^* K_\alpha = K_{\varphi(\alpha)}$ and the following observations.

LEMMA 3.2. (Shapiro-Shields). *If $\{z_k\} \subset \mathbf{D}$ is an interpolating sequence for H^∞, then $\{\sqrt{1-|z_k|^2}K_{z_k}\}$ is a basic sequence in H^2 equivalent to an orthonormal set; that is, the series $\sum \alpha_k\sqrt{1-|z_k|^2}K_{z_k}$ converges if and only if $\sum|\alpha_k|^2 < \infty$.*

Proof. This is the content of Theorem 2 of [26, page 521]; see [7, page 23] for details of this equivalence. ▨

COROLLARY 3.3. *If $\{z_k\}$ is a B-sequence with $\frac{1-|z_k|}{1-|z_{k-1}|} \leqslant r < 1$ for all k, then the conclusion of Lemma 3.2 holds. In particular, this is the case when $\{z_k\}$ converges to the B-point b and $\varphi'(b) > 1$.*

Proof. From the comments following the definition of B-sequence, we see that $\lim_{k\to\infty} \frac{1-|z_k|}{1-|z_{k-1}|} \leqslant \varphi'(b)^{-1}$ when $\{z_k\}$ converges to the B-point b. Hayman and Newman showed [16, page 203] that if $\frac{1-|z_k|}{1-|z_{k-1}|} \leqslant r < 1$ for all k, then $\{z_k\}$ is an interpolating sequence and Lemma 3.2 applies. ▨

We are now ready to develop the analogy between shifts and C_φ^*. If z_k is a B-sequence, for the remainder of this section, let $v_k = (1-|z_k|^2)^{1/2}K_{z_k}$. Thus

$$C_\varphi^* v_k = (1-|z_k|^2)^{1/2}K_{z_{k-1}} = \left(\frac{1-|z_k|^2}{1-|z_{k-1}|^2}\right)^{1/2} v_{k-1}.$$

THEOREM 3.4. *Let φ be analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$. If $\{z_k\}_{k=0}^\infty$ is a B-sequence for φ and $\lim_{k\to\infty} \frac{1-|z_k|}{1-|z_{k-1}|} = r < 1$, then $C_\varphi^* - \lambda$ is not bounded below, and $\lambda \in \sigma_e(C_\varphi)$, for $|\lambda| = r^{1/2}$.*

Proof. By Corollary 3.3, the set $\{v_k\}_{k=0}^\infty$ is a basic sequence in H^2 equivalent to an orthonormal set, that is, there are positive constants m and M so that

$$m \sum_{k=0}^\infty |\alpha_k|^2 \leqslant \|\sum_{k=0}^\infty \alpha_k v_k\|^2 \leqslant M \sum_{k=0}^\infty |\alpha_k|^2,$$

whenever (α_k) is in ℓ^2.

Let n be a positive integer. Now for $|\lambda| < r^{1/2}$, let $w_\lambda = \sum_{k=n+1}^\infty \alpha_k v_k$ where $\alpha_k = \lambda^{k-1}\left(\frac{1-|z_n|^2}{1-|z_k|^2}\right)^{1/2}$. (Since

$$\lim_{k\to\infty} \frac{|\alpha_k|^2}{|\alpha_{k-1}|^2} = \lim_{k\to\infty} |\lambda|^2 \left(\frac{1-|z_{k-1}|^2}{1-|z_k|^2}\right) = |\lambda|^2/r < 1,$$

the series for w_λ converges.) Thus

$$C_\varphi^* w_\lambda = \sum_{k=n+1}^\infty \alpha_k \left(\frac{1-|z_k|^2}{1-|z_{k-1}|^2}\right) v_{k-1} = v_n + \lambda w_\lambda.$$

It follows that for $|\lambda_0| = r^{1/2}$ and $|\lambda| < r^{1/2}$, we have

$$\|(C_\varphi^* - \lambda_0)(w_\lambda/\|w_\lambda\|)\| \leqslant \|(C_\varphi^* - \lambda)(w_\lambda/\|w_\lambda\|)\| + |\lambda - \lambda_0| =$$

$$= \|v_n\|/\|w_\lambda\| + |\lambda - \lambda_0| = \|w_\lambda\|^{-1} + |\lambda - \lambda_0|.$$

If $\lambda \to \lambda_0$, since $\sum|\alpha_k|^2 \to \infty$, we also have

$$\|w_\lambda\| \to \infty, \text{ which means } \|(C_\varphi^* - \lambda_0)(w_\lambda/\|w_\lambda\|)\| \to 0.$$

Therefore $(C_\varphi^* - \lambda_0)$ is not bounded below on the subspace spanned by $\{v_k : k \geq n\}$. Since $\{v_k\}$ is equivalent to an orthonormal set, $v_k \to 0$ weakly as $k \to \infty$. Since n was arbitrary this implies $A(C_\varphi^* - \lambda_0) \neq I + K$ for any operators A and K with K compact. That is, $\lambda_0 \in \sigma_e(C_\varphi^*)$. ▧

THEOREM 3.5. *Let φ be analytic in* **D** *with $\varphi(\mathbf{D}) \subset \mathbf{D}$ but not a Möbius transformation mapping* **D** *onto* **D**. *Suppose for each $j = 0,1,2,\ldots$, the sequence $\{z_k^j\}_{k=0}^\infty$ is a B-sequence with* $\limsup_{k\to\infty} \dfrac{1-|z_k^j|}{1-|z_{k-1}^j|} = r_j < 1$, *and suppose the set $\{z_k^j : j = 0,1,2,\ldots, k = 0,1,2,\ldots\}$ is not a Blaschke sequence. If for each $j = 0,1,2,\ldots$,* $\rho^2 < \liminf_{k\to\infty} \dfrac{1-|z_k^j|}{1-|z_{k-1}^j|}$, *then $\sigma(C_\varphi) \cap \{\lambda : |\lambda| = \rho\} \neq \varnothing$.*

Proof. Given non-negative integers j and k_0, let $\{z_n\}$ be the B-sequence

$$z_n = z_{n+k_0}^j, \quad \text{and} \quad v_n = (1-|z_n|^2)^{1/2} K_{z_n}.$$

Since $\limsup_n \dfrac{1-|z_n|}{1-|z_{n-1}|} < 1$, the sequence $\{v_n\}$ is a basic sequence in H^2 equivalent to an orthonormal set (Corollary 3.3). Given λ with $|\lambda| = \rho$, let $w(\lambda) = \sum_{n=1}^\infty \lambda^{n-1} \left(\dfrac{1-|z_0|^2}{1-|z_n|^2}\right)^{1/2} v_n$. Since $\rho_2 < \liminf_{n\to\infty} \dfrac{1-|z_n|}{1-|z_{n-1}|}$, as in the proof of 3.4, the series for w_λ converges and $(C_\varphi^* - \lambda)w(\lambda) = v_0$.

If $\sigma(C_\varphi) \cap \{\lambda : |\lambda| = \rho\} = \varnothing$, then $(C_\varphi^* - \rho e^{i\theta})^{-1}$ exists for each θ and we define Q by $Q = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} (C_\varphi^* - \rho e^{i\theta})^{-1}\, d\theta$. Now

$$Qv_0 = \frac{1}{2\pi}\int_0^{2\pi} (C_\varphi^* - \rho e^{i\theta})^{-1} v_0\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} w(\rho e^{i\theta})\, d\theta =$$

$$= \sum_{n=1}^\infty \frac{1}{2\pi}\int_0^{2\pi} \rho^{n-1} e^{i(n-1)\theta} \left(\frac{1-|z_0|^2}{1-|z_n|^2}\right)^{1/2} v_n\, d\theta = \left(\frac{1-|z_0|^2}{1-|z_1|^2}\right)^{1/2} v_1.$$

That is, $QK_{z_0} = K_{z_1}$, which means $C_\varphi^* Q(K_{z_0}) = K_{z_0}$. From the definition of Q, we see $C_\varphi^* Q = QC_\varphi^*$ and since z_0 was an arbitrary element of a non-Blaschke set, this means $C_\varphi^* Q = QC_\varphi^* = I$. But φ is not a Möbius transformation mapping **D** onto **D**, so C_φ is not invertible. This contradiction means

$$\sigma(C_\varphi) \cap \{\lambda : |\lambda| = \rho\} \neq \emptyset.$$

COROLLARY 3.6. *Let φ be analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$, but not a Möbius transformation of $\mathbf{D}$ onto $\mathbf{D}$. If $b_1, \ldots, b_n$ are fixed points of φ on $\partial\mathbf{D}$ with φ analytic in a neighborhood of each b_j, and $\varphi'(b_j) > 1$, then $\sigma(C_\varphi) \supset \bigcup_{j=1}^{n} \{\lambda : |\lambda| = \varphi'(b_j)^{-1/2}\}$ and $\sigma(C_\varphi)$ intersects every circle of radius ρ centered at 0 with $\rho < \max\{\varphi'(b_j)^{-1/2} : j = 1,2, \ldots, n\}$.*

Proof. By Lemma 3.1, for each $j = 1,2, \ldots, n$ there are uncountably many B-sequences converging to the B-point b_j with $\lim_{k\to\infty} \frac{1 - |z_k|}{1 - |z_{k-1}|} = \varphi'(b_j)^{-1}$. The corollary is now an immediate consequence of Theorems 3.4 and 3.5.

It is not difficult to see that if φ is not univalent then $\ker C_\varphi^* \neq (0)$. The following result extends this idea.

THEOREM 3.7. *Let φ be analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$. Suppose $\{z_k\}$ and $\{z_k'\}$ are B-sequences for φ with $z_0 = z_0'$ but $\lim_{k\to\infty} z_k = b \neq b' = \lim_{k\to\infty} z_k'$. If*

$$|\lambda|^2 < \lim_{k\to\infty} \frac{1 - |z_k|}{1 - |z_{k-1}|} \leqslant \lim_{k\to\infty} \frac{1 - |z_k'|}{1 - |z_{k-1}'|} < 1,$$

then λ is an eigenvalue of C_φ^*.

Proof. Since $b \neq b'$, we may assume without loss of generality that $z_1 \neq z_1'$. The growth conditions on the sequences ensure that each separately is an interpolating set, and since $b \neq b'$, we find $\{z_k\} \cup \{z_k'\}$ is an interpolating set (see [16]). Letting $v_k = \sqrt{1 - |z_k|^2} K_{z_k}$ and $v_k' = \sqrt{1 - |z_k'|^2} K_{z_k'}$, Lemma 3.2 implies that $\{v_k\} \cup \{v_k'\}$ is a basic sequence for H^2 equivalent to an orthogonal set.

Now let

$$w_\lambda = \sum_{k=1}^{\infty} \lambda^{k-1} \left(\frac{1 - |z_n|^2}{1 - |z_k|^2}\right)^{1/2} v_k - \sum_{k=1}^{\infty} \lambda^{k-1} \left(\frac{1 - |z_0'|^2}{1 - |z_k'|^2}\right)^{1/2} v_k'.$$

As before, the series for w_λ converge and an easy computation gives $(C_\varphi^* - \lambda)(w_\lambda) = 0$.

This theorem can be applied to finite Blaschke products (order $\geqslant 2$) but the results of Section 5 are considerably better.

So far, we have used the shift analogy to investigate the point spectrum of C_φ^*. The next theorem uses it to limit the size of the point spectrum of C_φ.

THEOREM 3.8. *Let φ be analytic in* **D** *with $\varphi(\mathbf{D}) \subset \mathbf{D}$, and suppose $\{z_k\}$ is a B-sequence for φ with $\liminf_{k\to\infty} \frac{1-|z_k|}{1-|z_{k-1}|} = r$. If there is f in H^2 with $f(z_0) \neq 0$ and $C_\varphi f = \lambda f$, then $|\lambda| \geqslant r^{1/2}$.*

Proof. Zero is an eingenvalue for C_φ only if φ is constant, so we may assume $r > 0$ and $|\lambda| > 0$. Since $f(z_{k-1}) = (C_\varphi f)(z_k) = \lambda f(z_k)$, we get $f(z_{m+k}) = \lambda^{-k} f(z_m)$ for all m, $k \geqslant 0$. In particular, $f(z_k) \neq 0$ for all k.

Since $v_k \to 0$ weakly in H^2, we must have $(1-|z_k|^2)^{1/2} f(z_k) \to 0$ as $k \to \infty$. Given $\tilde{r} < r$, choose M so that $m > M$ implies $\frac{1-|z_m|}{1-|z_{m-1}|} \geqslant \tilde{r}$. For $k \geqslant 0$, we have

$$(1-|z_{m+k}|^2)^{1/2}|f(z_{m+k})| =$$

$$= (1+|z_{m+k}|)^{1/2}(1-|z_m|)^{1/2} \prod_{j=1}^{k} \left[\frac{1-|z_{j+m}|}{1-|z_{j+m-1}|}\right]^{1/2} |\lambda|^{-k}|f(z_m)| \geqslant C\left(\frac{\tilde{r}^{1/2}}{\lambda}\right)^k$$

where C is a non-zero constant. Since the sequence on the left converges to zero, we must have $\tilde{r}^{1/2} < |\lambda|$. Since $\tilde{r}$ was an arbitrary number less than r, we find $|\lambda| \geqslant r^{1/2}$, as was to be proved. ▨

This results is best possible: let φ be the Möbius transformation $\varphi(z) = [(1+2i)z - 1][z - 1 + 2i]^{-1}$. Then $z_k = \{(1+2i/k)^{-1}\}_{k=1}^{\infty}$ is a B-sequence for φ and $\lim_{k\to\infty} \frac{1-|z_k|}{1-|z_{k-1}|} = 1$. Nordgren showed [21, page 447] that every λ with $|\lambda| = 1$ is an eigenvalue of infinite multiplicity.

4. THE SPECTRUM OF C_φ: NON-INNER FUNCTIONS

In this section, we combine information about $\sigma(C_\varphi)$ gained in the last section with information about eigenvalues of C_φ to get a more complete picture of the spectrum. The information about eigenvalues is based on a model for iteration developed in [8]. In some sense, that paper provides a complete solution to the eigenvalue equation $f \circ \varphi = \lambda f$ *where the solutions f are allowed to be arbitrary analytic functions*. Our difficulty is in determining which solutions are in H^2.

We may paraphrase the main results of [8] for our purposes as follows. If φ is analytic in **D** with $\varphi(\mathbf{D}) \subset \mathbf{D}$, there is a function σ analytic in **D** and a linear fractional transformation Φ such that $\Phi \circ \sigma = \sigma \circ \varphi$ and f, analytic in **D**, is a solution of $f \circ \varphi = \lambda f$ if and only if $f = F \circ \sigma$ where F is a solution of $F \circ \Phi = \lambda F$, [8, Theorem 3.2 and Lemma 4.1]. There are four essentially different cases depending on, among other things, the location of the Denjoy-Wolff point a and the value of $\varphi'(a)$. We obtain the most information when $|a| = 1$ and $\varphi'(a) < 1$, the least when $|a| = 1$ and F-sequences for φ are not interpolating sequences (which means $\varphi'(a) = 1$).

We begin with the case $|a| < 1$. The following result is a slight extension of Theorem 3 of [4, page 129] which dealt with power compact operators.

THEOREM 4.1. *Suppose φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ (φ non-constant and not a Möbius transformation mapping $\mathbf{D}$ onto $\mathbf{D}$), and suppose the Denjoy-Wolff point a is in $\mathbf{D}$. Then each of the numbers $\overline{\varphi'(a)^n}$, $n = 0,1,2, \ldots$, is an eigenvalue of C_φ^*. If $\varphi'(a) \neq 0$, then for some N, a positive integer or infinity, λ is an eigenvalue of C_φ if and only if $\lambda = \varphi'(a)^n$ where n is an integer, $0 \leqslant n < N$. If $\varphi'(a) = 0$, the only eigenvalue of C_φ is $\lambda = 1$. In either case, all eigenvalues of C_φ have multiplicity one.*

Proof. Let $\psi(z) = (z + a)(1 - \bar{a}z)^{-1}$. Then $C_\psi C_\varphi C_\psi^{-1} = C_{\psi^{-1}\circ\varphi\circ\psi}$, zero is the Denjoy-Wolff point of $\psi^{-1} \circ \varphi \circ \psi$ and $(\psi^{-1} \circ \varphi \circ \psi)'(0) = \varphi'(a)$. Since $\sigma(C_\varphi) = \sigma(C_{\psi^{-1}\circ\varphi\circ\psi})$, we may assume $a = 0$. When $\varphi(z) = \alpha z + \ldots$, we see that for each integer n, the subspace spanned by $1, z, \ldots, z^n$ is invariant for C_φ^*, and that with respect to this basis, the matrix for C_φ^* is upper triangular with diagonal entries $1, \bar{\alpha}, \ldots \bar{\alpha}^n$. Thus, for each n, $\bar{\alpha}^n$ is an eigenvalue of C_φ^*.

If $\varphi'(a) \neq 0$, Theorems 3.3 and 4.1 of [8, pages 81 and 88] show that there is σ analytic on $\mathbf{D}$ so that $\sigma \circ \varphi = \varphi'(a)\sigma$ and $f \circ \sigma = \lambda f$ if and only if $\lambda = \varphi'(a)^n$ for some $n = 0,1,2, \ldots$ and $f = c\sigma^n$ for some constant c. Since $\sigma^n \in H^2$ implies $\sigma^k \in H^2$ for $0 \leqslant k \leqslant n$, the assertions for this case follow.

If $\varphi'(a) = 0$ easy power series calculations show $f \circ \varphi = \lambda f$ implies $\lambda = 1$ and f is constant. ▨

The number N above can be estimated by using Theorem 3.8 if φ satisfies the hypotheses of Corollary 2.5, we can give a very precise estimate. Corollary 2.5 gives the essential spectral radius of C_φ, which we denote by ρ_e, and we know that if $\lambda \in \sigma(C_\varphi)$ with $|\lambda| > \rho_e$ then λ is an eigenvalue of C_φ. On the other hand, since ρ_e^N, under the conditions of Corollary 2.5, could be attained by a B-sequence for φ_N, Theorem 3.8 shows that in this case $|\lambda|^N < \rho_e^N$ implies λ^N is not an eigenvalue of C_φ^N so λ is not an eigenvalue of C_φ. Under the additional hypothesis that φ be analytic in a neighborhood of $\mathbf{D}$, Kamowitz computed the spectrum of C_φ [17, page 149].

If the Denjoy-Wolff point a of φ is in $\partial\mathbf{D}$ and $s = \varphi'(a) < 1$, then there is σ analytic in $\mathbf{D}$ with $\sigma(\mathbf{D}) \subset \mathbf{D}$ such that $\Phi \circ \sigma = \sigma \circ \varphi$ with

$$\Phi(z) = [(1 + s)z + (1 - s)]\,[(1 - s)z + (1 + s)]^{-1}$$

(Corollary, page 81 of [8]). The symbols Φ and σ will denote these maps for the next few paragraphs.

PROPOSITION 4.2. *If φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ and has Denjoy-Wolff point a in $\partial\mathbf{D}$ with $\varphi'(a) < 1$, then every F-sequence for φ is an interpolating sequence.*

In particular, each F-sequence for φ gives rise to an invariant subspace of C_φ^* on which it is similar to a weighted shift.

Proof. Let $\{z_k\}$ be an F-sequence (possibly doubly infinite). Since $\sigma(z_{k+n}) = \sigma(\varphi_n(z_k)) = \Phi_n(\sigma(z_k))$ for all integers k and all non-negative integers n, and since Φ is univalent, we see that $\sigma(z_k) = \Phi_k(\sigma(z_0))$ for all integers k. Now

$$\{\Phi_k(\sigma(z_0))\} = \{[(1+s^k)\sigma(z_0)+1-s^k][(1-s^k)\sigma(z_0)+(1+s^k)]^{-1}\}$$

which satisfies the growth condition of Hayman and Newman [16, page 203] both for $k \geqslant 0$ and $k < 0$. Since $\lim_{k\to\infty} \Phi_k(\sigma(z_0)) = +1$ and $\lim_{k\to-\infty} \Phi_k(\sigma(z_0)) = -1$, this means $\{\Phi_k(\sigma(z_0))\}$ is an interpolating set in $\mathbf{D}$.

Now suppose (α_k) is a bounded sequence. Since $\{\Phi_k(\sigma(z_0))\}$ is an interpolating set, there is a bounded analytic function F such that $F(\Phi_k(\sigma(z_0))) = \alpha_k$. But, since $\Phi_k(\sigma(z_0)) = \sigma(z_k)$, this means $F\circ\sigma$ is a bounded analytic function with $F\circ\sigma(z_k) = \alpha_k$. The sequence (α_k) was an arbitrary bounded sequence, so we conclude that $\{z_k\}$ is an interpolating set.

If $\{z_k\}$ is an F-sequence, let $v_k = \sqrt{1-|z_k|^2}K_{z_k}$ and let V be the closed subspace spanned by $\{v_k\}$. Lemma 3.2 asserts that $\{v_k\}$ is a basic sequence in V equivalent to an orthogonal basis for V, which means, since

$$C_\varphi^* v_k = \sqrt{1-|z_k|^2}K_{z_{k+1}} = \left(\frac{1-|z_k|^2}{1-|z_{k+1}|^2}\right)^{1/2} v_{k+1},$$

that $C_\varphi^*|V$ is similar to a shift with weights $\left\{\left(\frac{1-|z_k|^2}{1-|z_{k+1}|^2}\right)^{1/2}\right\}$. ▨

The following theorem carries the shift analogy somewhat further. It is interesting to note that the operator that implements the similarity is an analytic Toeplitz operator.

THEOREM 4.3. *If φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ and has Denjoy-Wolff point a in $\partial\mathbf{D}$ with $\varphi'(a) < 1$, then for θ real the operator C_φ is similar to the operator $e^{i\theta}C_\varphi$. In particular, $\lambda \in \sigma(C_\varphi)$ implies $e^{i\theta}\lambda \in \sigma(C_\varphi)$.*

Proof. Given θ real, let $\beta = \theta(\log \varphi'(a)^{-1})^{-1}$ and let $F(z) = \exp(i\beta \operatorname{Log}[(1+z)(1-z)^{-1}])$. Now $F(\Phi(z)) = e^{i\theta}F(z)$ and for all z in $\mathbf{D}$, $e^{-\beta\pi/2} \leqslant |F(z)| \leqslant e^{\beta\pi/2}$. Let $f = F\circ\sigma$ so that $f\circ\varphi = e^{i\theta}f$ and f and $1/f$ are in H^∞.

We see that

$$(T_f^{-1}C_\varphi T_f)(h) = (T_f)^{-1}((f\circ\varphi)(h\circ\varphi)) = e^{i\theta}T_f^{-1}(T_fC_\varphi h) = (e^{i\theta}C_\varphi)(h)$$

for all h in H^2 (where T_f is the operator of multiplication by f). ▨

COROLLARY 4.4. *If φ satisfies the hypotheses of Theorem 4.3, then each eigenvalue of C_φ has infinite multiplicity.*

Proof. The proof of Theorem 4.3 gives infinitely many linearly independent bounded analytic functions, f_k, with $f_k \circ \varphi = f_k$. If h is in $\ker(C_\varphi - \lambda)$, with $h \neq 0$, then $(C_\varphi - \lambda)f_k h = f_k(h \circ \varphi - \lambda h) = 0$ also, so $\ker(C_\varphi - \lambda)$ is infinite dimensional. ▨

We use the functions Φ and σ in a similar way to find an annulus of eigenvalues for C_φ.

THEOREM 4.5. *Suppose φ is analytic in* $\mathbf{D}$ *with* $\varphi(\mathbf{D}) \subset \mathbf{D}$ *and suppose φ has Denjoy-Wolff point a with $\varphi'(a) < 1$. If $\varphi'(a)^{1/2} < |\lambda| < \varphi'(a)^{-1/2}$, then λ is an eigenvalue of C_φ of infinite multiplicity.*

Proof. For $-1/2 < r < 1/2$, the function $g_0(z) = (1+z)^r(1-z)^{-r}$ is in H^2. For β real, the function

$$F(z) = \exp((r + i\beta)\ \mathrm{Log}[(1+z)(1-z)^{-1}])$$

is also in H^2 since $e^{-|\beta|\pi/2}|g_0(z)| \leqslant |F(z)| \leqslant e^{|\beta|\pi/2}|g_0(z)|$. Since σ is analytic in $\mathbf{D}$ and $\sigma(\mathbf{D}) \subset \mathbf{D}$, it also defines a bounded composition operator on H^2, so that $F \circ \sigma$ is in H^2.

It is easy to check that

$$C_\varphi(F \circ \sigma) = F \circ \sigma \circ \varphi = F \circ \Phi \circ \sigma = \exp(-(r + i\beta)\mathrm{Log}\varphi'(a))F \circ \sigma.$$

Now, for any λ with

$$\varphi'(a)^{1/2} < |\lambda| < \varphi'(a)^{-1/2},$$

there are infinitely many $r + i\beta$ with $\lambda = \exp(-(r + i\beta)\ \mathrm{Log}\ \varphi'(a))$, so λ is an eigenvalue of C_φ of infinite multiplicity. ▨

Under a somewhat stronger hypothesis, we can get a sharper result. Comparison with Theorem 3.8 suggests this is close to best possible: if there is a B-sequence converging *in a Stolz angle* to the appropriate fixed point of φ, then it is best possible. We note that if a is the Denjoy-Wolff point of φ and b is another fixed point, then by Theorem 3.1 of [9], $\varphi'(b)^{-1} < \varphi'(a)$ so the annulus of eigenvalues in Theorem 4.6 is larger than that of Theorem 4.5.

THEOREM 4.6. *Suppose φ is analytic in* $\mathbf{D}$ *with* $\varphi(\mathbf{D}) \subset \mathbf{D}$ *and φ' is continuous on* $\bar{\mathbf{D}}$.

Suppose also that $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\} = \{a, b_1, b_2, \ldots, b_k\}$, where a is the Denjoy-Wolff point of φ, the points $b_1, \ldots, b_k$ are other fixed points of φ, and $\varphi'(a) < 1$. If

$$\max\{\varphi'(b_j)^{-1/2} : j = 1, 2, \ldots, k\} < |\lambda| < \varphi'(a)^{-1/2},$$

then λ is an eigenvalue of C_φ of infinite multiplicity.

Proof. Let $r_0 = \min\{\varphi'(b_j) : j = 1,2, \ldots, k\}$, let $s = \varphi'(a)$, and let Φ and σ be as above. Examining the proofs of Theorems 4.3 and 4.4 reveals that for each μ with $|\mu| = 1$, there are infinitely many bounded analytic functions f_μ with $f_\mu \circ \varphi = \mu f_\mu$. From this and Corollary 4.4, it is sufficient to show that each positive number λ with $r_0^{-1/2} < \lambda < 1$ is an eigenvalue.

For such a λ let $x = \log\lambda/\log s$ so that $\lambda = s^x$. Now

$$\left(\frac{1 - \sigma(\varphi(z))}{1 + \sigma(\varphi(z))}\right)^x = \left(\frac{1 - \Phi(\sigma(z))}{1 + \Phi(\sigma(z))}\right)^x = s^x\left(\frac{1 - \sigma(z)}{1 + \sigma(z)}\right)^x$$

so we need only to show that $\left(\frac{1 - \sigma(z)}{1 + \sigma(z)}\right)^x$ is in H^2. In fact, we will show

$$\left|\frac{1 - \sigma(z)}{1 + \sigma(z)}\right|^x \leqslant M \prod_{j=1}^{k} |b_j - z|^{-p}$$

where $p < 1/2$, and the latter function is in H^2. (Theorem 3.8 of [8, page 87], extended to cover several fixed points, shows $\sigma(z) \to -1$ as $z \to b_j$.) Since $\sigma(z) \to 1$ as $z \to a$ and $|\sigma(e^{i\theta})| < 1$ if $|\varphi(e^{i\theta})| < 1$, it suffices to consider the growth of $|1 + \sigma(z)|^{-x}$ as z approaches a single fixed point $b \in \{b_1, \ldots, b_k\}$.

Choose r so that $r_0^{-1/2} < r^{-1/2} < \lambda = s^x$ so that $x(\log s^{-1})/\log r < 1/2$. Choose δ so that if $|b - z| < \delta$ then $|\varphi'(b) - \varphi'(z)| < r_0 - r$, and so that if $0 < |b - e^{i\theta}| \leqslant \delta$ then $|\varphi(e^{i\theta})| < 1$. Let $K = \{\zeta : |b - \zeta| \geqslant \delta$ and $\zeta = \varphi(w)$ for some w in $\bar{\mathbf{D}}$ with $|w - b| \leqslant \delta\}$. Now K is a compact subset of $\mathbf{D}$ with the property that if z is in $\mathbf{D}$ and $|b - z| < \delta$ then either $|b - \varphi(z)| < \delta$ or $\varphi(z) \in K$, that is $\varphi_n(z) \in K$ for some $n = 1,2, \ldots$. Let $\delta' = \sup\{|b - \zeta| : \zeta \in K\}$.

If n is a positive integer so that $\varphi_n(z)$ is in K, then $\sigma(\varphi_n(z)) = \Phi_n(\sigma(z))$ and, since Φ is univalent, $\sigma(z) = \Phi_{-n}(\sigma(\varphi_n(z))) \in \Phi_{-n}(\sigma(K))$. Since $\sigma(K)$ is a compact subset of $\mathbf{D}$, this means $|1 + \sigma(z)|^{-1} \leqslant M_1 s^{-n}$ where M_1 is some constant. We will estimate the integer n for which $\varphi_n(z)$ is in K.

If $|b - z| < \delta$, then

$$|b - \varphi(z)| = \left|\int_0^1 \varphi'(tb + (1 - t)z)\,(b - z)\,\mathrm{d}t\right| =$$

$$= |b - z|\left|\varphi'(b) - \int_0^1 \varphi'(b) - \varphi'(tb + (1 - t)z)\,\mathrm{d}t\right| \geqslant$$

$$\geqslant |b - z|(\varphi'(b) - (r_0 - r)) \geqslant r|b - z|.$$

Similarly, if $|b - \varphi(z)| < \delta$ then

$$|b - \varphi(\varphi(z))| \geqslant r|b - \varphi(z)| \geqslant r^2|b - z|.$$

Thus, if n is the least integer so that $|b - \varphi_n(z)| \geqslant \delta$, which is the least integer with $\varphi_n(z)$ in K, we have $\delta' \geqslant |b - \varphi_n(z)| \geqslant r^n|b - z|$. Rewriting this, we find $n \leqslant \log(\delta'|b - z|^{-1})/\log r$.

Finally

$$\left|\frac{1 - \sigma(z)}{1 + \sigma(z)}\right|^x \leqslant 2^x|1 + \sigma(z)|^{-x} \leqslant M_2 s^{-nx} = M_2 \exp(nx \log s^{-1}) \leqslant$$

$$\leqslant M_2 \exp(x \log s^{-1} \log(\delta'|b - z|^{-1})/\log r) =$$

$$= M_3|b - z|^{-x \log s^{-1}/\log r}$$

where M_2 and M_3 are constants. Since $x \log s^{-1}/\log r < 1/2$ and b was an arbitrary element of $\{b_1, \ldots, b_k\}$, this completes the proof. ▨

We may extend this result somewhat. If φ has Denjoy-Wolff point a in $\partial\mathbf{D}$ with $\varphi'(a) < 1$, for some $\delta > 0$, we have φ' continuous on $\{z: |z - a| < \delta$ and $|z| \leqslant 1\}$ and the set $\{\varphi(z): z \in \mathbf{D}$ and $|z - a| \geqslant \delta\}$ is contained in a compact subset of $\mathbf{D}$ then λ is an eigenvalue of infinite multiplicity whenever $0 < |\lambda| < \varphi'(a)^{-1/2}$. This follows because in this case $\left(\frac{1 - \sigma(z)}{1 + \sigma(z)}\right)^x$ is bounded for all $x > 0$.

Combining Theorem 4.6 and Corollary 3.6 we are able to determine the spectrum of a class of composition operators.

THEOREM 4.7. *Suppose φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ and has Denjoy-Wolff point a in $\partial\mathbf{D}$ with $\varphi'(a) < 1$. If there is some positive integer N so that φ_N' is continuous on $\overline{\mathbf{D}}$, the set $\{e^{i\theta}: |\varphi_N(e^{i\theta})| = 1\} = \{a, b_1, \ldots, b_k\}$ where $b_1, b_2, \ldots, b_k$ are fixed points of φ_N and φ_N is analytic in a neighborhood of each b_j then $\sigma(C_\varphi) = \{\lambda : |\lambda| \leqslant \varphi'(a)^{-1/2}\}$.*

Proof. The function φ_N satisfies the hypotheses of both Corollary 3.5 and Theorem 4.6. Thus $\sigma(C_{\varphi_N})$ intersects the circle of radius ρ centered at the origin for

$$0 < \rho < \max \varphi_N'(b_j)^{-1/2} \quad \text{and} \quad \max \varphi_N'(b_j)^{-1/2} < \rho < \varphi_N'(a)^{-1/2} = \varphi'(a)^{-N/2}.$$

Since $C_{\varphi_N} = C_\varphi^N$, the spectral mapping theorem implies $\sigma(C_\varphi)$ intersects every circle of radius ρ centered at the origin with $0 < \rho < \varphi'(a)^{-1/2}$. But Theorem 4.3 implies the spectrum is radially symmetric, so $\sigma(C_\varphi) = \{\lambda : |\lambda| \leqslant \varphi'(a)^{-1/2}\}$. ▨

COROLLARY 4.8. *Suppose φ, not a finite Blaschke product, is analytic in a neighborhood of $\overline{\mathbf{D}}$ and has $\varphi(\mathbf{D}) \subset \mathbf{D}$. If the Denjoy-Wolff point a of φ is in $\partial\mathbf{D}$ and $\varphi'(a) < 1$, then*

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leqslant \varphi'(a)^{-1/2}\}.$$

Proof. Since φ is analytic in a neighborhood of $\bar{\mathbf{D}}$ and not a finite Blaschke product, $\{e^{i\theta}: |\varphi(e^{i\theta})| = 1\}$ is finite. For some N, it follows that $\{e^{i\theta}: |\varphi_N(e^{i\theta})| = 1\}$ consists entirely of fixed points of φ_N, so Theorem 4.7 applies. ▨

In the next section, we will determine the spectrum of C_φ for φ an inner function and we will find the hypothesis "φ not a finite Blaschke product" can be replaced by "φ not a Möbius transformation of $\mathbf{D}$ onto $\mathbf{D}$." In the latter case, the spectrum was determined by Nordgren [21, page 448], so $\sigma(C_\varphi)$ is known for all φ analytic in the closed disk with $|a| = 1$ and $\varphi'(a) < 1$.

When $\varphi'(a) = 1$, the situation is much more difficult: indeed, there are two distinct cases that can best be distinguished by the kind of intertwinning relation φ satisfies. In one case (case 4 of [8, page 80]), there is σ analytic in $\mathbf{D}$ with $\sigma(\mathbf{D}) \subset \mathbf{D}$ so that $\Phi \circ \sigma = \sigma \circ \varphi$ where $\Phi(z) = [(1 \pm 2i)z - 1][z - 1 \pm 2i]^{-1}$ and in the other case (case 2 of [8, page 80]), there is σ analytic in $\mathbf{D}$ so that $\sigma \circ \varphi = \sigma + 1$ and $\bigcup_{n=-\infty}^{\infty} (\sigma(\mathbf{D}) + n) = \mathbf{C}$. In general the best we can say about $\sigma(C_\varphi)$ is the result of Theorem 2.1, that $\sigma(C_\varphi) \subset \bar{\mathbf{D}}$. In Section 6, we will give examples for which the inclusion is proper. The following proposition provides a (not necessarily practical) procedure for distinguishing the cases. In case 4 of [8, page 80], we can say more about $\sigma(C_\varphi)$.

PROPOSITION 4.9. *Let φ, analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$, have Denjoy-Wolff point a in $\partial\mathbf{D}$ with $\varphi'(a) = 1$. Then the following are equivalent.*

(i) *φ is in case 4 of* [8, page 80].

(ii) *every F-sequence for φ is an interpolating sequence.*

(iii) $\inf\left\{\left|\dfrac{z_k - z_{k+1}}{1 - z_k\bar{z}_{k+1}}\right| : k = 0, 1, 2, \ldots\right\} > 0$ *for some F-sequence of φ.*

Proof. (i) $\Rightarrow$ (ii). If φ is in case 4 then $\Phi \circ \sigma = \sigma \circ \varphi$ where and Φ and σ are as above. An easy computation gives $\Phi_n(z) = \left[\left(1 \pm \frac{2}{n}i\right)z - 1\right]\left[z - 1 \pm \frac{2}{n}i\right]^{-1}$.
We claim $\{\Phi_n(0)\}_{-\infty}^{\infty} = \left\{\left(1 - \frac{2}{n}i\right)^{-1}\right\}_{-\infty}^{\infty}$ is an interpolating sequence. We verify the Carleson conditon: $\{w_k\}$ is interpolating if and only if $\prod_{k \neq j}\left|\dfrac{w_k - w_j}{1 - \bar{w}_k w_j}\right| \geq \delta > 0$ for each k, [16, page 196]. Now

$$\left|\frac{\left(1 - \frac{2}{k}i\right)^{-1} - \left(1 - \frac{2}{j}i\right)^{-1}}{1 - \left(1 - \frac{2}{k}i\right)^{-1}\left(1 + \frac{2}{j}i\right)^{-1}}\right|^2 = \frac{(k-j)^2}{4 + (k-j)^2} = 1 - \frac{4}{4 + (k-j)^2}.$$

Thus for each k, the product is

$$\left(\prod_{k\neq j}\left(1-\frac{4}{4+(k-j)^2}\right)\right)^{1/2}=\left(\prod_{n\neq 0}\left(1-\frac{4}{4+n^2}\right)\right)^{1/2}$$

which is positive since $\sum_{n=0}\frac{4}{4+n^2}<\infty$.

For any w_0 in $\mathbf{D}$, $\{\Phi_n(w_0)\}_{-\infty}^{\infty}$ is the image under a Möbius transformation of $\{\Phi_n(0)\}_{-\infty}^{\infty}$, so it also is an interpolating sequence. We conclude that any F-sequence of φ is interpolating by the same reasoning as in Proposition 4.2.

(ii) $\Rightarrow$ (iii) This is an easy consequence of the Carleson condition since $\inf\left\{\left|\frac{z_k-z_{k+1}}{1-z_k\bar{z}_{k+1}}\right|\right\}\geqslant\inf\left\{\prod_{k\neq j}\left|\frac{z_k-z_j}{1-z_k\bar{z}_j}\right|\right\}$.

(iii) $\Rightarrow$ (i) Either φ in is case 4 or case 2, so we will show that (iii) cannot happen when φ is in case 2. Suppose φ is in case 2 with σ, $\Omega=\mathbf{C}$, and V as in [8, page 80] that is σ is analytic in $\mathbf{D}$ with $\sigma\circ\varphi=\sigma+1$. Let $\{z_k\}$ be an F-sequence for φ and $\delta>0$. Since $\sigma(V)$ is a fundamental set for $\Phi(w)=w+1$ on $\mathbf{C}$ (see [8, page 70]), we can find n large enough that $\sigma(\mathbf{D})$ includes a schlicht disk with center $\sigma(z_n)$ and radius $2/\delta$. Now define $f\colon\mathbf{D}\to\mathbf{D}$ by $(w)=\sigma^{-1}(2\delta^{-1}w+\sigma(z_n))$. Since $\sigma(z_{n+1})=\sigma(\varphi(z_n))=\sigma(z_n)+1$, we have, by Pick's inequality, $\left|\frac{z_n-z_{n+1}}{1-z_n\bar{z}_{n+1}}\right|=\left|\frac{f(0)-f(\delta/2)}{1-\overline{f(0)}f(\delta/2)}\right|\leqslant\delta/2<\delta$.

But δ was arbitrary, so $\inf\left|\frac{z_n-z_{n+1}}{1-z_n\bar{z}_{n+1}}\right|=0$ which contradicts (iii). ▨

THEOREM 4.10. *Let φ, analytic in $\mathbf{D}$ with $\varphi(\mathbf{D})\subset\mathbf{D}$, have Denjoy-Wolff point a in $\partial\mathbf{D}$ with $\varphi'(a)=1$. If there is an F-sequence $\{z_k\}$ for φ for which* $\inf\left\{\left|\frac{z_k-z_{k+1}}{1-z_k\bar{z}_{k+1}}\right|: k=0,1,2,\dots\right\}>0$, *then each λ with $|\lambda|=1$ is an eigenvalue of C_φ of infinite multiplicity.*

Proof. By Proposition 4.9, φ is in case 2 of [8, page 80]. Suppose σ is analytic in $\mathbf{D}$ with $\sigma(\mathbf{D})\subset\mathbf{D}$ and $\Phi\circ\sigma=\sigma\circ\varphi$ where (say) $\Phi(z)=[(1+2i)z-1]\times\times[z-1+2i]^{-1}$ (the case with $-$ is similar). Let $f(z)=\exp(-\theta(\sigma(z)+1)(\sigma(z)-1)^{-1})$. One can easily verify that $|f(z)|\leqslant 1$ for all z in $\mathbf{D}$, so f is in H^2 and, using the commutation relation $\Phi\circ\sigma=\sigma\circ\varphi$, we find $f(\varphi(z))=e^{i\theta}f(z)$, so f is an eigenvector with eigenvalue $e^{i\theta}$. ▨

Using the properties of V from Theorem 3.2 of [8, page 78], it is not difficult to show that the eigenfunction f constructed above is never bounded below. We are therefore unable to get similarity results by the technique used in the proof of Theorem 4.3, although some results along these lines carry over. For example, if λ is an eigenvalue of C_φ, then $e^{i\theta}\lambda$ is an eigenvalue of C_φ of infinite multiplicity: multiply the eigenfunction by the appropriate bounded eigenfunctions found above.

5. THE SPECTRUM OF C_φ: INNER FUNCTIONS

In [21], Nordgren showed that if φ is an inner function and has Denjoy-Wolff point in $\mathbf{D}$, then C_φ is similar to an isometry and he found the Wold decomposition for the isometry. In particular, in this case, $\sigma(C_\varphi) = \bar{\mathbf{D}}$ when φ is not a Möbius transformation, and $\sigma(C_\varphi) = \overline{\{\varphi'(a)^n\}}$ when φ is a Möbius transformation. When φ is a Möbius transformation with Denjoy-Wolff point a in $\partial\mathbf{D}$, he showed $\sigma(C_\varphi) = \{\lambda : \varphi'(a)^{1/2} \leqslant |\lambda| \leqslant \varphi'(a)^{-1/2}\}$. On the other hand, Nordgren did not treat the case φ a non-Möbius inner function with $|a| = 1$. In this section, we will consider this case.

THEOREM 5.1. *Let φ be an inner function, not a Möbius transformation, with Denjoy-Wolff point a in $\partial\mathbf{D}$. If $|\lambda| < \varphi'(a)^{1/2}$, then λ is an eigenvalue of C_φ^* of infinite multiplicity.*

Proof. We will show that if $|\lambda| = r < \varphi'(a)^{1/2}$, then $C_\varphi - \bar{\lambda}$ is left invertible but not Fredholm. Thus $C_\varphi^* - \lambda$ is right invertible, but not Fredholm, which means λ is an eigenvalue of C_φ^* of infinite multiplicity.

In the proof of Theorem 2.1, we saw

$$\varphi'(a)^{1/2} = \lim_{n\to\infty} \left(\frac{1 - |\varphi_n(0)|}{1 + |\varphi_n(0)|}\right)^{1/2n}.$$

Since $r < \varphi'(a)^{1/2}$, we may choose n large enough that $r < \left(\frac{1-|\varphi_n(0)|}{1+|\varphi_n(0)|}\right)^{1/2n}$, that is, that $r^n < \left(\frac{1 - |\varphi_n(0)|}{1 + |\varphi_n(0)|}\right)^{1/2}$.

Now φ_n is an inner function and for all f in H^2,

$$\left(\frac{1 - |\varphi_n(0)|}{1 + |\varphi_n(0)|}\right)^{1/2} \|f\| \leqslant \|C_{\varphi_n} f\| = \|C_\varphi^n f\|$$

(see [21, pages 443, 444]). Let P be the orthogonal projection onto $R = \operatorname{range} C_\varphi^n$ and let A be the inverse of C_φ^n on R, that is, $AC_\varphi^n = I$ and $C_\varphi^n A = I_R$. The above estimate shows $\|AP\| \leqslant \|A\| \leqslant \left(\frac{1 + |\varphi_n(0)|}{1 - |\varphi_n(0)|}\right)^{1/2}$ so the series $L_\theta = \sum_{k=0}^{\infty} r^{nk} e^{ink\theta}(AP)^{k+1}$ converges absolutely for each real θ.

It is easily checked that L_θ is a left inverse of $C_\varphi^n - r^n e^{in\theta}$. On the other hand, if v is in $\ker P$, which is infinite dimensional since C_φ is left invertible but not Fredholm, then $L_\theta v = 0$, so $C_\varphi^n - r^n e^{in\theta}$ is not Fredholm. Thus $\sigma_e(C_\varphi^n)$ includes the circle $|\mu| = r^n$.

Since

$$\prod_{k=1}^{n} (C_\varphi - re^{i(\theta+2\pi kn^{-1})}) = C_\varphi^n - r^n e^{in\theta},$$

the operator $\widetilde{L}_\theta = L_\theta \prod_{k=1}^{n-1} (C_\varphi - re^{i(\theta+2\pi k n^{-1})})$ is a left inverse for $C_\varphi - re^{i\theta}$. Since $\sigma_e(C_\varphi^n)$ includes the circle of radius r^n, the spectral mapping theorem implies $\sigma_e(C_\varphi)$ intersects the circle of radius r. If $\sigma_e(C_\varphi)$ did not include this circle, there would be λ_0 on the boundary of $\sigma_e(C_\varphi)$ with $|\lambda_0| = r$. This would mean, as λ approaches λ_0 with $|\lambda| = r$, that $\|(C_\varphi - \lambda)^{-1}\|_e \to \infty$. But $\|\widetilde{L}_\theta\| \leqslant \|A\|(1 - r^n\|A\|)^{-1}(\|C_\varphi\| + r)^{n-1}$ for all real θ. Thus, the boundary of $\sigma_e(C_\varphi)$ does not intersect the circle $|\lambda| = r$, and $\sigma_e(C_\varphi)$ includes it. We conclude that $C_\varphi - re^{i\theta}$ is left invertible but not Fredholm for $0 \leqslant \theta \leqslant 2\pi$ as was to be proved. ▨

COROLLARY 5.2. *Let φ be an inner function, not a Möbius transformation, with Denjoy-Wolff point a in $\partial\mathbf{D}$. Then $\sigma(C_\varphi) = \sigma_e(C_\varphi) = \{\lambda : |\lambda| \leqslant \varphi'(a)^{-1/2}\}$.*

Proof. We have shown $\{\lambda : |\lambda| < \varphi'(a)^{1/2}\}$ is a set of eigenvalues of infinite multiplicity for C_φ^* and (Theorem 4.5) $\{\lambda : \varphi'(a)^{1/2} < |\lambda| < \varphi'(a)^{-1/2}\}$ is a set of eigenvalues of infinite multiplicity of C_φ (when $\varphi'(a) < 1$). Since (Theorem 2.1) $\sigma(C_\varphi) \subset \{\lambda : |\lambda| \leqslant \varphi'(a)^{-1/2}\}$ the result follows. ▨

We have found enough about the sets of eigenvalues of C_φ and C_φ^* to find $\sigma(C_\varphi)$, but we have not found these sets exactly. In some cases, we can say more. For example, if $\varphi'(a) = 1$ and φ is in case 4 of [8, page 80] (as all parabolic Möbius transformations mapping $\mathbf{D}$ onto $\mathbf{D}$ are) then the point spectrum of C_φ includes the unit circle (Theorem 4.10). If $\varphi'(a) = 1$ and φ is in case 2 of [8, page 80], for example $\varphi(z) = \left(\frac{z + 1/3}{1 + 1/3z}\right)^2$, then we have no information about the eigenvalues of C_φ. It seems likely, in this case, that the constant functions are the only eigenfunctions.

If φ is a finite Blaschke product, $|a| = 1$ and $\varphi'(a) < 1$, then λ is not an eigenvalue of C_φ if $|\lambda| < \varphi'(a)^{1/2}$. Let Φ and σ be as in (case 3) of Theorem 3.2 of [8, page 78 and 80]. Theorem 3.7 of [8, page 86] states that σ is an inner function, and Lemma 4.1 of [8, page 88] implies that $f \circ \varphi = \lambda f$ if and only if $F \circ \Phi = \lambda F$ where $f = F \circ \sigma$. Since Φ is a Möbius transformation with $\Phi(1) = 1$ and $\Phi'(1) = \varphi'(a)$, we have

$$\sigma(C_\Phi) = \{\lambda : \varphi'(a)^{1/2} \leqslant |\lambda| \leqslant \varphi'(a)^{-1/2}\},$$

so λ is not an eigenvalue of C_Φ. But, Theorem 3.2 of [28, page 262] says that, since σ is an inner function, $f \in H^2$ if and only if $F \in H^2$ so λ is not an eigenvalue of C_Φ either.

6. SOME EXAMPLES

In this section we look at some examples that illustrate the richness of $\sigma(C_\varphi)$ when $\varphi'(a) = 1$. In particular, we will answer Kamowitz' question "If $|a| = 1$ and $\varphi'(a) = 1$, does $\sigma(C_\varphi) = \overline{\mathbf{D}}$?" by giving an example for which the spectrum is the interval [0,1]. All of these examples belong to holomorphic semigroups, and this fact is the basis for computing the spectrum.

Given θ with $0 < \theta \leqslant \pi$, let G be the domain $G = \{\zeta : |\arg\zeta| < \theta\}$, and let σ be the map $\sigma(z) = \left(\frac{1+z}{1-z}\right)^{2\theta/\pi}$ which is a conformal map of $\mathbf{D}$ onto G. Let $\tau(G) = G$ if $\theta \leqslant \pi/2$ and $\tau(G) = \{t: |\arg t| < \pi - \theta\}$ if $\theta > \pi/2$ so that, in either case, $\zeta + t \in G$ whenever $\zeta \in G$ and $t \in \tau(G)$. Now, for $t \in \tau(G)$, define φ_t by $\varphi_t(z) = \sigma^{-1}(\sigma(z) + t)$ so that φ_t is analytic in $\mathbf{D}$ with $\varphi_t(\mathbf{D}) \subset \mathbf{D}$. The function φ_t has Denjoy-Wolff point 1 with $\varphi_t'(1) = 1$. We will write C_t for the composition operator C_{φ_t}. In the case $\theta = \pi/2$ (so G is a half plane) $\varphi_t(z) = [t+(2-t)z]\times \times[(2+t) - tz]^{-1}$ for $\operatorname{Re} t > 0$. In the case $\theta = \pi/4$, we find

$$\varphi_t(z) = [t^2 + 2t\sqrt{1-z^2} + (2-t^2)z][(2+t^2) + 2t\sqrt{1-z^2} - t^2 z]^{-1}$$

for $|\arg t| < \pi/4$.

The following result will sometimes allow us to identify the spectrum of C_t. The idea for its proof was suggested to the author by R. P. Kaufman.

THEOREM 6.1. *Let θ, G, $\tau(G)$, and C_t be as above. Then* $\sigma(C_t) \subset \{e^{-\beta t}: |\arg\beta| \leqslant \leqslant |\pi/2 - \theta|\} \cup \{0\}$ *for all t in $\tau(G)$.*

Proof. The set $\{C_t: t \in \tau(G)\}$ is a holomorphic semigroup of operators. Indeed, by Theorem 3.10.1 of [15, page 93], it is sufficient to check that $t \to \langle C_t(f), K_\alpha\rangle$ is holomorphic in $\tau(G)$ for each f in H^2 and each α in $\mathbf{D}$. But, $\langle C_t(f), K_\alpha\rangle = = f(\sigma^{-1}(\sigma(\alpha) + t))$ which is holomorphic in t because σ^{-1} and f are holomorphic. In particular, $t \to C_t$ is continuous and holomorphic in the norm topology for t in $\tau(G)$.

Let $\mathfrak{A}$ be the norm closed algebra of operators generated by $\{I\} \cup \cup \{C_t : t \in \tau(G)\}$. Thus $\mathfrak{A}$ is a commutative Banach algebra with identity and the Gelfand theory applies: the spectrum of C_t as an element of $\mathfrak{A}$, denoted $\sigma_{\mathfrak{A}}(C_t)$, is the set $\{\Lambda(C_t): \Lambda$ is a multiplicative linear functional on $\mathfrak{A}\}$.

For Λ a multiplicative linear functional on $\mathfrak{A}$, let $\lambda(t) = \Lambda(C_t)$ for $t \in \tau(G)$. Since Λ is norm continuous ($\|\Lambda\| = 1$) and C_t is a norm-holomorphic semigroup, we see that $\lambda(t)$ is a holomorphic function on $\tau(G)$ such that

$$\lambda(t_1 + t_2) = \Lambda(C_{t_1+t_2}) = \Lambda(C_{t_1})\,\Lambda(C_{t_2}) = \lambda(t_1)\lambda(t_2).$$

But this means, either $\lambda(t) \equiv 0$ or $\lambda(t) = e^{-\beta t}$ for some complex number β. In addition, we see that for every t in $\tau(G)$, we have

$$|e^{-\beta t}| = \lim_{n\to\infty} |e^{-\beta nt}|^{1/n} = \lim_{n\to\infty} |\Lambda(C_t^n)|^{1/n} \leqslant$$

$$\leqslant \lim_{n\to\infty} \|C_t^n\|^{1/n} = \varphi_t'(1)^{-1/2} = 1.$$

(We used Theorem 2.1 and the fact $\|\Lambda\| = 1$). The definition of $\tau(G)$ and this inequality imply $|\arg\beta| \leqslant |\pi/2 - \theta|$.

Putting all our information together, we have

$$\sigma(C_t) \subset \sigma_{\mathfrak{A}}(C_t) = \{\Lambda(C_t) \colon \Lambda \text{ is a multiplicative linear functional on } \mathfrak{A}\} \subset$$
$$\subset \{e^{-\beta t} : |\arg\beta| \leqslant |\pi/2 - \theta|\} \cup \{0\}$$

as was to be proved. ▨

COROLLARY 6.2. *Suppose* $0 < \theta \leqslant \pi/2$ *and* $G, \tau(G)$, *and* C_t *are as above. Then*

$$\sigma(C_t) = \{e^{-\beta t} : |\arg\beta| \leqslant \pi/2 - \theta\} \cup \{0\}$$

for all t *in* $\tau(G)$.

Proof. When $|\arg\beta| \leqslant \pi/2 - \theta$, the real part of $\beta\sigma(z)$ is positive, so $f(z) = \exp(-\beta\sigma(z))$ is in H^∞. Since $f(\varphi_t(z)) = \exp(-\beta\sigma(z) - \beta t) = e^{-\beta t} f(z)$, we see that $e^{-\beta t}$ is an eigenvalue of C_t, so $\sigma(C_t) \supset \{e^{-\beta t} \colon |\arg\beta| \leqslant \pi/2 - \theta\}$. Theorem 6.1 and the fact that $\sigma(C_t)$ is closed yield the conclusion. ▨

When $\theta = \pi/2$, each of these sets is a logarithmic spiral from 1 to 0. In particular, we find $\sigma(C_\varphi) = [0,1]$ for $\varphi(z) = (2 - z)^{-1}$ (this is the case $t = 2$).

When $\theta = \pi/4$ and $t = 1$, that is,

$$\varphi(z) = (1 + z + 2\sqrt{1 - z^2})(3 - z + 2\sqrt{1 - z^2})^{-1},$$

we find $\sigma(C_\varphi)$ is the heart shaped region $\{e^{-\beta} : |\arg\beta| \leqslant \pi/4\} \cup \{0\}$.

When $\pi/2 < \theta \leqslant \pi$, Theorem 6.1 gives a hint as to what the spectrum might be, but there are no obvious eigenvalues, so we are unable to go further.

The techniques in this section depend heavily on the special semi-group properties of the operators. While these are indeed quite rare, the situation might not be completely hopeless: Section 5 of [8] shows that if φ is analytic in $\mathbf{D}$ with $\varphi(\mathbf{D}) \subset \mathbf{D}$ and $\varphi'(a) \neq 0$, then for each z in $\mathbf{D}$, there is a positive number t_z so that $\varphi_t(z)$ is defined for all $t > t_z$ and the dependence is holomorphic in z, real analytic in t in an appropriate open set. It remains to be seen whether these partial semi-groups can help in finding $\sigma(C_\varphi)$.

7. QUESTIONS AND CONJECTURES

The results (and omissions) of this paper suggest several questions for further study. Among the non-compact composition operators, we have the best understanding of C_φ when φ has Denjoy-Wolff point a in $\partial\mathbf{D}$ and $\varphi'(a) < 1$. The weighted shift analogy is strongest in this case: indeed for every α in $\mathbf{D}$, the kernel K_α belongs to an invariant subspace on which C_φ^* acts like a weighted shift. Since the K_α are a determining set for H^2, C_φ^* is apparently "put together" only from weighted shifts. Can this analogy be made precise? Is C_φ^* similar to a direct sum (or integral) of weighted shifts?

Much remains mysterious about C_φ when the Denjoy-Wolff point of φ is in $\mathbf{D}$, but the following conjecture seems reasonable.

CONJECTURE 1. *If* $\varphi(z) = bz + \ldots$ *is analytic in* $\mathbf{D}$ *with* $\varphi(\mathbf{D}) \subset \mathbf{D}$ *(and* $|b| < 1$*) then there is a number* B *so that* $C_\varphi - \lambda$ *is left invertible for* $|\lambda| < B$ *and* $\sigma_e(C_\varphi) = \{\lambda : |\lambda| \leqslant B\}$. *In particular, if* $\{e^{i\theta} : \varphi(e^{i\theta}) = 1\}$ *is a set of fixed points of* φ, *then* $B = \sup\{\varphi'(e^{i\theta})^{-1/2} : \varphi(e^{i\theta}) = e^{i\theta}\}$.

It might even be the case that C_φ is essentially similar to μC_φ for $|\mu| = 1$.

The situation in case the Denjoy-Wolff point a has $|a| = 1$ and $\varphi'(a) = 1$ is even more intractable, partly because it disguises two cases. Nevertheless, the following conjectures seem plausible (although evidence for the last is small).

CONJECTURE 2. *If* φ *satisfies the hypotheses of Theorem* 4.10, *then* $\sigma(C_\varphi) = \sigma_e(C_\varphi) = \bar{\mathbf{D}}$ *and the point spectrum of* C_φ *is an annulus or the unit circle.*

(In fact, there is some reason to believe that either the point spectrum of C_φ is $\partial\mathbf{D}$ or it is $\bar{\mathbf{D}}\diagdown\{0\}$ and that the latter case is impossible if φ has a B-sequence.)

CONJECTURE 3. *If* φ *is an inner function and is in case* 2 *of* [8, page 80] *then the only eigenfunctions of* C_φ *are the constants.*

CONJECTURE 4. *If* φ *is analytic in* $\mathbf{D}$ *with* $\varphi(\mathbf{D}) \subset \mathbf{D}$ *and is in case* 2 *of* [8, page 80], *then for some* θ_1, θ_2 *with* $-\pi/2 \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi/2$ *we have*

$$\sigma(C_\varphi) = \{e^{-\beta} : \theta_1 \leqslant \arg\beta \leqslant \theta_2\} \cup \{0\}.$$

The most glaring omission in this paper is illustrated by the gap in Theorem 4.7 and Corollary 5.2: the gap between assuming $|\varphi(e^{i\theta})| < 1$ a.e. and assuming $|\varphi(e^{i\theta})| = 1$ a.e.. Many of our results indicate that the structure of C_φ depends principally on the boundary behaviour of φ, but we have not treated the case when both $\{e^{i\theta} : |\varphi(e^{i\theta})| < 1\}$ and $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$ are large.

One would expect that assuming φ to be univalent would make all questions concerning C_φ easier to answer. Considering C_φ for univalent φ would be especially helpful if some non-trivial relations could be found between C_φ and $C_\psi C_\varphi$ when φ is univalent.

As we can see, there is much to be learned about this rich and interesting class of operators.

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