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HARDY AND RELLICH INEQUALITIES IN NON-INTEGRAL DIMENSION

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In [1], Davies asked the following question. Let $H_0 = -\Delta$ on $L^2(\mathbf{R}^3)$. Let

(1) $$V(x) = c|x|^{-\alpha}$$

with $c > 0$, $\alpha > 0$. $H = H_0 + V$ can always be defined as a self-adjoint operator via a form sum [4,5]. When is it true that

(2) $$D(H) = D(H_0) \cap D(V)\ ?$$

Davies found the striking answer: (2) holds if $\alpha < 3/2$ or $\alpha > 2$ but fails if $3/2 \leqslant \alpha < 2$. For the borderline case, $\alpha = 2$, he found that (2) is false if $0 < c \leqslant 3/4$ and true if $c > 3/2$. He left open the case $\alpha = 2$, $3/4 < c \leqslant 3/2$. Our original goal was to fill in this gap, and we will give an analysis of all the $\alpha = 2$ situations. We will concentrate on the question of whether the operator sum $H_0 + V$ is a closed operator on $D(H_0) \cap D(V)$. That $D(H_0) \cap D(V)$ is a core for H in the cases in this paper where it is closed is known (see e.g. [5], vol. II, p. 161) so (2) will be established by this closure result. It is an observation of Glimm — Jaffe [2] that $H_0 + V$ is closed on $D(H_0) \cap D(V)$ if and only if for some $a, b \geqslant 0$:

(3) $$\|V\varphi\| \leqslant a\|H\varphi\| + b\|\varphi\|.$$

In fact for the case

(4) $$V = c|x|^{-2}, \quad H = -\Delta + V$$

we will prove (3) with optimal constants:

THEOREM 1. *On* $\mathbf{R}^3$, (3) *holds with* V, H *given by* (4), *if and only if* $c > 3/4$ *and in that case, it holds with* $b = 0$ *and* $a = c(c - 3/4)^{-1}$ *and is false with any smaller constant.*

The first observation is that V and H have the same properties under scaling; this will be critical later also. That is, if

$$(u(\lambda)f)(x) = \lambda^{3/2} f(\lambda x),$$

then

$$u(\lambda)Vu(\lambda)^{-1} = \lambda^{-2}V; \quad u(\lambda)Hu(\lambda)^{-1} = \lambda^{-2}H.$$

Thus, if (3) holds with any b, it also holds with the same a, b for $\lambda^{-2}V$ and $\lambda^{-2}H$ or equivalently for the same a but for b replaced by $b\lambda^2$. Taking λ to zero, we find that if (3) holds, it holds for $b = 0$. Thus, we are reduced to seeing if

$$(3') \qquad \|V\varphi\| \leqslant a\|H\varphi\|.$$

But since H has no kernel, this is equivalent to asking if

$$(3'') \qquad \|VH^{-1}\| \leqslant a,$$

i.e. we must see if VH^{-1} is bounded, and, if so, to compute its norm. Since VH^{-1} computes with rotations and dilations, this will be easy.

In the usual way, VH^{-1} is a direct sum of operators on subspaces of the form $\{rg(r)Y_{lm} \mid g \in L^2(0, \infty; dr)\}$. Thus, we define

$$(5) \qquad h(L) = -\frac{d^2}{dr^2} + \frac{L(L+1)}{r^2}$$

on $\ell^2(0, \infty; dr)$ and we want to know if

$$\sup_{l=0,1,2\ldots} \| Vh(L(c, l))^{-1}\| < \infty$$

where $L(c, l)$ solves $L(L+1) = l(l+1) + c$. Theorem 1 will thus follow from the following (since $(L - 1/2)(L + 3/2) = L(L+1) - 3/4$):

THEOREM 1′. *$r^{-2}h(L)^{-1}$ is bounded if and only if $L > 1/2$ and*

$$(6) \qquad \|r^{-2}h(L)^{-1}\| = 1/(L - 1/2)\,(L + 3/2).$$

$r^{-2}h(L)^{-1}$ commutes with dilations, so following Herbst [3], we change variables to $r = e^x$ in which case, the new operator on $L^2(-\infty, \infty)$ is guaranteed to commute with translations and so be a convolution operator. Explicitly, let u: $L^2(0, \infty; dr) \to \to L^2(-\infty, \infty; dx)$ by $(uf)(x) = e^{x/2}f(e^x)$ so $(u^{-1}g)(r) = r^{-1/2}g(\ln r)$. Direct calculation shows that if K has an integral kernel $K(r, r')$ on $L^2(0, \infty; ds)$, then $\tilde{K} = uKu^{-1}$ has integral kernel

$$\tilde{K}(x, x') = e^{x/2}e^{x'/2}K(e^x, e^{x'}).$$

By standard ODE techniques and the fact that $h(L)u = 0$ is solved by $u = r^{L+1}$ and $u = r^{-L}$, we see that $r^{-2}h(L)^{-1}$ has integral kernel

$$K(r, r') = r^{-2}(2L+1)^{-1}r_<^{L+1}r_>^{-L}$$

where $r_> = \max(r, r')$, $r_< = \min(r', r)$. In particular, $r_</r_> = \exp(-|x - x'|)$. Thus, with $\theta(\cdot)$ the characteristic function of $[0, \infty)$:

$$\tilde{K}(x, x') = (2L + 1)^{-1} \{\theta(x' - x) \exp[-(L - 1/2)|x - x'|] + \\ + \theta(x - x') \exp[-(L + 3/2)|x - x'|]\}$$

so $\tilde{K}$ is convolution by a positive function, f, on R. This is bounded on L^2 if and only if $f \in L^1$ and its norm is then $\int_{-\infty}^{\infty} f(x)dx$. This yields (6).

A similar, but more direct, calculation shows that

$$\|r^{-1}h(L)^{-1}r^{-1}\| = 1/(L + 1/2)^2 \quad \text{if } L > -1/2, \tag{7}$$

or more generally, for $0 \leqslant \gamma \leqslant 1$,

$$\|r^{-1-\gamma}h(L)\, r^{-1+\gamma}\| = 1/[(L + 1/2)^2 - \gamma^2] \quad \text{if } L > \gamma - 1/2.$$

We close with two remarks:

(1) If one looks at $r^{-2}(-\Delta)^{-1}$ on $L^2(\mathbf{R}^n)$, it is a direct sum of $r^{-2}h(L)$ with $L = (1/2)(n - 3) + l$; $l = 0, 1, \ldots$. Thus Theorem 1′ implies that $r^{-2}(-\Delta)^{-1}$ is bounded if and only if $n \geqslant 5$ and

$$\|r^{-2}(-\Delta)^{-1}\| = 4/n(n - 4).$$

This is an equality of Rellich (see e.g. [6]). Thus (6) is a kind of Rellich inequality for non-integral n. It says the proper condition is not $n \geqslant 5$ but $n > 4$. Similarly, (7) is a kind of Hardy inequality (Hardy says $\|r^{-1}(\Delta)^{-1}r^{-1}\| = 4/(n - 2)^2$) for non-integral n.

(2) If $n \geqslant 4$, the above analysis shows that for all $c \geqslant 0$, $H_0 + cr^{-\alpha}$ is closed on $D(H_0) \cap D(V)$ in $L^2(\mathbf{R}^n)$ when $\alpha = 2$. Davies' methods [1] show this is true for all $\alpha \neq 2$; i.e., if one restricts to $V = cr^{-\alpha}$, the phenomenon of $D(H_0) \cap D(V) \neq \neq D(H)$ does not occur. For $n = 4$, it is easy to find other V with $D(H) \neq D(H_0) \cap \cap D(V)$, e.g. $V = 2/r^2(\ln r)$ for which $D(H)$ has vectors ψ, behaving as $\ln r$ for small r, so $V\psi \notin L^2$. For $n \geqslant 5$, it is not easy to construct such V. In fact we coniecture:

Conjecture. Let $V(x) = V(|x|)$ on $\mathbf{R}^n$; $n \geqslant 5$. Suppose

(i) $V \geqslant 0$;

(ii) V is bounded on any closed set avoiding 0.

Then $H_0 + V$ is closed on $D(H_0) \cap D(V)$.

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Additional note. The problem raised in this paper, namely, when is $-\Delta + c|x|^{-2}$ closed operator on $D(-\Delta)\cap D(|x|^{-2})$ is raised already in the paper by D. Robinson, *Ann. Inst. H. Poincaré*, **21**(1974), 185–215. In this paper, Robinson proves things are closed if the constant c is larger than 3/2 (that is, the result which we attribute to Davies in the paper appears already in the paper of Robinson).