J. OPERATOR THEORY
9(1983), 107–142

A BEURLING-LAX THEOREM FOR THE LIE GROUP $U(m, n)$ WHICH CONTAINS MOST CLASSICAL INTERPOLATION THEORY

JOSEPH A. BALL and J. WILLIAM HELTON

INTRODUCTION

In this article we generalize the famous theorem of Beurling, Lax, and Halmos from the Hilbert space $H^2(\mathbf{C}^n)$ to a space with a signed Hermitian form. Our proof is an adaptation of Halmos' wandering subspace proof of the theorem [14] and of McEnnis' analysis of shifts on a space with an indefinite metric [23]. Our Beurling-Lax theorem for the Lie group $U(m, n)$ (as opposed to the classical one where $U(n) = U(n, 0)$) has very strong consequences for Nevanlinna-Pick, Carathéodory-Fejér, etc. interpolation theory. We obtain directly from our theory a simple linear fractional parameterization of all solutions in $\mathscr{B}H^\infty(M_{m,n})$ or $\mathscr{B}H_l^\infty(M_{m,n})$ of the most general interpolation problem for a finite number of points and strong results for infinitely many points. Moreover we obtain a test to determine if any solution to a particular interpolation problem exists. Finally in the last section we apply an extended form of our Beurling-Lax theorem to the setting of the Sz.-Nagy—Foiaş commutant lifting theorem.

Here $\mathscr{B}H_l^\infty(M_{m,n})$ denotes the closed unit ball of $m\times n$ matrix valued functions on the unit circle with meromorphic continuations onto the unit disk with at most l poles there; multiplicity must be counted carefully — see [16]. As usual $U(m, n)$ denotes the group of $(m+n)\times(m+n)$ matrices g which leave the form

$$[x \oplus y,\ x \oplus y]_{\mathbf{C}^{m,n}} \equiv \langle x, x\rangle_{\mathbf{C}^m} - \langle y, y\rangle_{\mathbf{C}^n}$$

($x \oplus y \in \mathbf{C}^{m+n}$, $x \in \mathbf{C}^m$, $y \in \mathbf{C}^n$, where $\langle\cdot\,,\cdot\rangle$ is the usual Euclidean inner product) invariant.

The linear fractional parameterization and the test for existence was obtained for $l = 0$ or for $m = n = 1$ by Adamjan, Arov and Kreĭn [1], [2]. The test for existence was obtained in general by Ball [5] and very refined results due to Arsene, Ceauşescu and Foiaş [3] when $l = 0$ are also available. Also Nudelman has recently obtained such results [27], see also [26]. Further strong results are also due to T. S. Ivanchenko [19], [20]. S. V. Kung obtained a set of solutions to the general $l, m = n$ problem in [30]. However the theorem in this paper is appealing not only because of its generality but also because of the relative simplicity of the proof. This simplicity permits many easy applications [7] and suggests many extensions [8], [9]. The subsequent article [7] uses this method to obtain the Wiener-Hopf factorization of a (not positive) self-adjoint matrix function (due to Nikolaicuk and Spitkovskii), Potopov's symplectic inner-outer factorization, and Darlington's theorem. While in this article and in [7] we have refrained from the great generality needed in our treatise [10] on the mathematics of amplifier design, these methods generalize trivially to that case and the authors think of the $U(m, n)$ Beurling-Lax theorem as a single result from which most of the tools developed in [10] follow. In a completely different vein the forthcoming articles [8], [9] deal with the classical Lie groups (other than $U(m, n)$). We prove a Beurling-Lax theorem for them and give applications to mathematics and to theoretical engineering.

The results of this paper were announced in [6]. The authors are grateful to P. DeWilde for encouragement regarding the engineering value of a complete theory of shift invariant subspaces of $L^2(\mathbf{C}^m)$ with signed bilinear form. Such spaces arose in his studies of Darlington synthesis for multiports.

1. PRELIMINARIES ON INDEFINITE INNER PRODUCT SPACES

We begin with some preliminaries on indefinite inner product spaces. A comprehensive reference for indefinite inner product spaces is Bognar's book [11], but we shall depart slightly from his notation and terminology. We shall be working with complex vector spaces $\mathscr{K}$ having a Hermitian bilinear form, denoted by $[\ ,\ ]$ or $[\ ,\ ]_{\mathscr{K}}$, which induces an inner product on $\mathscr{K}$ which is not necessarily positive-definite. If in addition $\mathscr{K}$ can be written as a direct sum $\mathscr{K} = \mathscr{K}_+ \dotplus \mathscr{K}_-$ where $(\mathscr{K}_+, \langle\ ,\ \rangle_{\mathscr{K}_+})$ and $(\mathscr{K}_-, \langle\ ,\ \rangle_{\mathscr{K}_-})$ are Hilbert spaces, and the inner product on $\mathscr{K}$ has the form

$$[x, y] = \langle x_+, y_+\rangle_{\mathscr{K}_+} - \langle x_-, y_-\rangle_{\mathscr{K}_-}$$

where $x = x_+ + x_-$, $y = y_+ + y_-$ $(x_+, y_+ \in \mathscr{K}_+,\ x_-, y_- \in \mathscr{K}_-)$, then $\mathscr{K}$ is said to be a Kreĭn space. Given a Kreĭn space $\mathscr{K} = \mathscr{K}_+ \dotplus \mathscr{K}_-$, it is also a Hilbert space in the inner product

$$\langle x, y\rangle = \langle x_+, y_+\rangle_{\mathscr{K}_+} + \langle x_-, y_-\rangle_{\mathscr{K}_-}$$

If $P_+ : x \to x_+$ is the projection of $\mathscr{K}$ onto $\mathscr{K}_+$ along $\mathscr{K}_-$ and $P_- = I - P_+ : x \to x_-$ is the projection onto $\mathscr{K}_-$ along $\mathscr{K}_+$, then $J = P_+ - P_-$ is said to be a *fundamental* *symmetry* for $\mathscr{K}$, and the above Hilbert space inner product can also be written as

$$\langle x, y\rangle = [Jx, y] \quad (= \langle x, y\rangle_J)$$

with norm

$$\|x\|_J^2 = [Jx, x]$$

(or simply $\|x\|^2$ if the choice of J is understood). While the associated Hilbert space norm $\| \ \|_J$ depends on the choice of fundamental symmetry J, the induced norm topology is independent of J, and thus intrinsic to $(\mathscr{K}, [\ ,\ ])$.

Certain general geometrical properties which we now discuss arise in the context of any Kreĭn space $\mathscr{K}$. If $(\ ,\ )$ is a Hermitian form on $\mathscr{K}$, two vectors x and y are said to be $(\ ,\ )$-orthogonal if $(x, y) = 0$. If $\mathscr{M}$ and $\mathscr{N}$ are two subspaces of $\mathscr{K}$ such that $\mathscr{M} \cap \mathscr{N} = \{0\}$, $\mathscr{M} + \mathscr{N}$ is closed and $[x, y] = 0$ for all x in $\mathscr{M}$ and y in $\mathscr{N}$, we write $\mathscr{M} \boxplus \mathscr{N}$ for $\mathscr{M} + \mathscr{N}$; if $\mathscr{M}$ and $\mathscr{N}$ are closed subspaces with $\langle x, y\rangle_J = 0$ for $x \in \mathscr{M}$ and $y \in \mathscr{N}$, we write $\mathscr{M} \oplus_J \mathscr{N}$ for $\mathscr{M} + \mathscr{N}$. Any subspace $\mathscr{M}$ has a closed $[\ ,\ ]$-*orthogonal complement*

$$\mathscr{M}' = \{x : [x, y] = 0 \text{ for all } y \text{ in } \mathscr{M}\};$$

the $\langle\ ,\ \rangle_J$-orthogonal complement of $\mathscr{M}$ is denoted

$$\mathscr{M}^{\perp J} = \{x : \langle x, y\rangle_J = 0 \text{ for all } y \in \mathscr{M}\}$$

or sometimes simply $\mathscr{M}^{\perp}$ if the J is understood. Note that the $[\ ,\ ]$-orthogonal complement $\mathscr{K}'$ of a Kreĭn space $\mathscr{K}$ is $\{0\}$; however, a subspace $\mathscr{M}$ and the restriction of $[\ ,\ ]$ to it need not have this property. The subspace $\mathscr{M}$ is called *nondegenerate* if no x in $\mathscr{M}$ is $[\ ,\ ]$-orthogonal to $\mathscr{M}$ (i.e., $\mathscr{M} \cap \mathscr{M}' = \{0\}$), and *regular* if there is no sequence $\{x_n\} \subset \mathscr{M}$ such that

$$\lim_{n\to\infty} \sup_{y \in \mathscr{M}} \frac{|[x_n, y]|}{\|x_n\| \, \|y\|} = 0.$$

Equivalently $\mathscr{M}$ is nondegenerate if and only if $\mathscr{M} + \mathscr{M}'$ is dense in $\mathscr{K}$, and is regular if and only if in addition $\mathscr{M} + \mathscr{M}'$ is closed (and thus $\mathscr{K} = \mathscr{M} \boxplus \mathscr{M}'$). It is an easy corollary of [11], Theorem V.3.5 that $\mathscr{M}'$ is regular (in our terminology) if and only if $\mathscr{M}$ is regular. Also $\mathscr{M}$ is regular if and only if the restriction of the Hermitian form $[\ ,\ ]$ of $\mathscr{K}$ to $\mathscr{M}$ makes $\mathscr{M}$ a Kreĭn space in its own right. If $\mathscr{M}$ is merely nondegenerate, at best one can only decompose $\mathscr{M}$ as $\mathscr{M} = \mathscr{M}_+ + \mathscr{M}_-$ where the restriction of $[\ ,\ ]$ to $\mathscr{M}_+$ and of $-[\ ,\ ]$ to $\mathscr{M}_-$ respectively make $\mathscr{M}_+$ and $\mathscr{M}_-$ pre-Hilbert spaces.

We say that the subspace $\mathscr{M}$ of a Kreĭn space $\mathscr{K}$ is *pseudo-regular* if $\mathscr{M} + \mathscr{M}'$ is closed. For an arbitrary subspace $\mathscr{M}$, it is always the case that $\mathscr{M} + \mathscr{M}'$ is dense in $(\mathscr{M} \cap \mathscr{M}')'$; thus $\mathscr{M}$ is pseudo-regular if and only if we have the equality $\mathscr{M} + \mathscr{M}' =$

$= (\mathcal{M} \cap \mathcal{M}')'$. Clearly $\mathcal{M}$ is pseudo-regular if and only if $\mathcal{M}'$ is pseudo-regular. Equivalently $\mathcal{M}$ is pseudo-regular if and only if $\mathcal{M}$ is of the form $\mathcal{M} = \mathcal{M}_1 \dotplus \mathcal{M}_0$ where $\mathcal{M}_1$ is a regular subspace of $\mathcal{K}$ and $\mathcal{M}_0 = \mathcal{M} \cap \mathcal{M}'$ is a null subspace ($[x, y] = 0$ for all x and y in $\mathcal{M}_0$). Thus in this case the form $[\ ,\ ]$ on $\mathcal{K}$ induces a Kreĭn space structure on the quotient space $\mathcal{M}/(\mathcal{M} \cap \mathcal{M}')$.

A subspace $\mathcal{M}$ of an indefinite inner product space $(\mathcal{K}, [\ ,\ ])$ is said to be *positive* provided $[x, x] \geqslant 0$ for each x in $\mathcal{M}$, *strictly positive* if in addition $[x, x] = 0$ for some x in $\mathcal{M}$ implies $x = 0$; by the Cauchy-Schwarz inequality, for positive subspaces this is equivalent to the condition $[x, y] = 0$ for all y in $\mathcal{M}$ implies $x = 0$. A positive subspace is said to be *maximal positive* (with respect to $\mathcal{K}$) or *$\mathcal{K}$-maximal positive* if it is not contained in any larger subspace of $\mathcal{K}$ which is also positive. The term *$\mathcal{K}$-maximal strictly positive* is defined similarly. We define the conditions *negative, strictly negative, $\mathcal{M}$-maximal negative* and *$\mathcal{M}$-maximal strictly negative* for a subspace $\mathcal{M}$ analogously. By the *negative signature* of a subspace $\mathcal{M}$ of the Kreĭn space $\mathcal{K}$ we mean the dimension l $(0 \leqslant l \leqslant \infty)$ of any $\mathcal{M}$-maximal strictly negative subspace of $\mathcal{M}$. It turns out that this dimension is independent of which particular $\mathcal{M}$-maximal strictly negative subspace one chooses, and thus negative signature is well-defined; if $\mathcal{M}$ is nondegenerate, this quantity is also the dimension of any $\mathcal{M}$-maximal negative subspace. The *negative cosignature* of the negative subspace $\mathcal{N}$ is the codimension of $\mathcal{N}$ as a subspace of some maximal negative subspace $\mathcal{N}_1$ of $\mathcal{K}$; this quantity also is well-defined, that is, is independent of the choice of maximal negative subspace $\mathcal{N}_1$ containing $\mathcal{N}$.

The following general lemma is basic for our analysis of interpolation problems to come in § 3.

LEMMA 1.1. *Suppose $\mathcal{M}$ is a pseudo-regular subspace of $\mathcal{K}$. Then each $\mathcal{M}$-maximal negative subspace of $\mathcal{M}$ has negative cosignature l equal to the negative signature of $\mathcal{M}'$.*

Proof. The space $\mathcal{X} \equiv \mathcal{M}'$ has a $[\ ,\ ]$-orthogonal decomposition

$$\mathcal{X} = \mathcal{X}_+ \boxplus \mathcal{X}_- \boxplus \mathcal{X}_0$$

into a strictly positive subspace $\mathcal{X}_+$, a strictly negative subspace $\mathcal{X}_-$ and a null space $\mathcal{X}_0$ $(= \mathcal{X} \cap \mathcal{X}')$, where $\dim \mathcal{X}_-$ is the negative signature of $\mathcal{X}$. (If $\mathcal{X}$ is pseudo-regular, $\mathcal{X}_+$ and $\mathcal{X}_-$ are Hilbert spaces in $[\ ,\ ]$ and $-[\ ,\ ]$ respectively; in general they are only pre-Hilbert spaces.) Suppose $\mathcal{N}$ is a $\mathcal{M}$-maximal negative subspace $(\subset \mathcal{X}')$; we claim that $\mathcal{N} + \mathcal{X}_-$ is $\mathcal{K}$-maximal negative. From this it follows that any such $\mathcal{N}$ has codimension equal to $\dim \mathcal{X}_-$ in a $\mathcal{K}$-maximal negative subspace, and the lemma follows.

To prove that $\mathcal{N} + \mathcal{X}_-$ is maximal negative, we first observe that since $\mathcal{N}$ and $\mathcal{X}_-$ are each negative and are $[\ ,\ ]$-orthogonal, clearly $\mathcal{N} + \mathcal{X}_-$ is negative. To prove $\mathcal{N} + \mathcal{X}_-$ is maximal negative, we need only show that $(\mathcal{N} + \mathcal{X}_-)' = \mathcal{N}' \cap \mathcal{X}'_- =$

$= \mathcal{N}' \cap (\mathcal{M} + \mathcal{X}_+)$ is positive. Since $\mathcal{N} = \mathcal{M}$, $\mathcal{N}'$ splits as $\mathcal{N}' = (\mathcal{N}' \cap \mathcal{M}) + \mathcal{X}$ and thus

$$\mathcal{N}' \cap (\mathcal{M} + \mathcal{X}_+) = (\mathcal{N}' \cap \mathcal{M}) + \mathcal{X}_+ .$$

Since $\mathcal{N}$ is $\mathcal{M}$-maximal negative, $\mathcal{N}' \cap \mathcal{M}$ is positive. By orthogonality and the positivity of $\mathcal{X}_+$, it next follows that $(\mathcal{N}' \cap \mathcal{M}) + \mathcal{X}_+$ is positive, as claimed.

Finally, in the sequel we shall need the angle operator-graph correspondence for negative and positive subspaces of a Kreĭn space $\mathcal{K}$. Suppose $\mathcal{K} = \mathcal{K}_+ \boxplus \mathcal{K}_-$ is a $[\,,]$-orthogonal decomposition of the Kreĭn space $\mathcal{K}$ into a maximal positive subspace $\mathcal{K}_+$, and a maximal negative subspace $\mathcal{K}_-$; then any maximal positive subspace $\mathcal{S}_+$ is of the form

$$\mathcal{S}_+ = \{x \boxplus T_+ x \mid x \in \mathcal{K}_+\}$$

for some operator $T_+ : \mathcal{K}_+ \to \mathcal{K}_-$ which is a contraction when $(\mathcal{K}_+, [\,,])$ and $(\mathcal{K}_-, -[\,,])$ are considered as Hilbert spaces. The operator T_+ is said to be the *angle operator* for $\mathcal{S}_+$ (with respect to the decomposition $\mathcal{K}_+ \boxplus \mathcal{K}_-$ for $\mathcal{K}$) and $\mathcal{S}_+$ is said to be the *graph of* T_+. Similarly, a maximal negative subspace $\mathcal{S}_-$ is of the form

$$\mathcal{S}_- = \{T_- y \boxplus y \mid y \in \mathcal{K}_-\}$$

for a contraction operator $T_- : \mathcal{K}_- \to \mathcal{K}_+$.

2. REPRESENTATIONS OF SHIFT INVARIANT SUBSPACES

The most concrete instances of Kreĭn spaces arise as follows. The vector space $\mathbf{C}^N$ naturally decomposes as

$$\mathbf{C}^N \cong \mathbf{C}^m \oplus \mathbf{C}^n$$

where $N = m + n$; define the Hermitian form $[\,,]_{\mathbf{C}^{m,n}}$ on it by

$$[u, v]_{\mathbf{C}^{m,n}} = \langle u_m, v_m \rangle_{\mathbf{C}^m} - \langle u_n, v_n \rangle_{\mathbf{C}^n}$$

if $u = u_m \oplus u_n$ and $v = v_m \oplus v_n$ where $u_m, v_m \in \mathbf{C}^m$, $u_n, v_n \in \mathbf{C}^n$. Here $\langle\,,\rangle_{\mathbf{C}^n}$ denotes the usual Euclidean inner product on $\mathbf{C}^n$. The set of $([\,,]_{\mathbf{C}^{m_1,n_1}}, [\,,]_{\mathbf{C}^{m,n}})$-isometric mappings will be denoted $U(m_1, n_1; m, n)$. Note $U(m_1, n_1; m, n)$ is empty unless $m_1 \leqslant m$ and $n_1 \leqslant n$. In all this discussion, we may take any of the integers m, n, m_1, n_1 to be $+\infty$; one then interprets $\mathbf{C}^\infty$ as the Hilbert space ℓ^2 in the obvious way. With this concrete class of Kreĭn spaces we associate a class of functional Kreĭn spaces as follows. Let $L^2(\mathbf{C}^N)$ be the Hilbert space of measurable $\mathbf{C}^N$-valued functions on the unit circle $\{|z|=1\}$ square-integrable in norm, and let $H^2(\mathbf{C}^N)$ be its subspace

$$\left\{ f\in L^2(C^N) : \int_0^{2\pi} \langle f(e^{it}), x\rangle e^{ikt}dt = 0 \text{ for } x\in \mathbf{C}^N,\ k=1,2,\dots \right\},$$

the usual Hardy space $H^2 \otimes \mathbf{C}^N$. If $n+m=N$ we can define a Hermitian form on $L^2(\mathbf{C}^N)$ and $H^2(\mathbf{C}^N)$ by

$$[f,g]_{L^2(\mathbf{C}^{m,n})} = \frac{1}{2\pi}\int_0^{2\pi} [f(e^{it}), g(e^{it})]_{\mathbf{C}^{m,n}}dt$$

in addition to the usual form

$$\langle f,g\rangle_{L^2(\mathbf{C}^N)} = \frac{1}{2\pi}\int_0^{2\pi} \langle f(e^{it}),\ g(e^{it})\rangle_{\mathbf{C}^N}dt.$$

Frequently we suppress the subscripts $L^2(\mathbf{C}^{m,n})$ and $L^2(\mathbf{C}^N)$ on $[\ ,\ ]$ and $\langle\ ,\ \rangle$. The operator $S\colon L^2(\mathbf{C}^N)\to L^2(\mathbf{C}^N)$ defined by

$$[Sf](e^{it}) = e^{it} f(e^{it})$$

is called the forward shift operator, or sometimes multiplication by e^{it} $(M_{e^{it}})$. Its restriction to $H^2(\mathbf{C}^N)$ is also denoted by S. In any case it is an isometry in both the indefinite $L^2(\mathbf{C}^{m,n})$-inner product and the Hilbert space $L^2(\mathbf{C}^N)$ inner product.

We next describe the general notion of "phase function" and "inner function" appropriate for our Beurling-Lax theorem. First we call a measurable function Ξ on $\{|z|=1\}$ with values in $U(m_1, n_1; m, n)$, a $(m_1, n_1; m, n)$ phase function; if in addition $\Xi(e^{it})$ is the boundary value a.e. of a function $\Xi(z)$ analytic on the disk $\{|z|<1\}$, we say that Ξ is an *analytic* $(m_1, n_1; m, n)$*-phase*. A *full range subspace* of $H^2(\mathbf{C}^N)$ is one with the property that at some z_0 in the disk $\{|z|<1\}$, we have $\{f(z_0) : f\in\mathcal{M}\}=\mathbf{C}^N$. It is easy to check that if this happens at one z_0 then it happens at all but an isolated set of z_0 (see [15]). We shall only be concerned with phase functions Ξ such that $\Xi(e^{it})x$ is in $L^2(\mathbf{C}^N)$ for any x in $\mathbf{C}^{N_1}$; thus any such Ξ has $\int \|\Xi(e^{it})\|^2_{\mathcal{L}(\mathbf{C}^{N_1},\mathbf{C}^N)}dt<\infty$. Finally a closed subspace $\mathcal{M}$ of $L^2(\mathbf{C}^N)$ is said to be *simply invariant* if it is invariant under S and $\bigcap_{k\geqslant 0} S^k\mathcal{M}=\{0\}$. For example every closed shift-invariant subspace $\mathcal{M}$ of $H^2(\mathbf{C}^N)$ is simply invariant since $\bigcap_{k\geqslant 0} S^k\mathcal{M}\subset \bigcap_{k\geqslant 0} S^k H^2(\mathbf{C}^N)=\{0\}$. Our main theorem is:

THEOREM 2.1. *If $\mathcal{M}$ is a regular simply invariant subspace of $L^2(\mathbf{C}^{m,n})$, then there are nonnegative integers $m_1 \leqslant m$ and $n_1 \leqslant n$ and an $(m_1, n_1; m, n)$-phase function Ξ such that*

$$\mathcal{M} = [\Xi H^\infty(\mathbf{C}^{m_1, n_1})]^-.$$

Moreover,

(1) *Ξ is analytic $\Leftrightarrow \mathcal{M} \subset H^2(\mathbf{C}^{m,n})$*

(2) *Suppose $\mathcal{M} \subset H^2(\mathbf{C}^{m,n})$. Then $\mathcal{M}$ is a full range invariant subspace $\Leftrightarrow$ $\Leftrightarrow m_1 = m$ and $n_1 = n$.*

(3) *Suppose $\mathcal{M} \subset H^2(\mathbf{C}^{m,n})$ and is full range. Then Ξ is a rational function $\Leftrightarrow$ $\Leftrightarrow H^2(\mathbf{C}^{m,n}) \cap \mathcal{M}'$ is finite dimensional.*

Proof. The idea of the proof is to adapt Halmos' wandering subspace argument for the proof of the Beurling-Lax theorem for the definite case [14], an idea already used by McEnnis [23] for studying shifts in an indefinite metric. Set $\mathcal{L} = \mathcal{M} \cap$ $\cap (M_{e^{it}} \mathcal{M})'$. Since $\mathcal{M}$ is regular, so is $\mathcal{L}$; furthermore, the spaces $S^k\mathcal{L} = M_{e^{ikt}} \mathcal{L}$ are mutually $[\,,\,]$-orthogonal, and by regularity, the $[\,,\,]$-orthogonal decomposition

$$\mathcal{M} = \mathcal{L} \boxplus S\mathcal{L} \boxplus \dots \boxplus S^q \mathcal{L} \boxplus S^{q+1}\mathcal{M}$$

holds for all $q = 0,1, \dots$. Thus any vector $[\,,\,]$-orthogonal to all the spaces $S^k\mathcal{L} (k = 0, 1, 2, \dots)$ is in $\bigcap_{k \geqslant 0} S^k \mathcal{M} = \{0\}$; hence

$$M_0 \equiv \bigcup_{q \geqslant 0} \{\mathcal{L} \boxplus S\mathcal{L} \boxplus \dots \boxplus S^q \mathcal{L}\}$$

is dense in $\mathcal{M}$.

Now the $[\,,\,]$-inner product restricted to $\mathcal{L}$ makes $\mathcal{L}$ a Kreĭn space (since $\mathcal{L}$ is regular), and so $\mathcal{L}$ has a fundamental decomposition

$$\mathcal{L} = \mathcal{L}_+ \boxplus \mathcal{L}_-$$

where $\mathcal{L}_+$ is a positive subspace and $\mathcal{L}_-$ is negative. If $m_1 = \dim \mathcal{L}_+$ and $n_1 =$ $= \dim \mathcal{L}_-$, we can use this decomposition to construct a $([\,,\,]_{\mathbf{C}^{m_1, n_1}}, [\,,\,]_{\mathcal{L}})$-unitary operator $\Xi \colon \mathbf{C}^{m_1, n_1} \to \mathcal{L}$. We then can extend Ξ in a unique fashion to the polynomials in $H^2(\mathbf{C}^{m_1, n_1})$ with range equal to $\mathcal{M}_0$ by demanding

$$\Xi M_{e^{it}} = S\Xi.$$

By orthogonality, this extended operator is $([\,,\,]_{H^2(\mathbf{C}^{m_1, n_1})}, [\,,\,]_{H^2(\mathbf{C}^{m,n})})$-isometric. If we define Ξ (e^{it}) a.e. by

$$\Xi(e^{it})x = (\Xi x)(e^{it}), \quad x \in \mathbf{C}^{m,n},$$

then we see that Ξ is the operator $M_{\Xi(e^{it})}$ of multiplication by the matrix function $\Xi(e^{it})$.

Since Ξ is a $([\,,\,]_{H^2(\mathbf{C}^{m_1,n_1})}, [\,,\,]_{H^2(\mathbf{C}^{m,n})})$-isometry, it follows in a standard way [25] that the boundary values $\Xi(e^{it})$ are $([\,,\,]_{\mathbf{C}^{m_1,n_1}}, [\,,\,]_{\mathbf{C}^{m,n}})$-isometric a.e., that is Ξ is a $(m_1, n_1; m, n)$-phase. This then forces $m_1 \leqslant m$ and $n_1 \leqslant n$. Since Ξ maps $\mathbf{C}^{m_1,n_1}$ onto $\mathcal{L} \subset H^2(\mathbf{C}^{m,n})$, it follows that the matrix entries of Ξ are square integrable, and thus Ξ extends by continuity to $H^\infty(\mathbf{C}^{m,n})$. Since $\Xi H(\mathbf{C}^{m_1,n_1})$ contains $\mathcal{M}_0$, we see that $[\Xi H^\infty(\mathbf{C}^{m_1,n_1})]^- = \mathcal{M}$.

Clearly Ξ is analytic if and only if $\mathcal{M} \subset H^2(\mathbf{C}^{m,n})$. Suppose $\mathcal{M} \subset H^2(\mathbf{C}^{m,n})$ and is full range; that is $\{f(z_0)\,|\,f \in \mathcal{M}\} = \mathbf{C}^{m,n}$ for some $z_0 \in \{|z| < 1\}$. Thus $\operatorname{Ran}\Xi(z_0) = \mathbf{C}^{m,n}$ which forces $m_1 + n_1 \geqslant m + n$. Since it was previously shown that $m_1 \leqslant m$ and $n_1 \leqslant n$, we see that $m_1 = m$ and $n_1 = n$. Conversely if $m_1 = m$ and $n_1 = n$, then there is a $z_0 \in \{|z| < 1\}$ for which $\operatorname{Ran}\Xi(z_0) = \mathbf{C}^{m,n}$ and thus $\mathcal{M}$ is full range. If $\mathcal{M} \subset H^2(\mathbf{C}^{m,n})$ is full range, then Ξ is rational if and only if $P_{H^2(\mathbf{C}^{m,n})^\perp}\Xi^+ H^2(\mathbf{C}^{m,n})$ is finite-dimensional; it is not difficult to see that this is equivalent to $H^2(\mathbf{C}^{m,n}) \cap \mathcal{M}'$ being finite dimensional.

Our next task is to obtain a useful representation for simply invariant subspaces $\mathcal{M}$ of $L^2(\mathbf{C}^{m,n})$ which are pseudo-regular. By an $(m_1, n_1, p_1; m, n)$-*phase function* we shall mean a square-integrable $M_{m+n,\, m_1+n_1+p_1}$ matrix-valued function $\Xi(e^{it})$ which is injective for a.e. t and such that

$$[\Xi(e^{it})(u \oplus v \oplus w), \Xi(e^{it})(u \oplus v \oplus w)]_{\mathbf{C}^{m,n}} = \langle u, u\rangle_{\mathbf{C}^{m_1}} - \langle v, v\rangle_{\mathbf{C}^{n_1}} \quad \text{for a.e. } t$$

where

$$u \oplus v \oplus w \in \mathbf{C}^{m_1} \oplus \mathbf{C}^{n_1} \oplus \mathbf{C}^{p_1} \cong \mathbf{C}^{m_1+n_1+p_1}.$$

(Necessarily $m_1 + p_1 \leqslant m$ and $n_1 + p_1 \leqslant n$.)

To set up notation we define a degenerate inner product $[\,,\,]_{\mathbf{C}^{m_1,n_1,p_1}}$ on $\mathbf{C}^{m_1+n_1+p_1}$ by $[u \oplus v \oplus w, u \oplus v \oplus w]_{\mathbf{C}^{m_1,n_1,p_1}} = \langle u, u\rangle_{\mathbf{C}^{m_1}} - \langle v, v\rangle_{\mathbf{C}^{n_1}}$, and use this to define a degenerate inner product $[\,,\,]_{L^2(\mathbf{C}^{m_1,n_1,p_1})}$ on $L^2(\mathbf{C}^{m_1+n_1+p_1})$ by integration.)

THEOREM 2.2. *Suppose $\mathcal{M}$ is a simply invariant pseudo-regular subspace of $L^2(\mathbf{C}^{m,n})$. Set $\mathcal{N} = \bigvee_{j \geqslant 0} S^j(\mathcal{M} \cap \mathcal{M}')$. Then there is a $(m_1, n_1, p_1; m, n)$-phase function Ξ (with $m_1 + p_1 \leqslant m$, $n_1 + p_1 \leqslant n$) such that*

$$\mathcal{M} \cap \mathcal{N}' = [\Xi H^\infty(\mathbf{C}^{m_1,n_1,p_1})]^-.$$

Also

i) *Ξ is analytic* $\Leftrightarrow \mathcal{M} \subset H^2(\mathbf{C}^{m,n})$,

ii) *if* $\mathcal{M} \subset H^2(\mathbf{C}^{m,n})$ *and* $H^2(\mathbf{C}^{m,n}) \cap \mathcal{M}'$ *is finite-dimensional, then* Ξ *is uniformly bounded and* $\mathcal{M} = \Xi H^2(\mathbf{C}^{m_1, n_1; p_1})$.

Proof. Since $\mathcal{M}$ is pseudo-regular and S is a bounded isometry, $S\mathcal{M}$ is also pseudo-regular, and thus $S\mathcal{M} + (S\mathcal{M})' = (S\mathcal{M} \cap (S\mathcal{M})')' = (S(\mathcal{M} \cap \mathcal{M}'))'$. Thus, if we set $\mathcal{L} = \mathcal{M} \cap (S\mathcal{M})'$, then from this it follows that

$$\mathcal{L} + S\mathcal{M} = \mathcal{M} \cap (S(\mathcal{M} \cap \mathcal{M}'))'. \tag{2.1}$$

In general we wish to establish the identity:

$$\begin{aligned} &\mathcal{L} + S\mathcal{L} + \ldots + S^{q-1}\mathcal{L} + S^q\mathcal{M} = \\ &= \mathcal{M} \cap \left[\bigvee_{k=0}^{q} S^k(\mathcal{M} \cap \mathcal{M}')\right]'. \end{aligned} \tag{2.2$_q$}$$

To prove $(2.2)_q$, we note

$$\left[\bigvee_{k=0}^{q} S^k(\mathcal{M} \cap \mathcal{M}')\right]' = \left[\bigvee_{k=0}^{q} S^k\mathcal{M} \cap (S^k\mathcal{M})'\right]' =$$

$$= \bigcap_{k=0}^{q} [(S^k\mathcal{M})' + (S^k\mathcal{M})]$$

(where we have used in the last step that $S^k\mathcal{M}$ is pseudo-regular for $k = 0, 1, \ldots, q$). Thus $(2.2)_q$ follows if we show

$$\bigvee_{k=0}^{q-1} S^k\mathcal{L} + S^q\mathcal{M} = \bigcap_{k=0}^{q} \{[\mathcal{M} \cap (S^k\mathcal{M})'] + S^k\mathcal{M}\}. \tag{2.3$_q$}$$

We know by (2.1) that $(2.3)_1$ holds. Assume inductively that $(2.3)_q$ holds and we wish to establish

$$\begin{aligned} &\bigvee_{k=0}^{q-1} S^k\mathcal{L} + S^q\mathcal{L} + S^{q+1}\mathcal{M} = \\ &= \left(\bigcap_{k=0}^{q} \{[\mathcal{M} \cap (S^k\mathcal{M})'] + S^k\mathcal{M}\}\right) \cap \{[\mathcal{M} \cap (S^{q+1}\mathcal{M})'] + S^{q+1}\mathcal{M}\}. \end{aligned} \tag{2.3$_{q+1}$}$$

Using $(2.3)_q$, we see that this is the same as

$$\bigvee_{k=0}^{q-1} S^k\mathcal{L} + S^q\mathcal{L} + S^{q+1}\mathcal{M} =$$

$$= \left(\bigvee_{k=0}^{q-1} S^k\mathcal{L} + S^q\mathcal{M}\right) \cap ([\mathcal{M} \cap (S^{q+1}\mathcal{M})'] + S^{q+1}\mathcal{M}).$$

This in turn holds if

$$(2.4)_{q+1} \qquad S^q\mathscr{L} = S^q\mathscr{M} \cap [\mathscr{M} \cap (S^{q+1}\mathscr{M})'].$$

Using the definition of $\mathscr{L}$ $(= \mathscr{M} \cap (S\mathscr{M})')$ and that S is a $[\ ,\ ]$-isometry, one can easily check directly that $(2.4)_{q+1}$ is true. This establishes $(2.2)_{q+1}$, and hence $(2.2)_q$ for all q. Letting q tend to ∞ in $(2.2)_q$, one gets

$$(2.5) \qquad \bigcap_{q>0} \{\mathscr{L} + S\mathscr{L} + \ldots + S^{q-1}\mathscr{L} + S^q\mathscr{M}\} = \mathscr{M} \cap \mathscr{N}'.$$

In order to establish

$$(2.6) \qquad \bigvee_{k>0} S^k\mathscr{L} = \mathscr{M} \cap \mathscr{N}',$$

it suffices to establish

$$(2.7) \qquad \bigvee_{k>0} S^k\mathscr{L} = \bigcap_{k>0} \{\mathscr{L} + S\mathscr{L} + \ldots + S^{k-1}\mathscr{L} + S^k\mathscr{M}\}.$$

First note that the containment $\subset$ in (2.7) is obvious. For the reverse containment, we note that $\mathscr{L}$ is pseudo-regular and thus has a decomposition $\mathscr{L} = \mathscr{L}_r [\dot{+}] \mathscr{L}_0$ where $\mathscr{L}_r$ is a regular subspace and $\mathscr{L}_0$ is $\mathscr{L} \cap \mathscr{L}'$. Thus $\bigvee_{k>0} S^k\mathscr{L}$ is spanned by $\underset{k\geqslant 0}{[+]} S^k\mathscr{L}_r$ and $\bigvee_{k\geqslant 0} S^k\mathscr{L}_0$. If a vector x in $\mathscr{M} \cap \mathscr{N}'$ were simultaneously $[\ ,\ ]$-orthogonal to $\underset{k\geqslant 0}{[+]} S^k\mathscr{L}_r$ and $\langle\ ,\ \rangle$-orthogonal to $\bigvee_{k>0} S^k\mathscr{L}_0$, then the decomposition (2.5) would imply $x \in \bigcap_{k\geqslant 0} S^k\mathscr{M} = \{0\}$ (since $\mathscr{M}$ is simply invariant). This implies (2.7) and hence also (2.6).

We now use the representation (2.6) for $\mathscr{M} \cap \mathscr{N}'$ to construct the desired $(m_1, n_1, p_1;\ m, n)$-phase function Ξ. We construct an (m_1, n_1)-phase function Ξ_1 such that

$$\mathscr{M}_1 \equiv \bigvee_{k>0} S^k\mathscr{L}_r = [\Xi_1 \cdot H^\infty(\mathbf{C}^{m_1, n_1})]^-$$

just as in the proof of Theorem 2.1. Also one can show $\mathscr{N} = \bigvee_{k\geqslant 0} S^k\mathscr{L}_0$, and from the decomposition (2.6), $\mathscr{M}_1 + \mathscr{N}$ is dense in $\mathscr{M} \cap \mathscr{N}'$. Since $\mathscr{N}$ is a closed simply invariant subspace, by the usual Beurling-Lax theorem there is a phase function $\Psi \in H^\infty(M_{m+n, p_1})$ (isometric in the usual Euclidean metrics) such that $\mathscr{N} = \Psi H^2(\mathbf{C}^{p_1})$. To see that $\Xi = [\Xi_1 \Psi]$ is the desired $(m_1, n_1, p_1;\ m, n)$-phase function, it remains only to show that Ξ is one-to-one. To see this, note that since $\Xi(e^{it})$ is an isometry from $\mathbf{C}^{m_1, n_1, p_1}$ into $\mathbf{C}^{m, n}$ in their respective indefinite metrics, the columns

$\xi_j(e^{it})$ ($j = 1, \ldots, m_1 + n_1 + p_1$) of $\Xi(e^{it})$ satisfy

$$[\xi_j(e^{it}), \xi_k(e^{it})]_{C^{m,n}} = \begin{cases} 1, & 1 \leqslant j = k \leqslant m_1 \\ -1, & m_1 < j = k \leqslant m_1 + n_1 \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\Xi(e^{it}) f(e^{it}) = 0$ a.e. for some f in $H^\infty(C^{m_1, n_1, p_1})$. Then

$$0 = [\xi_j(e^{it}), \Xi(e^{it}) f(e^{it})]_{C^{m,n}} =$$

$$= [\xi_j(e^{it}), \xi_j(e^{it})]_{C^{m,n}} \overline{f_j(e^{it})}$$

where we have written $\Xi f = \sum_k f_k \xi_k$. Therefore $f_j(e^{it})=0$ a.e. for $1 \leqslant j \leqslant m_1 + n_1$, so f has the form $0 \oplus 0 \oplus \hat{f}$ for an $\hat{f}$ in $H^\infty(C^{p_1})$ and $\Xi f = \Psi \hat{f}$. Now, since Ψ was the traditional Beurling-Lax representor for $\mathcal{N}$ (and hence in particular is one--to-one), it follows that $\hat{f}$, and thus also f, is 0.

Clearly, Ξ is analytic if and only if $\mathcal{M} \cap \mathcal{N}' \subset H^2(C^{m,n})$ or $\mathcal{M} \subset H^2(C^{m,n})$. If $\mathcal{M} \subset H^2(C^{m,n})$ and $H^2(C^{m,n}) \cap \mathcal{M}'$ is finite dimensional, then $\mathcal{X} = H^2(C^{m,n}) \cap \cap \mathcal{M}'$, being finite-dimensional invariant subspace for the backward shift operator, consists of uniformly bounded functions. Then since $\mathcal{L} \subset \mathcal{X} + S\mathcal{X}$, so also does $\mathcal{L}$. From this one can deduce that Ξ is uniformly bounded in norm.

Some final remarks might help with the computation of Ξ. First we thank Bruce Francis for pointing out that

$$\mathcal{L} \cap \mathcal{L}' = \mathcal{M} \cap \mathcal{M}' + S(\mathcal{M} \cap \mathcal{M}').$$

Thus $\mathcal{N} = \bigvee_{k \geqslant 0} S^k(\mathcal{M} \cap \mathcal{M}')$. This allows us to take *any* Ψ which maps $H^2(C^{p_2}$ onto $\mathcal{M} \cap \mathcal{M}'$ (not just the Beurling-Lax Ψ). Here $p_2 = \dim \mathcal{M} \cap \mathcal{M}'$. While generically $\det \Psi(z) \not\equiv 0$ it may happen that $\det \Psi(z) = 0$ for all z. In this case one modifies $\Psi(z)$ to $\Psi_0(z)$ by throwing out just enough columns of Ψ to make the remaining ones linearly independent.

3. APPLICATIONS TO INTERPOLATION

In this section we describe the connection of Lemma 1.1 and Theorem 2.1 to generalized matrix interpolation problems. We first describe a specific class of interpolation problems.

a. The general construction

Let $\mathbf{z} = \{z_j\}_{j=0}^{N} \cup \{z_j'\}_{j=1}^{N'}$ be points in the unit disk and $\mathbf{p} = \{p_j\}_{j=1}^{N} \cup \{p_j'\}_{j=1}^{N'}$ and $\mathbf{q} = \{q_j\}_{j=1}^{N} \cup \{q_j'\}_{j=1}^{N'}$ be vectors in $\mathbf{C}^m$ and $\mathbf{C}^n$ respectively. The problem of interest is to describe the set

$$N\text{-}P(\mathbf{z}, \mathbf{p}, \mathbf{q}) = \{F \in \mathscr{B}H^\infty(M_{m,n}) \mid F(z_j)^* p_j = q_j,\ j = 1, \ldots, N \text{ and } F(z_j')\, p_j' = q_j',\ j = 1, \ldots, N'\}.$$

Here $\mathscr{B}H^\infty(M_{m,n})$ is the set of all $(m\times n)$-matrix valued functions with analytic continuation to the unit disk $\{|z| < 1\}$ with $\|F\|_\infty \leqslant 1$. Similarly $\mathscr{B}L^\infty(M_{m,n})$ will denote the set of $(m\times n)$-matrix valued L^∞-functions F with $\|F\|_\infty \leqslant 1$. We include the possibility of m or $n = \infty$; then $\mathbf{C}^m$ stands for a separable Hilbert space of dimension m, $M_{m,n}$ stands for the set of bounded linear operators from $\mathbf{C}^n$ into $\mathbf{C}^m$. The classical solution to the problem of determining if $N\text{-}P(\mathbf{z}, \mathbf{p}, \mathbf{q})$ is non-empty for the case $N' = 0$ is: There exists a function F in $N\text{-}P(\mathbf{z}, \mathbf{p}, \mathbf{q})$ if and only if the matrix

$$\Lambda_{\{\mathbf{z}, \mathbf{p}, \mathbf{q}\}} = \left[\frac{\langle p_j, p_k\rangle - \langle q_j, q_k\rangle}{1 - \bar{z}_j z_k}\right]_{j, k=1, \ldots, N}$$

is positive-definite.

More generally, consider the class $\mathscr{B}H_l^\infty(M_{m,n})$ of functions F which have a representation of the form $F = G\theta^{-1}$ where $G \in \mathscr{B}H^\infty(M_{m,n})$ and θ is a matrix Blaschke product of degree at most l. Consider the problem of describing the larger set

$$N\text{-}P_l(\mathbf{z}, \mathbf{p}, \mathbf{q}) = \{F \in \mathscr{B}H_l^\infty(M_{m,n}) \mid F(z_j)^* p_j = q_j,\ j = 1, \ldots, N \text{ and}$$

$$F(z_j')\, p_j' = q_j',\ j = 1, \ldots, N'\}.$$

(If F happens to have a pole at z_j, interpret the condition $F(z_j)^* p_j = q_j$ as $G(z_j)^* p_j = \theta(z_j)^* q$ where $F = G\theta^{-1}$ is the representation for F mentioned above. Similarly one handles the condition $F(z_j')p_j' = q_j'$ if F has a pole at z_j'.) The solution of the existence problem (for the case $N' = 0$), given by Ball [5] in this generality, is $N\text{-}P_l(\mathbf{z}, \mathbf{p}, \mathbf{q})$ is nonempty if and only if the associated Pick matrix $\Lambda_{\{\mathbf{z}, \mathbf{p}, \mathbf{q}\}}$ has at most l negative eigenvalues.

To begin our analysis, we write the set $N\text{-}P_l(\mathbf{z}, \mathbf{p}, \mathbf{q})$ in a different form, which in turn will suggest a more general problem (that of "generalized interpolation" in the sense of Sarason). For the sake of simplicity in the present discussion, we assume that no point z_j is the same as some point z_j' in the disk. Let $H^2(M_{m,n})$ be the class of $(m\times n)$-matrix functions K analytic on the disk with boundary value function $K(e^{it})$ square-integrable in matrix operator norm; $H_l^2(M_{m,n})$ then consists of matrix functions K such that $K\Psi \in H^2(M_{m,n})$ for some inner Ψ in $\mathscr{B}H^\infty(M_{m,n})$ of degree at most l. Note that $\{f \in H^2(\mathbf{C}^m) \mid \langle p_j, f(z_j)\rangle = 0 \text{ for } j = 1, \ldots, N\}$ is an invariant subspace for the shift operator S on $H^2(\mathbf{C}^m)$, and hence by the classical Beurling-Lax theorem is of the form $\theta H^2(\mathbf{C}^m)$ for some matrix inner function $\theta \in \mathscr{B}H^\infty(M_m)$. Similarly there is a matrix inner function $\varphi \in \mathscr{B}H^\infty(M_n)$ such that

$$\tilde{\varphi}H^2(\mathbf{C}^n) = \{f \in H^2(\mathbf{C}^n) \mid \langle p'_j, f(z'_j)\rangle = 0 \text{ for } j = 1, \ldots, N'\},$$

where $\tilde{\varphi}(e^{it}) = \varphi(e^{-it})^*$. Then it is not difficult to see that

$$N\text{-}P_l(\mathbf{z}, \mathbf{p}, \mathbf{q}) = (F_0 + \theta H^2_l(M_{m,n})\varphi) \cap \mathscr{B}L^\infty(M_{m,n})$$

where F_0 is any function in $H^2(M_{m,n})$ which satisfies the interpolation conditions $F_0(z_j)^* p_j = q_j$ for $j = 1, \ldots, N$ and $F(z'_j)p'_j = q'_j$ for $j = 1, \ldots, N'$. The inner functions θ and φ arising from a set $N\text{-}P_l(\mathbf{z}, \mathbf{p}, \mathbf{q})$ in this way are very special; they are rational and have only simple zeros. Allowing F_0, θ and φ to be L^2-functions gives a more general problem without an interpretation as an interpolation problem as above. We say a function Ω in $\mathscr{B}L^\infty(M_{m,n})$ is a *phase function* if its values $\Omega(e^{it})$ are isometries a.e.. The general problem to be analyzed in this section is the following.

GENERALIZED INTERPOLATION PROBLEM. Describe the set

$$C_{K,\theta,\varphi}(l) \equiv (K + \theta H^2_l(M_{k,n})\,\varphi) \cap \mathscr{B}L^\infty(M_{m,n})$$

for any given $K \in L^2(M_{m,n})$ and phase functions θ and φ in $\mathscr{B}L^\infty(M_{m,k})$ and $\mathscr{B}L^\infty(M_n)$ respectively.

Our approach is to make use of the angle operator-graph correspondence between contraction operators and maximal negative subspaces of a Kreĭn space described in § 1 to obtain an equivalent more geometric version of the problem Thus we consider the space $L^2(\mathbf{C}^m) \oplus \varphi^* H^2(\mathbf{C}^{k'})$ with the Kreĭn space inner product inherited from $L^2(\mathbf{C}^{m,n})$. Form the span $\mathscr{M}$ of all subspaces which are graphs of multiplication operators with multiplier in $K + \theta H^2(M_{k,n})\varphi$:

$$\mathscr{M} = \mathscr{M}_{K,\theta,\varphi} = \text{the closure of } \left\{\begin{bmatrix} K \\ I \end{bmatrix}\varphi^* H^\infty(\mathbf{C}^{k'}) + \begin{bmatrix} \theta \\ 0 \end{bmatrix} H^2(\mathbf{C}^k)\right\}.$$

Since the angle operators defining the spaces are multiplication operators, it is clear that $\mathscr{M}$ is invariant under the shift operator S. The following is basic to our analysis.

LEMMA 3.1. *Let K be an element of $L^2(M_{m,n})$, let $\theta \in \mathscr{B}L^\infty(M_{m,k})$ and $\varphi \in \mathscr{B}L^\infty(M_n)$ be phase functions, and set $\mathscr{M}$ equal to $\mathscr{M}_{K,\theta,\varphi}$ as above. The angle operator-graph correspondence induces a one-to-one correspondence between $C_{K,\theta,\varphi}(l)$ and shift-invariant negative suspaces of $\mathscr{M}$ which have codimension of at most l as a subspace of some $L^2(\mathbf{C}^m) \oplus \varphi^* H^2(\mathbf{C}^n)$-maximal negative subspace. In particular, when $\mathscr{M}' \equiv [L^2(\mathbf{C}^m) \oplus \varphi^*(H^2(\mathbf{C}^n))] \boxminus \mathscr{M}$ has negative signature l, these are the shift invariant $\mathscr{M}$-maximal negative subspaces of $\mathscr{M}$.*

Proof. Suppose F is in $C_{K,\theta,\varphi}(l)$. Then F has a representation

$$F = K + \theta G \Psi^{-1} \varphi$$

where $G \in H^2(M_{k,n})$ and $\Psi \in \mathscr{B}H^\infty(M_n)$ is a matrix Blaschke product of degree at most l. Since G maps $H^\infty(\mathbf{C}^n)$ into $H^2(\mathbf{C}^k)$ and $\varphi\varphi^* = I$, this representation implies

$$\begin{bmatrix} F \\ I \end{bmatrix} \varphi^* \Psi H^\infty(\mathbf{C}^n) \subset \mathscr{M}.$$

By continuity,

$$\mathscr{G} \equiv \begin{bmatrix} F \\ I \end{bmatrix} \varphi^* \Psi H^2(\mathbf{C}^n) \subset \mathscr{M}.$$

Since $\|F\|_\infty \leqslant 1$, the subspace $\mathscr{G}$ is a negative subspace of $L^2(\mathbf{C}^m) \oplus \varphi^* H^2(\mathbf{C}^n)$. Since Ψ is a matrix Blaschke product of degree at most l, $\varphi^* \Psi H^2(\mathbf{C}^n)$ has codimension at most l as a subspace of $\varphi^* H^2(\mathbf{C}^n)$. Therefore $\mathscr{G}$ has codimension at most l in some $L^2(\mathbf{C}^m) \oplus \varphi^* H^2(\mathbf{C}^n)$-maximal negative subspace. Clearly also $\mathscr{G}$ is shift invariant.

Conversely, suppose $\mathscr{G}$ is a shift invariant negative subspace of $\mathscr{M}$ of codimension l in a $L^2(\mathbf{C}^m) \oplus \varphi^* H^2(\mathbf{C}^n)$-maximal negative subspace. That $\mathscr{G}$ is invariant and has codimension l in a maximal negative subspace means

$$\mathscr{G} = \begin{bmatrix} F \\ I \end{bmatrix} \varphi^* \Psi H^2(\mathbf{C}^n)$$

where F is in $\mathscr{B}L^\infty(M_{m,n})$ and $\Psi \in H^\infty(M_n)$ is a matrix Blaschke product of degree l. Since also $\mathscr{G} \subset \mathscr{M}$, we have for any $h \in H^\infty(\mathbf{C}^n)$

$$\begin{bmatrix} F \\ I \end{bmatrix} \varphi^* \Psi h = \lim_{q \to \infty} \left\{ \begin{bmatrix} K \\ I \end{bmatrix} \varphi^* h_1^{(q)} + \begin{bmatrix} \theta \\ 0 \end{bmatrix} h_2^{(q)} \right\}$$

for some $h_1^{(q)} \in H^\infty(\mathbf{C}^n)$ and $h_2^{(q)} \in H^2(\mathbf{C}^k)$. From this we get $\varphi^* \Psi h = \lim_{q \to \infty} \theta h_1^{(q)}$ and then $(F - K)\varphi^* \Psi h = \lim_{q \to \infty} \theta h_2^{(q)}$. Thus $(F - K)\varphi^* \Psi = \theta G$ where $G \in H^2(M_{k,n})$, so $F \in K + \theta H^2(M_{k,n}) \Psi^{-1} \varphi$. Since Ψ has degree l, we conclude $F \in K + \theta H_l^2(M_{k,n}) \varphi$. Since from the above we also have $\|F\|_\infty \leqslant 1$, we conclude that F is in $C_{K,\theta,\varphi}(l)$ as desired. The uniqueness in the correspondence can be arranged by demanding that G and Ψ are right coprime in the representation $F = K + \theta G \Psi^{-1} \varphi$.

Note that for any L^2-function K and phases θ and φ, the associated invariant subspace $\mathscr{M}_{K,\theta,\varphi}$ is simply invariant. When $\mathscr{M}_{K,\theta,\varphi}$ is regular, Theorem 2.1 and Lemma 3.1 can be combined directly to get information concerning the set $C_{K,\theta,\varphi}(l)$. When $\mathscr{M}_{K,\theta,\varphi}$ is only pseudo-regular, we first require the following.

LEMMA 3.2. *Suppose $\mathcal{M}$ is a pseudo-regular invariant subspace of $L^2(C^{m,n})$. Set $\mathcal{N}$ equal to $\bigvee \{S^k(\mathcal{M} \cap \mathcal{M}') \mid k = 0,1, \ldots\}$. Then a subspace $\mathcal{G}$ of $\mathcal{M}$ is invariant and $\mathcal{M}$-maximal negative if and only if $\mathcal{G} \subset \mathcal{M} \cap \mathcal{N}'$ and $\mathcal{G}$ is invariant and $(\mathcal{M} \cap \mathcal{N}')$-maximal negative.*

Proof. Suppose $\mathcal{G}$ is invariant and $\mathcal{M}$-maximal negative. Any $\mathcal{M}$-maximal negative subspace must contain $\mathcal{M} \cap \mathcal{M}'$; since $\mathcal{G}$ is also invariant, it follows that $\mathcal{N} \subset \mathcal{G}$. Since $\mathcal{G}$ is a negative space and $\mathcal{N}$ is a null space ($[x, x] = 0$ for $x \in \mathcal{N}$), this in turn forces $\mathcal{G} \subset \mathcal{N}'$; hence $\mathcal{G} \subset \mathcal{M} \cap \mathcal{N}'$. Since $\mathcal{G}$ is $\mathcal{M}$-maximal negative, a fortiori $\mathcal{G}$ is $(\mathcal{M} \cap \mathcal{N}')$-maximal negative.

The converse direction does not require that $\mathcal{G}$ be invariant. Thus, suppose only that $\mathcal{G}$ is $(\mathcal{M} \cap \mathcal{N}')$-maximal negative. Since $(\mathcal{M} \cap \mathcal{N}') \cap (\mathcal{M} \cap \mathcal{N}')' = \mathcal{N}$, then $\mathcal{G} \supset \mathcal{N}$. Therefore if $\mathcal{G}_1$ is a negative subspace of $\mathcal{M}$ containing $\mathcal{G}$ then $\mathcal{G}_1 \supset \mathcal{N}$; as in the first part of the proof, this forces $\mathcal{G}_1 \subset \mathcal{M} \cap \mathcal{N}'$, and thus $\mathcal{G}_1 = \mathcal{G}$ by the maximality of $\mathcal{G}$ in $\mathcal{M} \cap \mathcal{N}'$. Therefore $\mathcal{G}$ is $\mathcal{M}$-maximal negative.

We are now ready to use our symplectic Beurling-Lax theorem to parameterize the set of invariant $\mathcal{M}$-maximal negative subspaces for an invariant pseudo-regular subspace $\mathcal{M}$ of $H^2(C^{m,n})$.

LEMMA 3.3. *Suppose $\mathcal{M}$ is a simply invariant pseudo-regular subspace of $L^2(C^{m,n})$. Then there is an $(m_1, n_1, p_1; m, n)$-phase function $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix}$ such that the invariant $\mathcal{M}$-maximal negative subspaces $\mathcal{G}$ of $\mathcal{M}$ are precisely those of the form*

$$\mathcal{G} = \text{the closure of} \left\{ \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix} \begin{bmatrix} F & 0 \\ I & 0 \\ 0 & I \end{bmatrix} (H^\infty(\mathbf{C}^{n_1}) \oplus H^\infty(\mathbf{C}^{p_1})) \right\}$$

for some $F \in \mathcal{B}H^\infty(M_{m_1, n_1})$. (Here $m_1 + p_1 \leqslant m$, $n_1 + p_1 \leqslant n$, $m_1 + n_1 + p_1 \leqslant m + n$).

More generally, the invariant negative subspaces $\mathcal{G}$ of $\mathcal{M}$ with $\mathcal{M}$-negative cosignature $\leqslant l_1$ are exactly those of the form

$$\mathcal{G} = \left\{ \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix} \begin{bmatrix} F & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \mathcal{K} \cap (H^\infty(\mathbf{C}^{n_1}) \oplus H^\infty(\mathbf{C}^{p_1})) \right\}^-$$

where $\mathcal{K}$ is an invariant subspace of $H^2(\mathbf{C}^{n_1}) \oplus H^2(\mathbf{C}^{p_1})$ of codimension at most l_1 and $F: P_{H^2(\mathbf{C}^{n_1}) \oplus \{0\}} \mathcal{K} \to H^2(\mathbf{C}^{m_1})$ is a contractive multiplication operator. Thus F is in $\mathcal{B}H^\infty_{l_1}(M_{m_1, n_1})$.

Proof. We do only the case $l_1 = 0$; the general case easily reduces to this. By Theorem 2.2, there is an $(m_1, n_1, p_1; m, n)$-phase function $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix}$ such that $\mathcal{M} \cap \mathcal{N}' = \{\Xi \cdot H^\infty(\mathbf{C}^{m_1, n_1, p_1})\}^-$ where $\mathcal{N} = \bigvee_{k \geq 0} S^k(\mathcal{M} \cap \mathcal{M}')$. Suppose for the moment that Ξ is bounded, so $\mathcal{M} \cap \mathcal{N}' = \Xi \cdot H^2(\mathbf{C}^{m_1, n_1, p_1})$. Then multiplication by Ξ is a metric-preserving isomorphism from $H^2(\mathbf{C}^{m_1, n_1, p_1})$, with $\langle , \rangle_{\mathbf{C}^{m_1}} - \langle , \rangle_{\mathbf{C}^{n_1}}$ to $\mathcal{M} \cap \mathcal{N}'$; so a $(\mathcal{M} \cap \mathcal{N}')$-maximal negative subspace $\mathcal{G}$ has the form $\Xi \cdot \mathcal{K}$, where $\mathcal{K}$ is a maximal negative subspace of $H^2(\mathbf{C}^{m_1, n_1, p_1})$. Also since Ξ is a multiplication operator, invariant subspaces of $\mathcal{M} \cap \mathcal{N}'$ correspond with invariant subspaces of $H^2(\mathbf{C}^{m_1, n_1, p_1})$ in this way. Thus $\mathcal{G}$ is invariant and $(\mathcal{M} \cap \mathcal{N}')$-maximal negative if and only if $\mathcal{G} = \Xi\mathcal{K}$ for some invariant $H^2(\mathbf{C}^{m_1, n_1, p_1})$-maximal negative subspace $\mathcal{K}$. But one easily checks that the invariant maximal negative subspaces of $H^2(\mathbf{C}^{m_1, n_1, p_1})$ are those of the form $\begin{bmatrix} F & 0 \\ I & 0 \\ 0 & I \end{bmatrix} (H^2(\mathbf{C}^{n_1}) \oplus H^2(\mathbf{C}^{p_1}))$ for some F in $\mathcal{B}H^\infty(\mathbf{C}^{m_1, n_1})$. By Lemma 3.2, invariant $\mathcal{M}$-maximal negative subspaces are $(\mathcal{M} \cap \mathcal{N}')$-maximal negative. This proves the Lemma for the case where Ξ is bounded.

The proof for the general case involves the same ideas, but must be done with more care. Given any invariant subspaces $\mathcal{G}$ contained in $\mathcal{M} = [\Xi \cdot H^\infty(\mathbf{C}^{m_1, n_1, p_1})]^-$, one can argue that $\mathcal{G}_1 \equiv \mathcal{G} \cap \Xi \cdot H^\infty(\mathbf{C}^{m_1, n_1, p_1})$ is dense in $\mathcal{G}$. Then $\mathcal{K}_1 = \Xi^{-1}\mathcal{G}_1$ is a negative submanifold of $H^2(\mathbf{C}^{m_1, n_1, p_1})$; denote its closure in $H^2(\mathbf{C}^{m_1, n_1, p_1})$ by $\mathcal{K}$. We claim that $\mathcal{K}$ is an invariant maximal negative subspace of $H^2(\mathbf{C}^{m_1, n_1, p_1})$. Indeed, if $\mathcal{K}$ is not maximal negative, then there is a strictly larger negative subspace $\mathcal{R}$ *which is also invariant.* By the classical Beurling-Lax theorem, we can produce a *bounded* F which is in $\mathcal{R}$ but not in $\mathcal{K}$. Then the closure of $\{\Xi \cdot (\mathcal{R} \cap H^\infty(\mathbf{C}^{m_1, n_1, p_1}))\}$ in $H^2(\mathbf{C}^{m,n})$ is a negative subspace of $H^2(\mathbf{C}^{m,n})$ which is strictly larger than $\mathcal{G}$, a contradiction. By a similar argument, one can show conversely that a subspace $\mathcal{G}$ of the form

$$\mathcal{G} = \text{closure of } \{\Xi \cdot (\mathcal{K} \cap H^\infty(\mathbf{C}^{m_1, n_1, p_1}))\},$$

where $\mathcal{K}$ is invariant and maximal negative in $H^2(\mathbf{C}^{m_1, n_1, p_1})$, is invariant and $(\mathcal{M} \cap \mathcal{N}')$-maximal negative. Finally it is not difficult to see that a linear manifold

$\mathscr{L}$ is of the form $\mathscr{K} \cap H^\infty(\mathbf{C}^{m_1, n_1, p_1})$ for some invariant maximal negative subspace $\mathscr{K}$ of $H^2(\mathbf{C}^{m_1, n_1, p_1})$ if and only if

$$\mathscr{L} = \begin{bmatrix} F & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \cdot (H^\infty(\mathbf{C}^{n_1}) \oplus H^\infty(\mathbf{C}^{p_1}))$$

for some F in $\mathscr{B}H^\infty(M_{m_1, n_1})$. This completes the proof of the lemma.

To parameterize a set $\mathbf{C}_{K,\theta,\varphi}(l)$, by Lemma 3.1 it remains only to obtain a parameterization of the angle operators corresponding to negative subspaces of $\mathscr{M}_{K,\theta,\varphi}$ having a prescribed $\mathscr{M}_{K,\theta,\varphi}$-negative cosignature. To do this we need to introduce a certain linear fractional transformation associated with a $(m_1, n_1, p_1; m, n)$-phase function $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \chi & \gamma & \omega \end{bmatrix}$. Assume $n_1 + p_1 = n$ and let

$$i: H^2(\mathbf{C}^{n_1}) \to H^2(\mathbf{C}^{n_1}) \oplus \{0\} \subset H^2(\mathbf{C}^{n_1}) \oplus H^2(\mathbf{C}^{p_1}) \cong H^2(\mathbf{C}^n)$$

and

$$j: H^2(\mathbf{C}^{p_1}) \to \{0\} \oplus H^2(\mathbf{C}^{p_1}) \subset H^2(\mathbf{C}^{n_1}) \oplus H^2(\mathbf{C}^{p_1}) \cong H^2(\mathbf{C}^n)$$

be the natural inclusion maps. Define a mapping $\mathscr{G}_\Xi$ from $\mathscr{B}L^\infty(M_{m_1, n_1})$ into $\mathscr{B}L^\infty(M_{m,n})$ by

$$\mathscr{G}_\Xi(F) = (\alpha F i^* + \beta i^* + \psi j^*)(\varkappa F i^* + \gamma i^* + \omega j^*)^{-1}.$$

(We shall see below that $(\varkappa F i^* + \gamma i^* + \omega j^*)^{-1}$ always exists if $H \in \mathscr{B}L^\infty(M_{m_1, n_1})$ and $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix}$ is a $(m_1, n_1, p_1; m, n)$-phase function.) The maps $\mathscr{G}_\Xi$ can be used to parameterize the sets $\mathbf{C}_{K,\theta,\varphi}(l)$ as we now see.

THEOREM 3.4. *Suppose K is in $\mathscr{B}H^\infty(M_{m,n})$ and $\theta \in \mathscr{B}H^\infty(M_{m,k})$ and $\varphi \in \mathscr{B}H^\infty(M_n)$ are phase functions. In addition suppose the associated invariant subspace $\mathscr{M}_{K,\theta,\varphi}$ is pseudo-regular, and let l be the negative signature of $(\mathscr{M}_{K,\theta,\varphi})'$. Then there is an analytic $(m_1, n_1, p_1;\ m, n)$-phase function $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix}$ with $n_1 = p_1 = n$, such that*

$$\mathbf{C}_{K,\theta,\varphi}(l + l_1) = \mathscr{G}_\Xi(\mathscr{B}H^\infty_{l_1}(M_{m_1, n_1})).$$

Furthermore, if $l' < l$, then $\mathbf{C}_{K,\theta,\varphi}(l')$ is empty. Moreover one can take Ξ to be the phase function associated with the invariant subspace $\mathscr{M}_{K,\theta,\varphi}$ as in Theorem 2.2.

Proof. By Lemma 3.1 we know that the angle operator-graph correspondence sets up a one-to-one pairing between elements of $\mathbf{C}_{K,\theta,\varphi}(l')$ and shift-invariant negative subspaces of $\mathcal{M}_{K,\theta,\varphi}$ of codimension of most l' in a maximal negative subspace of $L^2(\mathbf{C}^m)\oplus\varphi^{-1}H^2(\mathbf{C}^n)$. By Lemma 1.1, such subspaces of $\mathcal{M}_{K,\theta,\varphi}$ cannot exist if $l'<l$, and for $l'\geqslant l$ coincide with invariant negative subspaces of $\mathcal{M}_{K,\theta,\varphi}$ of $\mathcal{M}_{K,\theta,\varphi}$-negative cosignature at most $l_1\equiv l'-l$. By Lemma 3.3, such subspaces of $\mathcal{M}_{K,\theta,\varphi}$ exist in abundance for $l'\geqslant l$; indeed if $\Xi=\begin{bmatrix}\alpha & \beta & \psi\\ \varkappa & \gamma & \omega\end{bmatrix}$ is the $(m_1, n_1, p_1; m, n)$-phase function associated with the invariant pseudo-regular subspace $\mathcal{M}_{K,\theta,\varphi}$ as in Theorem 2.2, then such subspaces are those of the form

$$\mathcal{G}=\left\{\begin{bmatrix}\alpha & \beta & \psi\\ \varkappa & \gamma & \omega\end{bmatrix}\begin{bmatrix}F & 0\\ I & 0\\ 0 & I\end{bmatrix}[\mathcal{K}\cap(H^\infty(\mathbf{C}^{n_1})\oplus H^\infty(\mathbf{C}^{p_1}))]\right\}^-,$$

where $\mathcal{K}$ is an invariant subspace of $H^2(\mathbf{C}^{n_1})\oplus H^2(\mathbf{C}^{p_1})$ of codimension at most l_1; since $P_{H^2(\mathbf{C}^n)\oplus\{0\}}\mathcal{K}$ therefore has codimension at most l_1 in $H^2(\mathbf{C}^n)\ominus\{0\}$, we must have that F is in $\mathcal{B}H^\infty_{l_1}(M_{m_1,n_1})$. Using the inclusion maps i and j defined above and identifying $\mathcal{K}$ as a subspace of $H^2(\mathbf{C}^{n_1+p_1})$, we can rewrite the form for $\mathcal{G}$ above as

$$\mathcal{G}=\left\{\begin{bmatrix}\alpha Fi^*+\beta i^*+\psi j^*\\ \varkappa Fi^*+\gamma i^*+\omega j^*\end{bmatrix}(\mathcal{K}\cap H^\infty(\mathbf{C}^{n_1+p_1}))\right\}^-.$$

For this negative subspace to have finite codimension in a maximal negative subspace of $H^2(\mathbf{C}^{m,n})$, we necessarily have $n_1+p_1=n$ and $(\varkappa Fi^*+\gamma i^*+\omega j^*)(e^{it})$ invertible for a.e. t. The angle operator associated with this subspace clearly is

$$H=(\alpha Fi^*+\beta i^*+\omega j^*)(\varkappa Fi^*+\gamma i^*+\omega j^*)^{-1}=\mathcal{G}_\Xi(F),$$

and is in $\mathcal{B}L^\infty(M_{m,n})$ since $\mathcal{G}$ is a negative subspace. Putting all the pieces together, we conclude that $C_{K,\theta,\varphi}(l+l_1)$ is the set $\mathcal{G}_\Xi(\mathcal{B}H^\infty_{l_1}(M_{m_1,n_1}))$ as claimed.

For this result to be useful it is crucial to be able to compute the negative cosignature of a space $(\mathcal{M}_{K,\theta,\varphi})'$ directly in terms of the given $K\in H^2(M_{m,n})$ and phases θ,φ. To do this let $\mathcal{H}$ be the subspace $L^2(\mathbf{C}^m)\ominus\theta H^2(\mathbf{C}^{m_1})$; if K is bounded we simply define $\Gamma_{\theta,\varphi}(K)^*\colon\mathcal{H}\to H^2(\mathbf{C}^n)$ by $\Gamma_{\theta,\varphi}(K)^*\colon f\to P_{\varphi\cdot H^2(\mathbf{C}^n)}K^*f$. If K is merely in $L^2(M_{m,n})$, use the same formula but insist as well that $K^*f\in L^2(\mathbf{C}^n)$. Note that $\Gamma_\theta(K)$ may be defined on all of $\mathcal{H}$ (and thus be bounded by the closed graph theorem) even if K is unbounded. Then one easily checks that

$$(\mathcal{M}_{K,\theta,\varphi})' = \begin{bmatrix} I \\ \Gamma_{\theta,\varphi}(K^*) \end{bmatrix} \mathcal{H}$$

From this the next lemma follows almost immediately.

LEMMA 3.5. *The negative signature l of a subspace $(\mathcal{M}_{K,\theta,\varphi})'$ is the dimension of the negative eigenspace of the self-adjoint operator $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ on $\mathcal{H} = L^2(\mathbf{C}^m) \ominus \theta H^2(\mathbf{C}^{m_1})$. Moreover, $\mathcal{M}_{K,\theta,\varphi}$ is regular if and only if $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ is invertible; and pseudo-regular if and only if $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ has closed range.*

Thus Theorem 3.4 combined with Lemma 3.5 not only proves that $C_{K,\theta,\varphi}(l')$ is nonempty if $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ has closed range and negative eigenspace of dimension at most l', but also effectively parameterizes the set $C_{K,\theta}(l')$, once we have a procedure for computing the associated phase function $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix}$. Later we shall illustrate how one can compute Ξ for a simple example. First we obtain the result on existence for the general case where $\mathcal{M}_{K,\theta,\varphi}$ may not be pseudo-regular.

THEOREM 3.6. *Suppose K is in $H^2(M_{m,n})$ and $\theta \in L^\infty(M_{m,k})$ and $\varphi \in L^\infty(M_{k',n})$ are phase functions. Then $C_{K,\theta,\varphi}(l)$ is nonempty if and only if the negative eigenspace of $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ has dimension at most l.*

Proof. Lemmas 1.1 and 3.1 give necessity of the dimension l condition immediately. If $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ is invertible (or even if only it has closed range), sufficiency follows from Theorem 3.4 and Lemma 3.5. To remove the invertibility assumption, we use an approximation argument of the type used by Adamjan, Arov and Kreĭn [1], [2]. Suppose $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ has negative eigenspace of dimension at most l and is not invertible. Replace the $L^2(\mathbf{C}^{m,n})$ inner product by $L^2(\mathbf{C}^{m,n_\varepsilon})$, where for $f \oplus g \in L^2(\mathbf{C}^m) \oplus L^2(\mathbf{C}^n) \cong L^2(\mathbf{C}^{m,n})$,

$$[f \oplus g, f \oplus g]_{L^2(\mathbf{C}^{m,n_\varepsilon})} = \langle f, f \rangle_{L^2(\mathbf{C}^m)} - \langle (1+\varepsilon) g, g \rangle_{L^2(\mathbf{C}^n)}.$$

With this new inner product, for $\varepsilon > 0$ sufficiently small, $\begin{bmatrix} I \\ \Gamma_{\theta,\varphi}(K)^* \end{bmatrix} \mathcal{H}$ is regular with negative signature l, and its $[\,,\,]_{L^2(\mathbf{C}^{m,n_\varepsilon})}$-orthogonal complement $\mathcal{M}^\varepsilon_{K,\theta,\varphi}$ contains a negative invariant subspace $\mathcal{G}_\varepsilon$ of codimension l in a $(L^2(\mathbf{C}^m) \oplus \varphi^* H^2(\mathbf{C}^n))$-maximal negative subspace. If we let $\mathcal{G}$ be any accumulation point of $\mathcal{G}_\varepsilon$ as $\varepsilon \searrow 0$, then the angle operator of $\mathcal{G}$ gives a function in $C_{K,\theta,\varphi}(l)$.

In applications it is sometimes useful to know that there is an element F in $C_{K,\theta,\varphi}(l)$ with boundary values $F(e^{it})$ which are isometries. The following is a fairly general sufficient condition.

COROLLARY 3.7. *Suppose K is in $H^2(M_{m,n})$, $\theta \in L^\infty(M_{m,k})$ and $\varphi \in L^\infty(M_n)$ are phases, and that $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ has closed range. Set l equal to the dimension of the negative eigenspace of $I - \Gamma_{\theta,\varphi}(K)\Gamma_{\theta,\varphi}(K)^*$ and assume $l < \infty$. Then if $m_1 \geq n_1$, $C_{K,\theta,\varphi}(l)$ contains an element with isometric boundary values.*

Proof. By Lemma 3.1, elements of $C_{K,\theta,\varphi}(l)$ correspond to invariant $\mathcal{M}_{K,\theta,\varphi}$--maximal negative subspaces of $\mathcal{M}_{K,\theta,\varphi}$; in this correspondence, isometric subspaces of $C_{K,\theta,\varphi}(l)$ correspond to invariant $(\mathcal{M}_{K,\theta,\varphi} \cap \mathcal{N}')$-maximal negative subspaces $(\mathcal{N} = \bigvee_{k\geq 0} S^k(\mathcal{M}_{K,\theta,\varphi} \cap \mathcal{M}'_{K,\theta,\varphi}))$ which are null spaces. Lemma 3.3 establishes a correspondence between invariant $(\mathcal{M}_{K,\theta,\varphi} \cap \mathcal{N}')$-maximal negative subspaces and invariant maximal negative subspaces of $H^2(\mathbf{C}^{m_1, n_1, p_1})$. Moreover, there exist invariant maximal negative subspaces of $H^2(\mathbf{C}^{m_1, n_1, p_1})$ which are null spaces if and only if $m_1 \geq n_1$. This establishes the corollary.

As an instructive special case, consider the scalar case ($m = n = 1$), and suppose $k = 1$. Then if $C_{K,\theta,1}(l)$ is nonempty, we must have either $m_1 = n_1 = 1$, $p_1 = 0$, or $m_1 = n_1 = 0$, $p_1 = 1$. In either case, *if $I - \Gamma_{\theta,1}(K)\Gamma_{\theta,1}(K)^*$ has closed range, then $C_{K,\theta,1}(l)$ contains a function of modulus 1 on the boundary if it contains any function at all.* Without the special topological assumption on the operator $I - \Gamma_{\theta,1}(K)\Gamma_{\theta,1}(K)^*$, it is known that this is no longer the case (see eg. [13]).

b. CONCRETE EXAMPLES

As an illustrative example of the above theory, consider the interpolation problem N-$P_l(\mathbf{z}, \mathbf{p}, \mathbf{q})$ mentioned at the beginning of this section for the case $N' = 0$. Set $C_{K,\theta,\varphi}(l)$ equal to $C_{K,\theta}(l)$ if $\varphi = I$, and similarly for $\mathcal{M}_{K,\varphi}$ and $\Gamma_\theta(K)$. While N-$P_l(\mathbf{z}, \mathbf{p}, \mathbf{q})$ corresponds to $C_{K,\theta}(l)$ for a certain choice of K and θ we need not actually compute them to see that $\mathcal{M}_{K,\theta}$ is the space $\mathcal{M}$ defined by

$$\mathcal{M} = \left\{ f \in H^2(\mathbf{C}^{m,n}) \;\middle|\; \left[\begin{bmatrix} p_j \\ q_j \end{bmatrix}, f(z_j) \right]_{\mathbf{C}^{m,n}} = 0 \text{ for } j = 1,2,\ldots,N \right\}.$$

Since the function $(1 - z\bar{w})^{-1}$ is a reproducing kernel for H^2, we have

$$\left[\begin{bmatrix} p_j \\ q_j \end{bmatrix}, f(z_j) \right]_{\mathbf{C}^{m,n}} = \left[(1 - z\bar{z}_j)^{-1} \begin{bmatrix} p_j \\ q_j \end{bmatrix}, f(z) \right]_{H^2(\mathbf{C}^m)},$$

which is 0 for $j = 1, \ldots, N$ if and only if f is in $\mathcal{M}$. Thus the space $\mathcal{M}'$ is spanned by the functions

$$(*) \qquad \left\{(1 - z\bar{z}_j)^{-1}\begin{bmatrix} p_j \\ q_j \end{bmatrix},\ j = 1, \ldots, N\right\}.$$

A typical element $k(z)$ of $\mathscr{M}'$ is

$$k(z) = \sum_{j=1}^{N} c_j(1 - z\bar{z}_j)^{-1}\begin{bmatrix} p_j \\ q_j \end{bmatrix}$$

from which we compute

$$[k, k]_{H^2(\mathbf{C}^{m,n})} = \sum_{l=1}^{N}\sum_{j=1}^{N} \bar{c}_l c_j(1 - z_l\bar{z}_j)^{-1}\left[\begin{bmatrix} p_j \\ q_j \end{bmatrix}, \begin{bmatrix} p_l \\ q_l \end{bmatrix}\right]_{\mathbf{C}^{m,n}} =$$

$$= \sum_{l=1}^{N}\sum_{j=1}^{N}\left\{\frac{\langle p_j, p_l\rangle_{\mathbf{C}^m} - \langle q_j, q_l\rangle_{\mathbf{C}^n}}{1 - \bar{z}_j z_l}\right\}\bar{c}_l c_j =$$

$$= \mathbf{c}^* \Lambda_{\{\mathbf{z}, \mathbf{p}, \mathbf{q}\}}\mathbf{c}$$

when $\mathbf{c} = [\bar{c}, \ldots, \bar{c}_N]^{\mathrm{T}}$. In other words, the Pick matrix is the Grammian of the basis $(*)$ for $\mathscr{M}$. Thus the negative signature of the space $\mathscr{M}'$ is precisely the number of negative eigenvalues of the Hermitian matrix $\Lambda_{\{\mathbf{z}, \mathbf{p}, \mathbf{q}\}}$. Therefore the matrix test for when $N\text{-}P_l(\mathbf{z}, \mathbf{p}, \mathbf{q})$ is nonempty mentioned at the beginning of this section is an immediate consequence of Theorem 3.6 and Lemma 3.5. Alternatively, one can check that the space $\mathscr{H} = H^2 \ominus \theta H^2$ is the span of

$$(**) \qquad \{(1 - z\bar{z}_j)^{-1} p_j \mid j = 1, \ldots, N\}$$

and $\Gamma_\theta(K)^*: (1 - z\bar{z}_j)^{-1}p_j \to (1 - z\bar{z}_j)^{-1}q_j$. Thus $\Lambda_{\{\mathbf{z}, \mathbf{p}, \mathbf{q}\}}$ can also be viewed as the matrix representation for $I - \Gamma_\theta(K)\Gamma_\theta(K)^*$ with respect to the basis $(**)$ for $\mathscr{H}$.

While the very geometric approach we take appears abstract, actually it serves very well to organize and simplify concrete computations. We now analyze the scalar case ($n = m = k = 1$) of $N\text{-}P_l(\mathbf{z}, \mathbf{p}, \mathbf{q})$ thoroughly. In this case the space $\mathscr{M}$ has the form

$$\mathscr{M} = \left\{\begin{bmatrix} f \\ g \end{bmatrix} \in H^2(\mathbf{C}^{1,1}) \mid w_j g(z_j) = f(z_j) \quad \text{for} \quad j = 1, \ldots, N\right\}$$

for N-tuples $\mathbf{z}=(z_1, \ldots, z_N)$ and $\mathbf{w}=(w_1, \ldots, w_N)$ of complex numbers. Note that $\mathscr{M}'$ is finite dimensional, so $\mathscr{M}$ is pseudo-regular. We shall show how to compute the phase Ξ associated with $\mathscr{M}$ as in Theorems 2.1 and 2.2 explicitly. We first consider the regular case ($\mathscr{M} \cap \mathscr{M}' = \{0\}$). The first step is to compute $\mathscr{L} \equiv \mathscr{M} \boxminus S\mathscr{M} =$

$\mathcal{M} \cap (S\mathcal{M})'$. One easily checks

$$(S\mathcal{M})' = (SH^2(\mathbf{C}^{1,1}))' + S\mathcal{M}'$$

and thus $(S\mathcal{M})'$ consists of the functions

$$f_{\alpha,a,b}(e^{it}) = \begin{bmatrix} a \\ b \end{bmatrix} + \sum_{j=1}^{N} \alpha_j e^{it}(1 - e^{it}\bar{z}_j)^{-1} \begin{bmatrix} 1 \\ \bar{w}_j \end{bmatrix}$$

for all a, b, α_j in $\mathbf{C}$. The functions in $\mathcal{L}$ satisfy in addition

$$0 = \left[f_{\alpha,a,b}, \frac{1}{1 - e^{it}\bar{z}_\nu} \begin{bmatrix} 1 \\ \bar{w}_\nu \end{bmatrix} \right]_{H^2(\mathbf{C}^{1,1})} =$$

$$= a - w_\nu b + \sum_{j=1}^{N} \alpha_j z_\nu \frac{(1 - \bar{w}_j w_\nu)}{1 - \bar{z}_j z_\nu}$$

for all $\nu = 1, \ldots, N$. So the prescription for arriving at a basis for $\mathcal{L}$ is to solve

$$\Lambda\alpha^1 = \begin{bmatrix} z_1^{-1} \\ \vdots \\ z_N^{-1} \end{bmatrix}, \quad \Lambda\alpha^2 = \begin{bmatrix} w_1 z_1^{-1} \\ \vdots \\ w_N z_N^{-1} \end{bmatrix},$$

where Λ is the Pick matrix $\Lambda = [(1 - \bar{w}_j w_k)/(1 - \bar{z}_j z_k)]$ (invertible, since $\mathcal{M}$ is regular). Define σ by

$$\sigma^s(e^{it}) = \sum_{j=1}^{N} \alpha_j^s \frac{e^{it}}{1 - \bar{z}_j e^{it}} \begin{bmatrix} 1 \\ \bar{w}_j \end{bmatrix} \quad (s = 1, 2).$$

Then the functions

$$f_1(e^{it}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sigma^1(e^{it}) \quad \text{and} \quad f_2(e^{it}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sigma^2(e^{it})$$

are a basis for $\mathcal{L}$.

The only remaining step is to [,] Gramm-Schmidt the vectors f_1, f_2 to obtain [,]-orthonormal positive and negative vectors l^+ and l^- on $\mathcal{L}$. Express these as $l^\pm = [l_1^\pm, l_2^\pm]^T$. Define

$$\Xi = \begin{pmatrix} l_1^+ & l_1^- \\ l_2^+ & l_2^- \end{pmatrix}.$$

It satisfies $\Xi H^2(\mathbf{C}^{1,1}) = \mathcal{M}$ and so by Theorem 3.4

$$F = (l_1^+ G + l_1^-)(l_2^+ G + l_2^-)^{-1}, \quad G \in \mathcal{B}H^\infty$$

gives all solutions to $(N\text{-}P)_l$ where l is the number of negative eigenvalues of Λ.

The case not covered is when $\mathcal{M} \cap \mathcal{M}' \neq \{0\}$, or Λ is not invertible. To settle it, note that $\mathcal{M} \cap \mathcal{M}'$ consists of all vectors $g_\alpha(e^{it}) = \sum_{j=1}^{N} \alpha_j(1 - e^{it}\bar{z}_j)^{-1} \begin{bmatrix} 1 \\ \bar{w}_j \end{bmatrix}$ such that

$$0 = \left[g_\alpha, \frac{1}{(1 - e^{it}\bar{z}_\nu)} \begin{bmatrix} 1 \\ \bar{w}_\nu \end{bmatrix} \right]_{H^2(C^{1,1})} =$$

$$= \sum_{j=1}^{N} \alpha_j \frac{1 - w_\nu \bar{w}_j}{1 - z_\nu \bar{z}_j},$$

that is $\boldsymbol{\alpha}$ is in the kernel of the Pick matrix Λ. For this special case ($m = n = 1$), $p_1 = 1 \Rightarrow m_1 = n_1 = 0$ and thus $\mathcal{M} \cap \mathcal{N}' = \mathcal{N}$ where

$$\mathcal{N} = \bigvee_{j \geqslant 0} \{e^{ijt} g_\alpha \mid \boldsymbol{\alpha} \in \ker \Lambda\},$$

and the solution F of $(N\text{-}P)_l$ (where l is the number of (strictly) negative eigenvalues of Λ) is unique. We obtain this unique inner function F as the quotient $\Psi_1 \Psi_2^{-1}$ of the components of a representation $\mathcal{N} = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} H^2(C^1)$. For example, take

$$F = \left[\sum_{j=1}^{N} \alpha_j(1 - e^{it}z_j)^{-1} \right] \left[\sum_{j=1}^{N} \alpha_j(1 - e^{it}\bar{z}_j)^{-1} \bar{w}_j \right]^{-1}$$

where $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_N]^T$ is any vector in the kernel of Λ. Thus computing the unique solution of $(N\text{-}P)_l$ is reduced to an eigenvalue problem, much in the spirit of Hintzman's work [18] on Hankel matrices. Higher order interpolation problems can be handled in much the same way by using the Pick matrix as in [28] or [5].

c. L^∞ APPROXIMATION FROM H^∞

A second special case of interest is when both phase functions θ and φ are the identity. This corresponds to supremum norm approximation of a given function by H^∞ functions. We are given an $L^2(M_{m,n})$ function K and wish to know when there exists a function F in the set

$$A_K(l) \equiv \{K + H_l^2(M_{m,n})\} \cap \mathcal{B}L^\infty(M_{m,n}) \; (= C_{K,I,I}(l))$$

and if possible, to parameterize $A_K(l)$ when it is nonempty. The set $A_K(l)$ can be regarded as the set of all error functions of L^∞ norm less than one obtainable by approximating K with the error functions in $H_l^2(M_{m,n})$. In this case the appropriate operator $\Gamma_{K,I,I}$ is the Hankel operator

$$\Gamma_{K,I,I} = \mathcal{H}_K : H^2(C^n) \to H^2(C^m)^\perp$$

defined by

$$\mathscr{H}_K : k \to P_{H^2(\mathbf{C}^m)^\perp}(Fk)$$

for $k \in H^2(\mathbf{C}^n)$. The following is simply a restatement of Theorems 3.4 and 3.6 for the special case $\theta = \varphi = I$.

THEOREM 3.8. *Suppose K is in $L^2(M_{m,n})$. Then the set $A_K(l)$ is non-empty if and only if the self-adjoint operator $I - \mathscr{H}_K \mathscr{H}_K^*$ has negative eigenspace of dimension at most l. Moreover, if $I - \mathscr{H}_K \mathscr{H}_K^*$ has closed range and l is the dimension of its negative eigenspace, then there is an $(m_1, n_1, p_1;\ m, n)$-phase function $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix}$ with $n_1 + p_1 = n$ such that*

$$A_K(l + l_1) = \mathscr{G}_\Xi(\mathscr{B}H^\infty_{l_1}(M_{m,n_1})).$$

d. BOUNDARY INTERPOLATION

Another application of invariant subspace techniques here are to boundary interpolation. This concerns the extension of a function f in L^∞_A (where A is a subset of the unit circle of positive Lebesgue measure) to one in $\mathscr{B}H^\infty$. Our treatment is in the spirit of a proof given by Rosenblum and Rovnyak [28] adapted to our setting. We actually give a result considerably more general than the classical one or the one in [28]. Suppose p and q are matrix-valued functions on A, namely, $p \in L^\infty_A(M_{n,k})$ and $q \in L^\infty_A(M_{m,k})$.

THEOREM 3.9. *There is a function F in $\mathscr{B}H^\infty_{l'}(M_{m,n})$ in the set*

$$I(p, q, A) \equiv \{F \in L^2(M_{m,n}) : F(e^{it})p(e^{it}) = q(e^{it}) \text{ for almost all } e^{it} \text{ in } A\}$$

if and only if the dimension l of the negative spectral space of the self-adjoint operator

$$\Lambda_{p,q,A} = M^*_{q\chi_A} P_{H^2(\mathbf{C}^n)} M_{p\chi_A} - M^*_{q\chi_A} P_{H^2(\mathbf{C}^m)} M_{q\chi_A}$$

acting on $L^2(\mathbf{C}^k)$ is at most l'. If $p(e^{it})$ is rank n on a set of positive measure, then the solution F with $l' = l$ is unique (whenever it exists) by analytic continuation. Here χ_A is the characteristic function of the set A and M_F is the operator of multiplication by the function F.

Proof. Set

$$\mathscr{M} = \left\{ f \in H^2(\mathbf{C}^{m,n}) \,\middle|\, \left[f(e^{it}), \begin{bmatrix} p \\ q \end{bmatrix} (e^{it})x \right]_{\mathbf{C}^{m,n}} = 0 \text{ for all } x \text{ in } \mathbf{C}^k \text{ and a.e. } e^{it} \text{ in } A \right\}.$$

Then $\mathcal{M}$ is an invariant subspace of $H^2(\mathbf{C}^{m,n})$ and if $F \in \mathcal{B}H^\infty(M_{m,n})$ satisfies $F(e^{it})^* p(e^{it}) = q(e^{it})$ a.e. on A, then $\begin{bmatrix} F \\ I \end{bmatrix} H^2(\mathbf{C}^n)$ is an invariant $H^2(\mathbf{C}^{m,n})$-maximal negative subspace of $\mathcal{M}$. Conversely, if $\mathcal{N}$ is an invariant $H^2(\mathbf{C}^{m,n})$-maximal negative subspace of $\mathcal{M}$, then its angle operator gives rise to an $F \in \mathcal{B}H^\infty(M_{m,n})$ satisfying $F(e^{it})^* p(e^{it}) = q(e^{it})$ a.e. on A. Analogous statements apply for solutions F in $\mathcal{B}H^\infty_{l'}(M_{m,n})$ and invariant negative subspaces of $\mathcal{M}$ having codimension at most l' in a $H^2(\mathbf{C}^{m,n})$-negative subspace. By Lemma 1.1, the existence of a solution F of $I(p, q, A)$ in $\mathcal{B}H^\infty_{l'}(M_{m,n})$ then implies that the negative signature l of $\mathcal{M}'$ is at most l'. Conversely, if $l_1 \equiv l' - l \geqslant 0$, then, again by Lemma 1.1, invariant negative subspaces of $\mathcal{M}$ with $\mathcal{M}$-negative cosignature l_1 are exactly the invariant negative subspaces with negative cosignature l' which are contained in $\mathcal{M}$. If $\mathcal{M}$ is pseudo-regular, such subspaces exist in abundance by Lemma 3.3; otherwise, we can perturb the metric and use an approximation argument as in the proof of Theorem 3.6 to still get the existence assertion. Thus Theorem 3.8 follows once we establish that the negative signature of the space $\mathcal{M}'$ is equal to the dimension of the negative eigenspace of $\Lambda_{p,q,A}$.

To see this, observe that an equivalent way to define $\mathcal{M}$ is

$$\mathcal{M} = \left\{ f \in H^2(\mathbf{C}^{m,n}) \,\middle|\, \left[f, P_{H^2(\mathbf{C}^{m,n})} \begin{bmatrix} p \\ q \end{bmatrix} \chi_A \varphi \right]_{H^2(\mathbf{C}^{m,n})} = 0 \text{ for all } \varphi \in L^2(\mathbf{C}^k) \right\}.$$

From this representation, it is clear that

$$\mathcal{M}' = \left\{ P_{H^2(\mathbf{C}^{m,n})} \begin{bmatrix} p \\ q \end{bmatrix} \chi_A \varphi \,\middle|\, \varphi \in L^2(\mathbf{C}^k) \right\}.$$

Thus we see that $\mathcal{M}'$ is parameterized by $L^2(\mathbf{C}^k)$ and the self-adjoint operator $\Lambda_{p,q,A}$ induces the inner product on $L^2(\mathbf{C}^k)$ equivalent to the $H^2(\mathbf{C}^{m,n})$-inner product restricted to $\mathcal{M}'$. Therefore the negative signature of $\mathcal{M}'$ is the dimension of the negative eigenspace of $\Lambda_{p,q,A}$ as desired.

If p is full rank on a set of positive measure, then $P_{H^2(\mathbf{C}^m)}\{p\chi_A \varphi \mid \varphi \in L^2(\mathbf{C}^k)\}$ is dense in $H^2(\mathbf{C}^m)$. This implies that if f_1 and f_2 are two elements of $\mathcal{M}$ with $P_{\{0\}\oplus H^2(\mathbf{C}^n)} f_1 = P_{\{0\}\oplus H^2(\mathbf{C}^n)} f_2$, then in fact $f_1 = f_2$. This implies that $\mathcal{M}$ is a graph space $\left(\mathcal{M} = \begin{bmatrix} H \\ I \end{bmatrix} \mathrm{Dom}(H) \text{ where } H : \mathrm{Dom}(H) \subset H^2(\mathbf{C}^n) \to H^2(\mathbf{C}^m) \text{ is a closed multiplication operator} \right)$ and the $\mathcal{M}$-maximal negative subspace $\mathcal{G}$ for $\mathcal{M}$ is unique $\left(\mathcal{G} = \begin{bmatrix} H \\ I \end{bmatrix} \mathcal{D}_0 \text{ where } \mathcal{D}_0 = \{x \in \mathrm{Dom}(H) \mid \|Hx\| \leqslant \|x\|\} \right)$. This proves the uniqueness assertion in Theorem 3.8 as well.

We conclude this section with a converse to Theorem 3.4. Theorem 3.4 characterized a generalized interpolation set $C_{K,\theta,\varphi}(l+l_1)$ (for the pseudo-regular case) as the image under a certain linear fractional map $\mathcal{G}_\Xi$ of a function disk $\mathcal{B}H_l^\infty(M_{m_1,n_1})$. The phase function Ξ and the index l in principle can be computed directly from K, θ and φ. Conversely, in the next theorem we start with the phase function Ξ and indicate how one can compute K, θ, φ and l directly from Ξ. This result thus computes the image of $\mathcal{B}H_l^\infty(M_{m_1,n_1})$ under a linear fractional map $\mathcal{G}_\Xi$ and also gives a cleaner less computational proof of a result of Helton [17].

THEOREM 3.10. *Let* $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix}$ *be a* $(m_1, n_1, p_1;\ m, n)$*-phase function with* $n_1+p_1=n$. *Suppose also that*

i) *the closure of* $\varkappa H^\infty(\mathbf{C}^{m_1}) + \gamma H^\infty(\mathbf{C}^{n_1}) + \omega H^\infty(\mathbf{C}^{p_1})$ *is a full range simply invariant subspace of* $L^2(\mathbf{C}^n)$,

ii) *the closure of* $\gamma H^\infty(\mathbf{C}^{n_1}) + \omega H^\infty(\mathbf{C}^{p_1})$ *has finite codimension in the closure of* $\varkappa H^\infty(\mathbf{C}^{m_1}) + \gamma H^\infty(\mathbf{C}^{n_1}) + \omega H^\infty(\mathbf{C}^{p_1})$, *and*

iii) $\mathcal{M} = [\Xi \cdot H^\infty(\mathbf{C}^{m_1+n_1+p_1})]$ *is a simply invariant subspace of* $L^2(\mathbf{C}^{m+n})$. *Then*

$$\mathcal{G}_\Xi(\mathcal{B}H^\infty(M_{m_1,n_1})) = C_{K,\theta,\varphi}(l)$$

where

a) $\varphi \in \mathcal{B}L^\infty(M_n)$ *is a phase function such that*

$$[P_{\{0\}\oplus L^2(\mathbf{C}^n)}\mathcal{M}]^- = \varphi^* H^2(\mathbf{C}^n);$$

b) $\theta \in \mathcal{B}L^\infty(M_{m,k})$ *is a phase function such that*

$$\mathcal{M} \cap [H^2(\mathbf{C}^m) \oplus \{0\}] = \theta H^2(\mathbf{C}^k) \oplus \{0\};$$

c) K *is any* $L^2(M_{m,n})$*-function such that*

$$\begin{bmatrix} K \\ I \end{bmatrix} \varphi^* H^\infty(\mathbf{C}^n) \subset \mathcal{M}$$

and

d) l *is the codimension of* $\{\gamma H^\infty(\mathbf{C}^{n_1}) + \omega H^\infty(\mathbf{C}^{p_1})\}^-$ *as a subspace of* $\{\varkappa H^\infty(\mathbf{C}^{m_1}) + \gamma H^\infty(\mathbf{C}^{n_1}) + \omega H^\infty(\mathbf{C}^{p_1})\}^-$.

Proof. By the classical Beurling-Lax theorem there exist phase functions θ and φ as prescribed in a) and b). Thus

$$\theta H^2(\mathbf{C}^k \oplus \{0\}) \subset \mathcal{M} \subset L^2(\mathbf{C}^m) \oplus \varphi^* H^2(\mathbf{C}^n).$$

Consider the space $\mathscr{K} \equiv L^2(\mathbf{C}^m) \oplus \varphi^* H^2(\mathbf{C}^n)$ as a Kreĭn space with inner product inherited from $L^2(\mathbf{C}^{m,n})$. By Lemma 3.3, invariant maximal negative subspaces of $H^2(\mathbf{C}^{m_1, n_1, p_1})$ match up with $\mathscr{M}$-maximal negative subspaces in $\mathscr{K}$ via multiplication by Ξ. A particular maximal negative subspace of

$$H^2(\mathbf{C}^{m_1, n_1, p_1}) \cong H^2(\mathbf{C}^{m_1}) \oplus H^2(\mathbf{C}^{n_1}) \oplus H^2(\mathbf{C}^{p_1})$$

is

$$\mathscr{N}_0 = \{0\} \oplus H^2(\mathbf{C}^{n_1}) \oplus H^2(\mathbf{C}^{p_1}).$$

The $\mathscr{M}$-maximal negative subspace in K corresponding to $\mathscr{N}_0$ as above is the subspace $\{\gamma H^\infty(\mathbf{C}^{n_1}) + \omega H^\infty(\mathbf{C}^{p_1})\}^-$. By d) and the definition of $\mathscr{K}$ and φ, the codimension of this space in a $\mathscr{K}$-maximal negative subspace is l, which is finite by condition ii). By Lemma 1.1 *any* $\mathscr{M}$-maximal negative subspace has negative cosignature l with respect to $\mathscr{K}$, or equivalently, $\mathscr{M}' \equiv \mathscr{K} \boxminus \mathscr{M}$ has negative signature equal to l.

We are now ready to show that a function K as prescribed in c) exists. Let $\mathscr{H}$ equal to $L^2(\mathbf{C}^m) \ominus \theta H^2(\mathbf{C}^k)$ and let $\mathscr{D} \subset H^2(\mathbf{C}^n)$ be the linear manifold $\mathscr{D} \equiv P_{\{0\} \oplus H^2(\mathbf{C}^m)} \mathscr{M}$. Thus $\mathscr{D}$ is dense in $\varphi^* H^2(\mathbf{C}^n)$ by definition. Then for each f in $\mathscr{D}$ there is unique Xf in $\mathscr{H}$ such that $Xf \oplus f$ is in $\mathscr{M}$. This defines a closed operator $X: \mathscr{D} \to \mathscr{H}$ such that

$$\mathscr{M} = \begin{bmatrix} X \\ I \end{bmatrix} \mathscr{D} + [\theta H^2(\mathbf{C}^k) \oplus \{0\}].$$

From this representation it is easy to check that

$$\mathscr{M}' = \begin{bmatrix} I \\ X^* \end{bmatrix} \mathscr{E}$$

where $\mathscr{E}$ is the domain of X^* and is dense in $\mathscr{H}$. We saw above that $\mathscr{M}'$ has negative signature $l < \infty$; it follows that the self-adjoint operator $I - XX^*$ has a negative eigenspace of dimension at most l, and hence must be bounded. Therefore X is bounded and the linear manifold $\mathscr{D}$ is in fact all of $\varphi^* H^2(\mathbf{C}^n)$.

Let X_0 be the restriction $X \mid \varphi^* \mathbf{C}^n$ of the operator X to the subspace $\varphi^* \mathbf{C}^n$ of $\varphi^* H^2(\mathbf{C}^n)$, and define an $M_{m,n}$-valued function K by

$$K(e^{it_0})x = X_0(\varphi^* \cdot \varphi(e^{it_0}) x_0)(e^{it_0})$$

for x in $\mathbf{C}^n$. Then K is an $L^2(M_{m,n})$-valued function which also satisfies

$$K(e^{it})\varphi(e^{it})^* x = (X_0 \varphi^* x)(e^{it})$$

for x in $\mathbf{C}^n$, and thus $\begin{bmatrix} K \\ I \end{bmatrix} \varphi^* \mathbf{C}^n \subset \mathcal{M}$. By the invariance of the space $\mathcal{M}$,

$$\begin{bmatrix} K \\ I \end{bmatrix} \varphi^* H^\infty(\mathbf{C}^n) \subset \mathcal{M}.$$

This establishes the existence of a function K as in c).

Now suppose K is any function as in c). Then if F is any function in $H^2(M_{m,n})$ such that $\begin{bmatrix} F \\ I \end{bmatrix} \varphi^* H^\infty(\mathbf{C}^n) \subset \mathcal{M}$, then

$$\begin{bmatrix} F - K \\ 0 \end{bmatrix} \varphi^* H^\infty(\mathbf{C}^n) \subset \begin{bmatrix} F \\ I \end{bmatrix} \varphi^* H^\infty(\mathbf{C}^n) + \begin{bmatrix} K \\ I \end{bmatrix} \varphi^* H^\infty(\mathbf{C}^n) \subset \mathcal{M},$$

and therefore $F - K \in \Theta H^2(M_{m,n})\varphi$. One can further check that $\mathcal{M}$ is spanned by such subspaces and therefore $\mathcal{M} = \mathcal{M}_{K,\theta,\varphi}$. That $\mathcal{G}_\Xi(\mathcal{B}H^\infty(M_{m_1,n_1})) = C_{K,\theta,\varphi}(l)$ now follows exactly as in the proof of Theorem 3.4.

4. THE COMMUTANT LIFTING THEOREM

In this section we indicate how the work of Arsene, Ceaușescu and Foiaș [4] on parametrizing the set of all contractive intertwining dilations of a given contraction intertwining two contractions can be put in the framework of this paper. Furthermore we provide a more general lifting theorem which is the abstract analogue of H_l^∞-interpolation. The fact that commutant lifting is intimately connected to interpolation is due to Sarason [29]. Thus the reader who is already familiar with this connection probably already seen how our development will unfold. For this reason we shall be brief.

Following the notation of [4], let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be Hilbert spaces, and $\mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})$ be the set of all (bounded linear) operators from $\mathcal{H}$ to $\tilde{\mathcal{H}}$; when $\mathcal{H} = \tilde{\mathcal{H}}$, we abbreviate $\mathcal{L}(\mathcal{H}, \mathcal{H})$ to $\mathcal{L}(\mathcal{H})$. We fix two contractions $T \in \mathcal{L}(\mathcal{H})$ and $\tilde{T} \in \mathcal{L}(\mathcal{H})$, and let $U \in \mathcal{L}(\mathcal{K})$ and $\tilde{U} \in \mathcal{L}(\tilde{\mathcal{K}})$ be their minimal isometric dilations. (See [25] for the geometric structure of isometric dilations.) It is known that

$$\mathcal{K} = \mathcal{H} \oplus M_+(\mathcal{L})$$

where $\mathcal{L} = [(U - T)\mathcal{H}]^-$ and $M_+(\mathcal{L}) = \bigoplus_0^\infty U^n \mathcal{L}$, and similarly, $(\tilde{\mathcal{K}} = \tilde{\mathcal{H}} \oplus M_+(\tilde{\mathcal{L}}))$. Consider a fixed operator $A \in \mathcal{I}(\tilde{T}, T)$ (i.e. $A \in \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}})$ and

$\tilde{T}A = AT$; we say that A intertwines T and $\tilde{T}$). An intertwining dilation of A is an operator $A_\infty \in \mathscr{L}(\mathscr{K}, \tilde{\mathscr{K}})$ such that $A_\infty \in \mathscr{I}(\tilde{U}, U)$ and $\tilde{P}A_\infty = AP$, where $P: \mathscr{K} \to \mathscr{H}$ and $\tilde{P}: \tilde{\mathscr{K}} \to \tilde{\mathscr{H}}$ are the orthogonal projections. When $\|A\| \leqslant 1$, it is the content of the commutant lifting theorem that there is a *contractive intertwining dilation* A_∞ of A (i.e. $\|A_\infty\| \leqslant 1$, $U\tilde{A}_\infty = A_\infty U$ and $\tilde{P}A_\infty = AP$). The goal of the work of Arsene, Ceauşescu and Foiaş is to describe via an explicit parameterization the set CID(A) of *all* contractive intertwining dilations of A.

Such a parameterization can be obtained via a slightly more abstract version of Theorems 2.1 and 2.2 as follows. Let $\hat{\mathscr{K}} = \tilde{\mathscr{K}} \boxplus \mathscr{K}$ be the Kreĭn space with inner product

$$[\tilde{k} \boxplus k, \tilde{k} \boxplus k]_{\hat{\mathscr{K}}} = [\tilde{k}, \tilde{k}]_{\tilde{\mathscr{K}}} - [k, k]_{\mathscr{K}}$$

and define a subspace $\mathscr{M}$ ($= \mathscr{M}(T, \tilde{T}; U, \tilde{U}; A)$) by

$$\mathscr{M} = \begin{bmatrix} A \\ I \end{bmatrix} \mathscr{H} \boxplus \begin{bmatrix} M_+(\tilde{\mathscr{L}}) \\ M_+(\mathscr{L}) \end{bmatrix}.$$

If we let $\hat{U}$ be the operator $\begin{bmatrix} \tilde{U} & 0 \\ 0 & U \end{bmatrix}$ on $\hat{\mathscr{K}}$, then $\hat{U}$ is isometric in the $[\,,\,]_{\hat{\mathscr{K}}}$-inner product, and since

$$\hat{U}\begin{bmatrix} A \\ I \end{bmatrix} = \begin{bmatrix} \tilde{U} & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} A \\ I \end{bmatrix} = \begin{bmatrix} \tilde{T} & 0 \\ 0 & T \end{bmatrix}\begin{bmatrix} A \\ I \end{bmatrix} + \begin{bmatrix} \tilde{U} - \tilde{T} & 0 \\ 0 & U - T \end{bmatrix}\begin{bmatrix} A \\ I \end{bmatrix} =$$

$$= \begin{bmatrix} A \\ I \end{bmatrix} T + \begin{bmatrix} (\tilde{U} - \tilde{T})A \\ U - T \end{bmatrix},$$

we see that $\mathscr{M}$ is invariant for $\hat{U}$. Furthermore, $A_\infty \in \mathscr{L}(K, \tilde{K})$ intertwines $\tilde{U}$ and U ($\tilde{U}A_\infty = A_\infty U$) $\Leftrightarrow$ its graph $\begin{bmatrix} A_\infty \\ I \end{bmatrix} \mathscr{K}$ is invariant under $\hat{U}$, $\tilde{P}A_\infty = AP \Leftrightarrow$ the graph $\begin{bmatrix} A_\infty \\ I \end{bmatrix} \mathscr{K}$ is contained in $\mathscr{M}$, and $\|A_\infty\| \leqslant 1 \Leftrightarrow$ the graph $\begin{bmatrix} A_\infty \\ I \end{bmatrix} \mathscr{K}$ is a negative subspace of $\hat{K}$. As we have seen in the previous sections, a $\hat{\mathscr{K}}$-maximal negative subspace is always the graph of a contraction operator. We have obtained

THEOREM 4.1. *The angle operator-graph correspondence establishes a one-to-one correspondence between the contractive intertwining dilations A_∞ of A (i.e. the set* CID(A)) *and $\hat{U}$-invariant, $\hat{\mathscr{K}}$-maximal negative subspaces of $\mathscr{M} = \mathscr{M}(T, \tilde{T}; U, \tilde{U}; A)$ as defined above.*

We next obtain the abstract analogue of H_l^∞-interpolation. We say $A_\infty \in \mathrm{CID}_l(A)$ if there is some U-invariant subspace $\psi\mathscr{K}$ of codimension at most l as a subspace of $\mathscr{K}$ such that $A_\infty\colon \psi\mathscr{K} \to \tilde{\mathscr{K}}$ satisfies $\|A_\infty\| \leqslant 1$, $A_\infty(U \mid \psi\mathscr{K}) = \tilde{U}A_\infty$ and $\tilde{P}A_\infty = AP \mid \psi\mathscr{K}$. (Thus $\tilde{P}A_\infty \mid \mathscr{H}$ agrees with A on a T-invariant subspace of codimension at most l.) Using the same analysis as above, we obtain the following refinement of Theorem 4.1.

THEOREM 4.1a. *The angle operator-graph correspondence establishes a one-to-one correspondence between the set* $\mathrm{CID}_l(A)$ *and* $\hat{U}$*-invariant negative subspaces of* $\mathscr{M} = \mathscr{M}(T, \tilde{T}\colon U, \tilde{U}; A)$ *of codimension at most* l *in a* $\hat{\mathscr{K}}$*-maximal negative subspace of* $\hat{\mathscr{K}}$.

Thus by Lemma 1.1, a necessary condition that the set $\mathrm{CID}_l(A)$ be nonempty is that the negative signature of $\mathscr{M}'$ be at most l, and conversely, once we establish that there exist $\mathscr{M}$-maximal negative subspaces of $\mathscr{M}$ which are also $\hat{U}$-invariant, we shall see that this condition is sufficient as well. Now it is easily checked that

$$\mathscr{M}' = \begin{bmatrix} I \\ A^* \end{bmatrix} \tilde{\mathscr{H}}$$

and thus the negative signature of $\mathscr{M}'$ is the dimension of the negative spectral subspace for the self-adjoint operator $I - AA^*$ on $\tilde{\mathscr{H}}$. Thus, modulo the gap mentioned above, we obtain the following generalization of the existence part of the Sz.-Nagy--Foiaş lifting theorem (which corresponds to the $l = 0$ case of the following).

THEOREM 4.2. *The set* $\mathrm{CID}_l(A)$ *is nonempty if and only if the negative spectral subspace of the self-adjoint operator* $I - AA^*$ *on* $\tilde{\mathscr{H}}$ *has dimension of at most* l.

In contradistinction to the previous sections of this paper, it may happen here that $\hat{U} \mid \mathscr{M}$ is *not* simply invariant (i.e. it may happen that $\bigcap_{k \geqslant 0} U^k \mathscr{M} \supsetneqq \{0\}$). For example, one may simply take $A = 0$ and let T be any completely nonunitary contraction which is not $C_{\cdot 0}$ (see [25]). For such an isometry we must generalize Theorems 1.1 and 1.2 and develop a Wold decomposition for the $[\,,\,]_{\hat{\mathscr{K}}}$-geometry.

Let $\hat{\mathbf{K}}$ be a Kreĭn space and suppose that $\hat{U}$ is an isometry on $\hat{\mathbf{K}}$ in the $[\,,\,]_{\hat{\mathbf{K}}}$-inner product. We shall say that $\hat{U}$ is *fundamentally reducible* if there is a fundamental symmetry J on $\hat{\mathbf{K}}$ which commutes with $\hat{U}$ (see [11]): equivalently $\hat{U}$ is fundamentally reducible if there is a decomposition $\hat{\mathbf{K}} = \tilde{\mathbf{K}} [+] \mathbf{K}$ of $\hat{\mathbf{K}}$ as the $[\,,\,]_{\hat{\mathbf{K}}}$-orthogonal direct sum of a regular positive subspace $\tilde{\mathbf{K}}$ and a regular negative subspace $\mathbf{K}$ such that with respect to this decomposition $\hat{U} = \begin{bmatrix} \tilde{U} & 0 \\ 0 & U \end{bmatrix}$ is diagonal. Thus such a U

is an isometry in both the Kreĭn space $[\ ,\ ]_{\mathbf{K}}$ and the Hilbert space $\langle \cdot, \cdot \rangle_J$ inner products. The next result is an extension of our Beurling-Lax theorem which handles isometries which are not necessarily shifts.

THEOREM 4.3. *Suppose* $\hat{U} = \begin{bmatrix} \tilde{U} & 0 \\ 0 & U \end{bmatrix}$ *is a fundamentally reducible isometry on the Kreĭn space* $\hat{\mathbf{K}} = \tilde{\mathbf{K}} \boxplus \mathbf{K}$ *and* $\mathbf{M}$ *is a pseudo-regular* $\hat{U}$*-invariant subspace of* $\hat{\mathbf{K}}$. *Set* $\mathbf{N} = \bigvee_{j \geqslant 0} \hat{U}^j(\mathbf{M} \cap \mathbf{M}')$. *Then there are a densely defined* $([\ ,\ ]_{\hat{\mathbf{K}}_1 \boxplus \hat{\mathbf{K}}_0}, [\ ,\ ]_{\hat{\mathbf{K}}})$*-isometry* $\Xi : \mathscr{D}(\Xi)\ (\subset \hat{\mathbf{K}}_1 \boxplus \hat{\mathbf{K}}_0) \to \hat{\mathbf{K}}$, *and a fundamentally reducible* $[\ ,\ ]_{\hat{\mathbf{K}}_1 \boxplus \hat{\mathbf{K}}_0}$*-isometry* $\begin{bmatrix} \hat{U}_1 & 0 \\ 0 & \hat{U}_0 \end{bmatrix}$ *on* $\hat{\mathbf{K}}_1 \oplus \hat{\mathbf{K}}_0$ *such that*

$$\Xi\left(\begin{bmatrix} \hat{U}_1 & 0 \\ 0 & \hat{U}_0 \end{bmatrix}\Big| \mathscr{D}(\Xi)\right) = \hat{U}\Xi$$

and

$$\mathbf{M} \cap \mathbf{N}' = (\Xi \cdot \mathscr{D}(\Xi))^{-}.$$

Here $\hat{\mathbf{K}}_1$ *is a Kreĭn space,* $\hat{\mathbf{K}}_0$ *is a Hilbert space and*

$$[k_1 \oplus k_0,\ \ k \oplus k_0]_{\hat{\mathbf{K}}_1 \boxplus \hat{\mathbf{K}}_0} = [k_1, k_1]_{\hat{\mathbf{K}}_1}$$

and

$$[k_1 \oplus k_0, k_1 \oplus k_0]_{\hat{\mathbf{K}}_1 \oplus \hat{\mathbf{K}}_0} = [k_1, k_1]_{\hat{\mathbf{K}}_1} + (k_0, k_0)_{\hat{\mathbf{K}}_0}.$$

Proof. As in the proof of Theorem 2.2, we get a generalized Wold decomposition for $\mathbf{M} \cap \mathbf{N}'$:

$$\mathbf{M} \cap \mathbf{N}' = \{\bigvee_{n \geqslant 0} \hat{U}^n \mathbf{L} + \bigcap_{n \geqslant 0} U^n \mathbf{M}\}^{-}$$

where $\mathbf{L} = \mathbf{M} \cap (\hat{U}\mathbf{M})'$. (Note that $\bigvee_{n \geqslant 0} \hat{U}^n \mathbf{L}$ and $\bigcap_{n \geqslant 0} \hat{U}^n \mathbf{M}$ are $[\ ,\ ]$-orthogonal, but may have nontrivial intersection.) Since $\mathbf{M}$ is pseudo-regular, $\mathbf{L}$ is also, and hence $\mathbf{L}$ as a $[\ ,\ ]$-orthogonal direct sum decomposition $\mathbf{L} = \mathbf{L}_1 \boxplus \mathbf{L}_0$ where $\mathbf{L}_1$ is regular and $\mathbf{L}_0$ is a null space. The space $\mathbf{M} \cap \mathbf{N}'$ has the space $\mathbf{N}$ as its isotropic subspace; $\mathbf{N}$ in turn has a Hilbert space Wold decomposition $\mathbf{N} = [\bigoplus_{n \geqslant 0} \hat{U}^n(\mathbf{N} \ominus \hat{U}^n \mathbf{N})] \oplus \oplus \bigcap_{n \geqslant 0} \hat{U}^n \mathbf{N} \equiv \mathbf{N}_s \oplus \mathbf{N}_u$. Note $\mathbf{N}_u$ is contained in $\bigcap_{n \geqslant 0} \hat{U}^n \mathbf{M}$ and is reducing for $\hat{U}$ and thus $\mathbf{M}_u = \bigcup_{n \geqslant 0} \hat{U}^n \mathbf{M} \ominus \mathbf{N}_u$ is also reducing for U. It is not difficult now to see that an equivalent form for the representation of $\mathbf{M} \cap \mathbf{N}'$ above is

$$\mathbf{M} \cap \mathbf{N}' = \{(\bigvee_{n \geqslant 0} \hat{U}^n \mathbf{L}_1) + (\mathbf{N}_s \boxplus \mathbf{N}_u \boxplus \mathbf{M}_u)\}^{-} \tag{4.1}$$

where each summand is invariant for $\hat{U}$. The theorem follows once we show that each summand has a representation $\{\Xi \cdot \mathscr{D}(\Xi)\}^-$ for some phase function Ξ. The summand $\bigvee_{n\geq 0} \hat{U}^n \mathbf{L}_1$ is handled as in Theorem 2.1; $\mathbf{N}_s$ and $\mathbf{N}_u$ are handled easily in a manner similar to that in the proof of Theorem 2.2, since they are null spaces. It remains only to consider $\mathbf{M}_u$.

The basic structural property of $\mathbf{M}_u$ is that it is a reducing (in the Hilbert space sense) subspace for the Hilbert space unitary operator $\hat{U}$ which is also nondegenerate in the indefinite metric. By the spectral theorem we can represent $\hat{U}$ as the multiplication operator $M_{e^{it}}$ on a direct integral Hilbert space $\mathscr{K} = \int_0^{2\pi} \oplus \mathscr{K}(t) \mathrm{d}m(t)$, where m is a scalar spectral measure for $\hat{U}$ and $\dim \mathscr{K}(t)$ is a multiplicity function for $\hat{U}$ (see [12]). By assumption there is a signature operator J which commutes with $\hat{U}$ and induces the Kreĭn space inner product on $\mathscr{K}$. Such a J must be represented as multiplication by a measureable field $\{J(t)\}_{0\leq t\leq 2\pi}$ of signature operators, where each $J(t)$ is a signature operator on $\mathscr{K}(t)$, which thus makes each fiber space $\mathscr{K}(t)$ also a Kreĭn space. Now since the subspace $\mathbf{M}_u$ is reducing for $\hat{U}$, it must be decomposable, that is, $\mathbf{M}_u$ has the form $\mathbf{M}_u = \int_0^{2\pi} \oplus \mathscr{M}(t) \mathrm{d}m(t)$ where each $\mathscr{M}(t)$ is some subspace of $\mathscr{K}(t)$. Since $\mathbf{M}_u$ is nondegenerate as a subspace of $\mathscr{K}$, a.e. $\mathscr{M}(t)$ must be nondegenerate as a subspace of the Kreĭn space $\mathscr{K}(t)$. We can thus produce a densely defined Kreĭn space isometry $\Psi(t)\colon X(t) \to \mathscr{K}(t)$ with range dense in $\mathscr{K}(t)$. If we do this in a measurable way, then the operator Ψ of multiplication by $\Psi(t)$ gives a densely defined isometry from the Kreĭn space $\int_0^{2\pi} \oplus X(t) \mathrm{d}m(t)$ onto a dense subset of $\int_0^{2\pi} \oplus \mathscr{M}(t) \mathrm{d}m(t) = \mathbf{M}_u$ which intertwines $M_{e^{it}}$. The assertion of the theorem follows.

As an aside, it is interesting to note that the Wold decomposition in the proof of Theorem 4.3 can be refined if the invariant subspace $\mathbf{M}$ is regular. Indeed we get the following:

PROPOSITION 4.4. *Suppose* $\hat{U} = \begin{bmatrix} \tilde{U} & 0 \\ 0 & U \end{bmatrix}$ *is a fundamentally decomposable isometry on the Kreĭn space* $\hat{\mathbf{K}} = \tilde{\mathbf{K}} \boxplus \mathbf{K}$ *and* $\mathbf{M}$ *is a regular* $\hat{U}$*-invariant subspace* *of* $\hat{\mathbf{K}}$. *Define subspaces* $\mathbf{L}$, $M_+(\mathbf{L})$ *and* $\mathbf{M}_\infty$ *by* $\mathbf{L} := \mathbf{M} \boxminus \hat{U}\mathbf{M}$, $M_+(\mathbf{L}) := \bigvee_{k \geq 0} \hat{U}^k\mathbf{L}$, $\mathbf{M}_\infty := \bigcap_{k \geq 0} \hat{U}^k\mathbf{M}$. *Then* $M_+(\mathbf{L})$ *and* $\mathbf{M}_\infty$ *are regular* $\tilde{U}$*-invariant subspaces of* $\hat{\mathbf{K}}$, $\hat{U} \mid M_+(\mathbf{L})$ *is a shift,* $\hat{U} \mid M_\infty$ *is unitary and the invariant subspace* $\mathbf{M}$ *has the Wold decomposition* $\mathbf{M} = M_+(\mathbf{L}) \boxplus \mathbf{M}_\infty$.

Proof. Since $\mathbf{M}$ is regular, the restriction of $[\,,\,]_{\hat{K}}$ to $\mathbf{M}$ makes $\mathbf{M}$ a Kreĭn space. Since $U \mid \mathbf{M}$ is also an isometry in the compatible Hilbert space topology on $\mathbf{M}$, the norms $\|(\hat{U} \mid \mathbf{M})^k\|$ $(k \geq 0)$ are uniformly bounded. The Wold decomposition of the proposition now follows immediately from a general result of McEnnis [24].

The next Theorem extends Theorem 3.8 to a more general setting, which for $l = 0$ should be compared to the results of [4]. To state it, we note that Ξ is as in Theorem 4.4, and we decompose $\hat{\mathbf{K}}$ into the maximal positive and negative subspaces $\hat{\mathbf{K}} = \tilde{\mathbf{K}} \boxplus \mathbf{K}$ which diagonalize $\hat{U}$ $\left(\hat{U} = \begin{bmatrix} \tilde{U} & 0 \\ 0 & U \end{bmatrix}\right)$, and similarly, write $\hat{\mathbf{K}}_1 = \tilde{\mathbf{K}}_1 \boxplus \mathbf{K}_1$ with $\hat{U}_1 = \begin{bmatrix} \tilde{U}_1 & 0 \\ 0 & U_1 \end{bmatrix}$, then these decompositions of $\hat{\mathbf{K}}$ and $\hat{\mathbf{K}}_1 \boxplus \hat{\mathbf{K}}_0$ induce an operator matrix representation

$$\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix}$$

for Ξ. In principle, all the objects in the following theorem are computable via our techniques. The proof is completely analogous to that of Theorem 3.4 and will be omitted.

THEOREM 4.5. *Suppose* T, $\tilde{T}$, U, $\tilde{U}$, *on* $\mathscr{H}$, $\tilde{\mathscr{H}}$, $\mathscr{K}$, $\tilde{\mathscr{K}}$, A *in* $\mathscr{L}(\mathscr{H}, \tilde{\mathscr{H}})$ *and* $\mathscr{M} = \mathscr{M}(T, \tilde{T}; U, \tilde{U}; A)$ *are as in Theorem 4.3, and suppose that* $\mathscr{M}$ *is pseudo--regular. Then there is a Kreĭn space* $\hat{\mathscr{K}}_1 = \tilde{\mathscr{K}}_1 \boxplus \mathscr{K}_1$ *and a Hilbert space* $\mathscr{K}_0$, *a fundamentally decomposable isometry* $\begin{bmatrix} \tilde{U}_1 & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & U_0 \end{bmatrix}$ *on* $\hat{\mathscr{K}}_1 \oplus \mathscr{K}_0 = \tilde{\mathscr{K}}_1 \boxplus \mathscr{K}_1 \oplus \mathscr{K}_0$, *a densely defined* $([\,,\,]_{\hat{\mathscr{K}}_1 \boxplus \mathscr{K}_0}, [\,,\,]_{\hat{\mathscr{K}}})$*-isometry* $\Xi = \begin{bmatrix} \alpha & \beta & \psi \\ \varkappa & \gamma & \omega \end{bmatrix} : \mathscr{D}(\Xi) \to \hat{\mathscr{K}}$, *such that* $\Xi((\tilde{U}_1 \oplus U_1 \oplus U_0) \mid \mathscr{D}(\Xi)) = (\tilde{U} \oplus U)\Xi$ *and*

$$\mathrm{CID}_l(A) = \{(\alpha H i^* + \beta i^* + \psi j^*)(\varkappa H i^* + \gamma i^* + \omega j^*)^{-1} \mid H : \mathscr{K}_1 \to \tilde{\mathscr{K}}_1, \|H\| \leq 1, HU_1 = \tilde{U}_1 H\}$$

where

$$i : \mathscr{K}_1 \to \mathscr{K}_1 \oplus \{0\} \subset \mathscr{K}_1 \oplus \mathscr{K}_0$$

and

$$j: \mathscr{K}_0 \to \{0\} \oplus \mathscr{K}_0 \subset \mathscr{K}_1 \oplus \mathscr{K}_0$$

are the natural injections.

The same conventions for $[\ ,\ ]_{\hat{\mathscr{K}}_1 \oplus \mathscr{K}_0}$ and $[\ ,\ ]_{\hat{\mathscr{K}}_1 \oplus \mathscr{K}_0}$ hold as in Theorem 4.4.

In the case where the invariant subspace $\mathscr{M} = \mathscr{M}(T, \tilde{T}; U, \tilde{U}; A)$ is regular, the degenerate subspace $\mathscr{K}_0$ can be dispensed with, and Theorem 4.3 guarantees the existence of many elements in $\mathrm{CID}_l(A)$, where l is the negative signature of $\mathscr{M}'$. If $\mathscr{M}$ is not regular (or even pseudo-regular) but $\mathscr{M}'$ has finite negative signature, $\mathscr{M}$ becomes regular after a slight perturbation in A; one can then use Theorem 4.1 and a continuity argument to complete the proof of Theorem 4.2.

Finally we consider the question of uniqueness. The first part of the following corollary refines a result of [3].

COROLLARY 4.6. *Suppose $A: \mathscr{H} \to \tilde{\mathscr{H}}$ intertwines contractions $T \in \mathscr{L}(\mathscr{H})$ and $\tilde{T} \in \mathscr{L}(\tilde{\mathscr{H}})$ as above and that $I - AA^* \geqslant 0$.*

(i) *If $\ker \mathscr{D}_A$ ($\mathscr{D}_A = (I - AA^*)^{1/2}$) is cyclic for the minimal isometric dilation U of T, then there is a unique A_∞ in $\mathrm{CID}(A)$ and furthermore this unique A_∞ is an isometry.*

(ii) *Assume 0 is an isolated point of the spectrum of $\mathscr{D}_A$. Then there is a unique A_∞ in $\mathrm{CID}(A)$ if and only if the subspace $\mathscr{L} \dotplus \mathscr{M}_\infty$ is either a positive or a negative subspace of $\hat{\mathscr{K}}$. Here $\mathscr{L} = \mathscr{M} \ominus \hat{U}\mathscr{M}$, $\mathscr{M}_\infty = \bigcap_{k \geqslant 0} \hat{U}^k$, where $\mathscr{M} = \mathscr{M}(T, \tilde{T}; U, \tilde{U}; A)$ as above.*

This unique A_∞ in $\mathrm{CID}(A)$ is isometric if and only if $\mathscr{L} \dotplus \mathscr{M}_\infty$ is positive.

Proof. (i) Let A_∞ be any element of $\mathrm{CID}(A)$. Since $\tilde{P}A_\infty = AP$ and $\|A_\infty\| \leqslant 1$, A_∞ has to be isometric on $\ker \mathscr{D}_A$; from the intertwining condition it follows that A_∞ is uniquely determined as an isometry on $\bigcup_{n \geqslant 0} U^n \ker \mathscr{D}_A$; the U cyclicity of $\ker \mathscr{D}_A$ implies that A_∞ then is unique. We thank the referee for pointing out that this result is elementary and does not depend on the machinery of our Beurling-Lax theorem.

(ii) If 0 is an isolated point of the spectrum of $I - AA^*$, then the space $\mathscr{M}$ is a regular subspace of $\hat{\mathscr{K}}$, and a Wold decomposition as in Proposition 4.4 holds for $\mathscr{M}$. By Theorem 4.1, we know that operators in $\mathrm{CID}(A)$ are in one-to-one correspondence with $\hat{U}$-invariant $\hat{\mathscr{K}}$-maximal negative subspaces of $\mathscr{M}$. Since $\hat{\mathscr{K}} \ominus \mathscr{M} = \begin{bmatrix} I \\ A^* \end{bmatrix} \tilde{\mathscr{H}}$ is a positive subspace, by Lemma 1.1 $\hat{\mathscr{K}}$-maximal negative subspaces of $\mathscr{M}$ are exactly the $\mathscr{M}$-maximal negative subspaces of $\mathscr{M}$. As in Lemma 3.2, invariant $\mathscr{M}$--maximal negative subspaces must be subspaces of $\mathscr{M} \cap \mathscr{N}'$ where $\mathscr{N} = \bigvee_{k=0} \hat{U}^k(\mathscr{M} \cap \mathscr{M}')$.

By the decomposition (4.1) for $\mathcal{M} \cap \mathcal{N}'$ we see that if $\mathcal{M} \cap \mathcal{N}'$ has a unique maximal negative subspace, it is automatically $\hat{U}$-invariant. Now it is easy to see that this uniqueness occurs if and only if either $\mathcal{M} \cap \mathcal{N}'$ is positive or $\mathcal{M} \cap \mathcal{N}'$ is negative. If $\mathcal{M} \cap \mathcal{N}'$ is positive, the unique maximal negative subspace of $\mathcal{M} \cap \mathcal{N}'$ is the isotropic subspace of $\mathcal{M} \cap \mathcal{N}'$ and the corresponding element A_∞ of CID(A) is isometric. If $\mathcal{M} \cap \mathcal{N}'$ is negative, then the maximal negative subspace for $\mathcal{M} \cap \mathcal{N}'$ is the whole space $\mathcal{M} \cap \mathcal{N}'$. By the decomposition (4.1) for $\mathcal{M} \cap \mathcal{N}'$, we see that $\mathcal{M} \cap \mathcal{N}'$ is positive (negative) if and only if $\mathcal{L} + \mathcal{M}_\infty$ is positive (negative) and the desired conclusion follows.

The main result of [3] is that CID(A) has a unique element if and only if at least one of the factorizations $\tilde{T} \cdot A = A \cdot T$ is a regular factorization. It would be interesting to have a direct proof for the equivalence of this condition and that of Corollary 4.6 (ii) for the case where $I - AA^*$ has closed range.

Both authors are partially supported by National Science Foundation grants.

REFERENCES

1. Adamjan, V. M.; Arov, D. Z.; Kreĭn, M. G., Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem (Russian), *Mat. Sb.*, **86**(128) (1971), 34–75; *Math. USSR. Sb.*, **15**(1971), 31–73.
2. Adamjan, V. M.; Arov, D. Z.; Kreĭn, M. G., Infinite Hankel block matrices and related extension problems (Russian), *Izv. Akad. Nauk. Armjan. SSR. Ser. Mat.*, **6**(1971), 87–112; *Transl. Amer. Math. Soc.* (2), **111**(1978), 133–156.
3. Ando, T.; Ceauşescu, Z.; Foiaş, C., On intertwining dilations. II, *Acta Sci. Math. (Szeged)*, **39**(1977), 3–14.
4. Arsene, Gr.; Ceauşescu, Z.; Foiaş, C., On intertwining dilations. VIII, *J. Operator Theory*, **4**(1980), 55–91.
5. Ball, J. A., Interpolation problems of Pick-Nevanlinna and Loewner types for meromorphic matrix functions, preprint.
6. Ball, J. A., A non-Euclidean Lax-Beurling theorem with applications to matricial Nevanlinna-Pick interpolation, *Proceedings of the Toeplitz Memorial Conference*, Birkhäuser Verlag, 1981, pp. 69–84.
7. Ball, J. A.; Helton J. W., Factorization results related to shifts in an indefinite metric, *Integral Equations and Operator Theory*, **5**(1982), 612–658.
8. Ball, J. A.; Helton, J. W., Beurling-Lax representations using the classical Lie groups with many applications. III: groups preserving forms, in preparation.
9. Ball, J. A.; Helton, J. W., Beurling-Lax representations using the classical Lie groups with many applications. IV: GL(n,R), U*(2n), SL(n, 4) and a solvable group, in preparation.
10. Ball, J. A.; Helton, J. W., Lie groups over the field of rational functions, signed spectral factorization, signed interpolation, and amplifier design, *J. Operator Theory*, **8**(1982), 19–64.
11. Bognár, J., *Indefinite inner product spaces*, Springer-Verlag, New York, 1974.

12. DIXMIER, J., *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.
13. GARNETT, J., Two remarks on interpolation by bounded analytic functions, in *Banach spaces of analytic functions*, Lecture Notes in Math., **604**(1977), Springer-Verlag, New York.
14. HALMOS, P. R., Shifts on Hilbert space, *J. Reine Angew. Math.*, **208**(1961), 102–112.
15. HELSON, H., *Lectures on invariant subspaces*, Academic Press, 1964.
16. HELTON, J. W., Orbit structure of the Möbius transformation semigroup acting in H^{∞} (broadband matching), in *Topics in functional analysis*, Advances in Mathematics Supplemental Studies, 3(1978), 129–157, Academic Press.
17. HELTON, J. W., The distance of a function to H^{∞} in the Poincaré metric; electrical power transfer, *J. Functional Analysis*, **38**(1980), 273–314.
18. HINTZMAN, W., Best uniform approximation via annihilating measures, *Bull. Amer. Math. Soc.*, **76**(1970), 1062–1066.
19. IVANCHENKO, T. S., The Nevanlinna-Pick problem in the case of indefinite metrics (Russian), *Dopovidi Akad. Nauk Ukrain. RSR Ser. A.*, **5**(1980), 8–13.
20. IVANCHENKO, T. S., The Schur problem in the case of indefinite metrics (Russian), *VINITI*, 1979, submitted.
21. IVANCHENKO, T. S., Nevanlinna-Pick problem with simple poles (Russian), *VINITI*, 1980, submitted.
22. JACOBSON, N., *Lectures in abstract algebra. Vol. II*, Springer-Verlag, New York.
23. MCENNIS, B., Shifts on indefinite inner product spaces, *Pacific J. Math.*, **81**(1979), 113–130.
24. MCENNIS, B., Shifts on indefinite inner product spaces. II, *Pacific J. Math.*, **100**(1982), 177–184.
25. SZ.-NAGY, B.; FOIAŞ, C., *Harmonic analysis of operators on Hilbert space*, American Elsevier New York, 1970.
26. NUDELMAN, A. A., On a new type of moment problem (Russian), *Dokl. Akad. Nauk SSSR.*, **233**: 5 (1977), 792–795.
27. NUDELMAN, A. A., On a generalization of classical interpolation problems (Russian), *Dokl. Akad. Nauk SSSR.*, **256**: 4 (1981), 790–793.
28. ROSENBLUM, M.; ROVNYAK, J., An operator-theoretic approach to theorems of the Pick-Nevanlinna and Loewner types. I, *Integral Equations and Operator Theory*, 3(1980), 408–436.
29. SARASON, D., Generalized interpolation in H^{∞}, *Trans. Amer. Math. Soc.*, **127**(1967), 179–203.
30. KUNG, S. Y., *IEEE Trans. Circuit Theory*.

JOSEPH A. BALL
Department of Mathematics,
Virginia Polytechnic Institute
and State University,
Blacksburg, Virginia 24061,
U.S.A.

J. WILLIAM HELTON
Department of Mathematics,
University of California at
San Diego
La Jolla, California 92093,
U.S.A.

Received November 12, 1981; revised June 1, 1982.