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(BCP)-OPERATORS AND ENRICHMENT OF INVARIANT SUBSPACE LATTICES

C. FOIAŞ and C. M. PEARCY

This paper is dedicated, with warm affection, to Professor Béla Sz.-Nagy, on the occasion of his seventieth birthday.

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $A \in \mathscr{L}(\mathscr{H})$, we denote by $\sigma(A)$ the spectrum of A, by $\sigma_e(A)$ the essential (i.e., Calkin) spectrum of A, and by $\sigma_{le}(A)$ and $\sigma_{re}(A)$ the left and right essential spectra of A, respectively. Moreover, we write $\mathrm{r}(A)$ for the spectral radius of A and $\mathrm{w}(A)$ for the numerical radius of A. Recall that an operator A in $\mathscr{L}(\mathscr{H})$ is a completely nonunitary contraction if $\|A\| \leqslant 1$ and there exists no nonzero reducing subspace $\mathscr{M}$ for A such that $A|\mathscr{M}$ is a unitary operator.

In this paper the Banach algebra $H^\infty = H^\infty(\mathbf{D})$ of bounded holomorphic functions h on the open unit disc $\mathbf{D} = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$, with supremum norm $\|h\|_\infty = \sup_{\lambda \in \mathbf{D}} |h(\lambda)|$, will be useful. In particular, there is an H^∞-functional calculus for any completely nonunitary contraction A, so that the operator $h(A)$ is defined for every h in H^∞ and has various properties reflecting those of A and h (cf. [20], Theorem III.2.1). Recall that a subset S of $\mathbf{D}$ is said to be *dominating* for the unit circle $C = \partial\mathbf{D}$ if

$$\sup_{\lambda \in S} |h(\lambda)| = \|h\|_\infty, \quad h \in H^\infty,$$

and that these subsets of $\mathbf{D}$ can be characterized by the property that almost every point of C is a nontangential limit point of S; cf. [5]. In analogy with this characterization, we say that a subset S of $\mathbf{D}$ is *dominating for a subset* s of C if almost every point of s is a nontangential limit point of S.

Let (BCP) denote the class of all completely nonunitary contractions A in $\mathscr{L}(\mathscr{H})$ for which $\sigma_e(A) \cap \mathbf{D}$ is dominating for C. We permit ourselves the indulgence of referring to such operators A as (BCP)-operators.

The class (BCP) was first studied in [6], where the existence of nontrivial invariant subspaces for (BCP)-operators was proved, and this study continued in [2], [3], [1], [16] and [18]. (In particular, we owe to Robel [18] the clarification of the definition of the class (BCP).) Thus we now have considerable knowledge about the structure of (BCP)-operators, including the fact that they are reflexive operators [1]. One point of this paper, which is a continuation of [10], is to show that the (positive) solution of the invariant subspace problem for either the class of square roots or the class of inverses of the invertible (BCP)-operators has as a consequence the solution of the invariant subspace problem for a class of operators that contains all operators A satisfying $r(A) = \|A\|$.

But another, perhaps equally important, consequence of the constructions we employ to prove these results is as follows.

Let us write, as usual, Lat(A) for the lattice of invariant subspaces of a given operator A in $\mathscr{L}(\mathscr{H})$. Then, it turns out (Corollaries 2.5 and 5.3) that there are two functions h_1 and h_2 in H^∞ such that if A is any completely nonunitary contraction with connected spectrum containing the point 1, then $h_1(A)$ and $h_2(A)$ are both invertible operators having the same properties as A just described, and having the additional properties that

a) $\mathrm{Lat}(h_1(A)^2)\setminus\mathrm{Lat}(A)$ and $\mathrm{Lat}(h_2(A)^{-1})\setminus\mathrm{Lat}(A)$ each contains a lattice isomorphic to the lattice of all subspaces of $\mathscr{H}$, while

b) $\mathrm{Lat}(h_1(A)^2) \cap \mathrm{Lat}(h_1(A)) = \mathrm{Lat}(A) = \mathrm{Lat}(h_2(A)^{-1}) \cap \mathrm{Lat}\,(h_2(A))$.

Thus, beginning with any operator A with the aforementioned properties, one can construct two sequences of operators

$$A, \quad A' = h_1(A)^2, \quad A'' = h_1(A')^2, \ldots$$

and

$$A, \quad \hat{A} = h_2(A)^{-1}, \quad \hat{\hat{A}} = h_2(\hat{A})^{-1}, \ldots$$

such that the corresponding lattices

$$\mathrm{Lat}(A), \quad \mathrm{Lat}(A'), \quad \mathrm{Lat}(A''), \ldots$$

and

$$\mathrm{Lat}(A), \quad \mathrm{Lat}(\hat{A}), \quad \mathrm{Lat}(\hat{\hat{A}}), \ldots$$

become progressively richer and richer. This seems to be a phenomenon worth further study.

Finally, in Section 6 we give a specific example that illustrates some of the difficulties one encounters in trying to resolve either the square root or the inverse problem for (BCP)-operators and shows also that there exist reflexive invertible operators A in $\mathscr{L}(\mathscr{H})$ such that A^{-1} is not reflexive.

To begin our program, let $(\mathcal{F})$ denote the set of all operators A in $\mathcal{L}(\mathcal{H})$ for which some two of the three numbers

$$\mathrm{r}(A) \leqslant \mathrm{w}(A) \leqslant \|A\|$$

coincide, and let $(\mathcal{P})$ denote the set of all completely nonunitary contractions A in $\mathcal{L}(\mathcal{H})$ such that $1 \in \sigma(A)$ and $\sigma(A)$ is connected. The following elementary proposition shows that the invariant subspace problem for the class $(\mathcal{F})$ reduces to that for the subclass $(\mathcal{P})$.

PROPOSITION 1.1. *If every operator in $(\mathcal{P})$ has a nontrivial invariant subspace, then so does every operator in $(\mathcal{F})$.*

Proof. Let B be any nonzero operator in $(\mathcal{F})$. We wish to show, operating under the hypothesis, that B has a nontrivial invariant subspace. If $\mathrm{r}(B) = \|B\|$, then there exists a complex number γ with $|\gamma| = 1$ such that $1 \in \sigma(\gamma B/\|B\|)$. Moreover B and $B' = \gamma B/\|B\|$ satisfy $\mathrm{Lat}(B) = \mathrm{Lat}(B')$, and if B' either has disconnected spectrum or is not completely nonunitary, then B' (and thus B) has a nontrivial hyperinvariant subspace for elementary reasons. Thus we may suppose that $B' \in (\mathcal{P})$, and that $\mathrm{Lat}(B) \neq \{(0), \mathcal{H}\}$ then follows from the hypothesis.

If $\mathrm{w}(B) = \|B\|$, it follows from [14, Problem 173] that $\mathrm{r}(B) = \|B\|$ also, so the result follows from the case already treated. Finally, if $\mathrm{r}(B) = \mathrm{w}(B)$, then, upon setting $B' = B/\mathrm{r}(B)$, we have $\mathrm{r}(B') = \mathrm{w}(B') = 1$. But, according to [20, Corollary II.8.2], any such operator B' is similar to an operator B'' satisfying $\mathrm{r}(B'') = \|B''\| = 1$, and the result follows as before.

2. SQUARE ROOTS

We have just seen that to show that every operator in $(\mathcal{F})$ has a nontrivial invariant subspace, it suffices to deal with the operators in the set $(\mathcal{P}) \subset (\mathcal{F})$. The main idea of this section is that every operator A in $(\mathcal{P})$ can be "traded off" (in a sense made precise in Theorem 2.1) for a second, invertible operator A' in $(\mathcal{P})$ whose left essential spectrum has been "blown up" to be large enough that $(A')^2 \in$ $\in$ (BCP). This implies (Corollary 2.3) that to solve the invariant subspace problem for the class $(\mathcal{F})$, it suffices to solve the "square root" problem for invertible (BCP)--operators.

We will need one more piece of notation. For any A in $\mathcal{L}(\mathcal{H})$, we denote by $\mathcal{A}(A)$ the smallest algebra that contains A and $1_{\mathcal{H}}$ and is closed in the weak operator topology, and we remark that if A_1 and A_2 satisfy $\mathcal{A}(A_1) = \mathcal{A}(A_2)$, then obviously $\mathrm{Lat}(A_1) = \mathrm{Lat}(A_2)$. The following theorem, whose proof is given in Section 4, is similar in many respects to the Theorem of [10].

THEOREM 2.1. *For every proper subarc E of C having 1 as its midpoint, there exists a function $g = g_E$ in H^∞ which maps* **D** *conformally into itself and is such that for $h = g \circ g$ and every A in $(\mathscr{P})$:*

(1) *$g(A)$ and $h(A)$ belong to $(\mathscr{P})$, $0 \notin \sigma(h(A))$,*

(2) *$\partial\sigma_e(g(A)) \cap C = E$,*

(3) *$\partial\sigma_e(h(A)) \cap \mathbf{D}$ is dominating for the arc E, and*

(4) *$\mathscr{A}(A) = \mathscr{A}(g(A)) = \mathscr{A}(h(A))$.*

REMARK. The arc E in Theorem 2.1 can clearly be characterized by an angle ε:

$$E = E_\varepsilon = \left\{ e^{it} : \left(\frac{\varepsilon}{2} - \pi\right) \leqslant t \leqslant \left(\pi - \frac{\varepsilon}{2}\right) \right\}, \quad \text{where } 0 < \varepsilon < 2\pi.$$

See Figure 1.

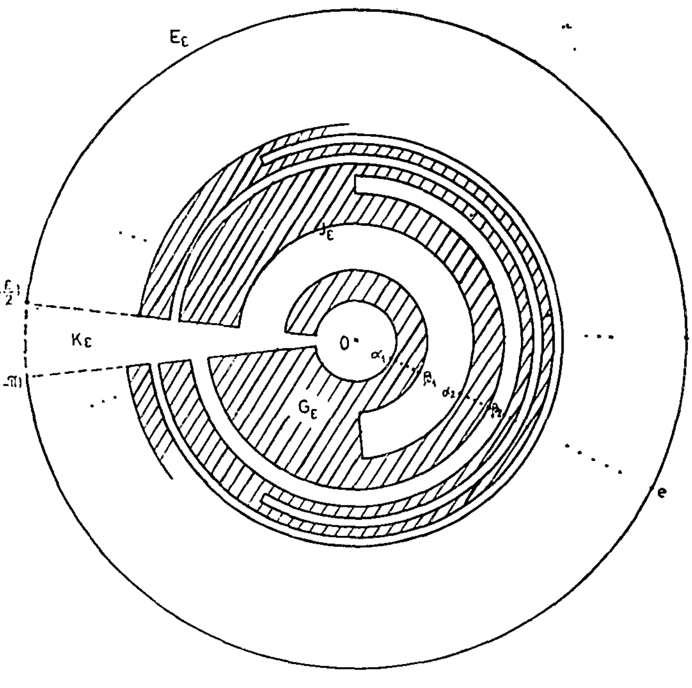

Fig. 1

COROLLARY 2.2. *There exists a nonconstant function k in H^∞ such that for every A in $(\mathscr{P})$, $k(A)$ is reflexive.*

Proof. Letting h be the function in Theorem 2.1 corresponding to the arc E_π, we observe from conclusion (3) that $\partial\sigma_e(h(A)) \cap \mathbf{D}$ is dominating for the arc E_π, and since for any operator B,

$$\partial\sigma_e(B) \subset \sigma_{le}(B) \cap \sigma_{re}(B),$$

we have that $\sigma_{1e}(h(A)) \cap \mathbf{D}$ is dominating for E_π. If we set $k = h^2$, then, since $\sigma_{1e}([h(A)]^2) = [\sigma_{1e}(h(A))]^2$, it follows that $\sigma_{1e}(k(A)) \cap \mathbf{D}$ is dominating for $E_\pi^2 = C$. Thus $k(A) \in$ (BCP) and is reflexive from [1].

The following corollary is the promised reduction of the invariant subspace problem for the class ($\mathscr{F}$) to the "square root" problem for invertible (BCP)-operators.

COROLLARY 2.3. *If every invertible operator A in ($\mathscr{P}$) such that $A^2 \in$ (BCP) has a nontrivial invariant subspace, then every operator in ($\mathscr{F}$) has a nontrivial invariant subspace.*

Proof. We must show that, under the hypothesis, every operator in ($\mathscr{F}$) has a nontrivial invariant subspace. But, according to Proposition 1.1, it suffices to show that every operator B in ($\mathscr{P}$) has a nontrivial invariant subspace. Let h be the function in H^∞ corresponding to the arc E_π in Theorem 2.1. Then, just as was shown in Corollary 2.2, $h(B)^2 \in$ (BCP), and from conclusion (1) of Theorem 2.1 we see that $h(B)$ is an invertible operator in ($\mathscr{P}$). Thus, from the hypothesis, it follows that $h(B)$ has a nontrivial invariant subspace, and since from conclusion (3) of Theorem 2.1 we know that $\mathscr{A}(B) = \mathscr{A}(h(B))$, it follows that $\mathrm{Lat}(B) = \mathrm{Lat}(h(B)) \neq \{(0), \mathscr{H}\}$, proving the corollary.

The following corollary, whose proof is almost the same as that of [10, Corollary 4] and is thus omitted, shows that if one considers hyperinvariant subspaces instead of invariant ones, then the role of the class (BCP) in Corollary 2.3 can be played by a much smaller class of operators.

COROLLARY 2.4. *If every completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$ whose left essential spectrum is the closed unit disc has a nontrivial hyperinvariant subspace, then every nonscalar operator in ($\mathscr{F}$) has a nontrivial hyperinvariant subspace.*

The following corollary of Theorem 2.1 shows that associated with every operator A in ($\mathscr{P}$) there are operators whose lattice has been "fattened up" considerably.

COROLLARY 2.5. *There exists a function h in H^∞ such that for every operator A in ($\mathscr{P}$),*

(1) *$h(A)$ is an invertible operator in ($\mathscr{P}$),*

(2) $\mathrm{Lat}(h(A)^2) \cap \mathrm{Lat}(h(A)) = \mathrm{Lat}(A)$, *and*

(3) $\mathrm{Lat}(h(A)^2)$ *contains a lattice L that is disjoint from* $\mathrm{Lat}(A)$ *and is isomorphic to the lattice of all subspaces of $\mathscr{H}$.*

Proof. Let $g = g_{E_\pi}$ and h be as in Theorem 2.1. It then follows from that theorem that $h(A)$ is an invertible operator in ($\mathscr{P}$) and that $\mathrm{Lat}(h(A)) = \mathrm{Lat}(A)$. Furthermore, since obviously $\mathrm{Lat}(A) \subset \mathrm{Lat}(h(A)^2)$, (2) is established. To prove (3), note that $h(A)^2 \in$ (BCP). It thus results from [3] that $\mathrm{Lat}(h(A)^2)$ contains a subspace

$\mathcal{M}$ such that $\dim(\mathcal{M} \ominus h(A)^2\mathcal{M}) = \aleph_0$. Thus we may consider the lattice L given by

$$L = \{h(A)^2 \mathcal{M} \oplus \mathcal{N} : \mathcal{N} \text{ a subspace of } \mathcal{M} \ominus h(A)^2\mathcal{M}\}.$$

It is obvious that L is isomorphic to the lattice of all subspaces of $\mathcal{H}$, and to show that no element of L belongs to $\mathrm{Lat}(h(A)) = \mathrm{Lat}(A)$ we note simply that since $\sigma(h(A))$ is contained in the set G_π^- defined in § 3, and since the function $(1/\zeta)$ can be approximated uniformly by polynomials $p(\zeta)$ on G_π^-, it follows that every subspace $\mathcal{L}$ in $\mathrm{Lat}(h(A))$ also belongs to $\mathrm{Lat}(h(A)^{-1})$, and hence is mapped onto itself by $h(A)^2$. Thus the proof is complete.

3. SOME CONFORMAL MAPS

The proof of Theorem 2.1 is similar to the proof of the Theorem of [10], but there are certain differences, and therefore we choose to give the proof in full detail. The construction needed to prove Theorem 2.1 involves certain conformal maps of $\mathbf{D}$, and we turn now to some definitions and notation in that area that we shall need. (We reproduce the following discussion from [10] for the reader's convenience.) A bounded, simply connected domain G in $\mathbf{C}$ is called a *Carathéodory domain* if the Carathéodory hull of G (cf. [19]) is identical with G. This is equivalent to saying that the boundary ∂G of G coincides with the outer boundary of G (where, by definition, the outer boundary of G is the boundary of the unbounded component of $\mathbf{C}\setminus G^-$). One knows from [19] that the Carathéodory domains are exactly those bounded, simply connected domains G in $\mathbf{C}$ with the property that every Riemann mapping function g of $\mathbf{D}$ onto G is a sequential weak* generator for H^∞, i.e., has the property that every function h in H^∞ is a weak* limit of a sequence $\{p_n \circ g\}$ of polynomials in g. It follows easily from this and the known facts about the H^∞-functional calculus (cf. [20, Theorem III.2.1]) that if G is a Carathéodory domain contained in $\mathbf{D}$, g is a Riemann map of $\mathbf{D}$ onto G, and A is any completely nonunitary contraction, then A is the limit in the weak operator topology of a sequence of polynomials $\{p_n(g(A))\}$. Since $g(A)$ is also the weak limit of a sequence of polynomials $\{q_n(A)\}$, it follows that $\mathcal{A}(A) = \mathcal{A}(g(A))$. Thus, in order to prove (4) of Theorem 2.1, it suffices to choose $g = g_{E_\varepsilon}$ to be a conformal mapping of $\mathbf{D}$ onto some Carathéodory domain G_ε contained in $\mathbf{D}$ and set $h = g \circ g$. (For, one knows from [20, Theorem III.2.1] that in this case $g(A)$ is a completely nonunitary contraction with the property that $(g \circ g)(A) = g(g(A))$. Moreover, from the above discussion one has that $\mathcal{A}(A) = \mathcal{A}(g(A))$, and, applying this fact with $g(A)$ replacing A, one concludes that $\mathcal{A}(g(A)) = \mathcal{A}(h(A))$.)

We next fix a subarc $E = E_\varepsilon$ of C, $0 < \varepsilon < 2\pi$, centered on the point 1. We associate with E_ε the domain G_ε defined by

$$G_\varepsilon = \mathbf{D}\setminus\left[K_\varepsilon \cup \left(\bigcup_0^\infty L_n\right)\right]$$

where

$$K_\varepsilon = \left\{re^{it} : 0 \leqslant r \leqslant 1,\ \pi - \frac{\varepsilon}{2} \leqslant t \leqslant \pi + \frac{\varepsilon}{2}\right\} \cup \{re^{it} : |r| \leqslant 1/10\},$$

$$L_n = \left\{re^{it} : \frac{2n+1}{2n+5} \leqslant r \leqslant \frac{2n+2}{2n+5},\ \frac{n+1}{n+2}\left(\frac{\varepsilon}{2} - \pi\right) \leqslant (-1)^n t \leqslant \pi\right\}.$$

For a sketch of G_ε see Figure 1.

Clearly G_ε is simply connected, and its boundary ∂G_ε is formed by the union of the subarc E_ε of C and a simple path J_ε contained in $\mathbf{D}$ (that is, J_ε is an open Jordan arc). Furthermore, it is clear from the geometry that G_ε is a Carathéodory domain.

Let $g = g_{E_\varepsilon}$ be a conformal mapping of $\mathbf{D}$ onto G_ε, and let $\tilde{g}$ be its Carathéodory extension to a homeomorphism of $\mathbf{D}^-$ onto the prime end compactification of G_ε. (See, for example, [7], [13, p. 44], and [8].) We may, without loss of generality, assume that g is normalized in such a way that the point 1 of $\mathbf{D}^-$ corresponds under $\tilde{g}$ to that prime end $\hat{E}_\varepsilon$ of G_ε whose "impression" (see, for example, [8]) is the set E_ε, that is, the prime end determined by the sequence of crosscuts consisting of the intervals of the real line $\left[\frac{2n}{2n+4}, \frac{2n+1}{2n+5}\right]$, $n = 1, 2, \ldots$. All of the other prime ends of G_ε have one point impressions lying on the path J_ε, and every point of J_ε is the impression of just one prime end. Stating things slightly differently, we have

a) $\tilde{g}$ is a homeomorphism of $\mathbf{D}^- \setminus \{1\}$ onto $G_\varepsilon \cup J_\varepsilon$,

b) the set of cluster points of all sequences $\{g(\lambda_n)\}$, where $\lambda_n \in \mathbf{D}$ and $\lambda_n \to 1$, is exactly the set E_ε, and

c) if a sequence $\{\lambda_n\}$ of points of $G_\varepsilon \cup J_\varepsilon$ converges to a point of E_ε, then the sequence $\{\tilde{g}^{-1}(\lambda_n)\}$ converges to 1.

In order to deduce one more fact, let us consider a point e belonging to the interior of E_ε (i.e., $e \in E_\varepsilon$ and is not an endpoint), and write $(e/10, e)$ for the line segment in $\mathbf{C}$ joining those two points. Let $l_n = (\alpha_n, \beta_n), n = 1, 2, \ldots$, be the sequence of line segments constituting the set $(e/10, e) \cap G_\varepsilon$ (where $|\alpha_n| < |\beta_n|$; see Figure 1). Observe from a), b), and c) above and the geometry of the domain G_ε that for n sufficiently large, the points α_n and β_n are situated on the path J_ε in the following order:

(I) $\ldots, \beta_{n+2}, \alpha_{n+1}, \beta_n, \alpha_{n-1}, \ldots, \alpha_1, \beta_1, \ldots, \beta_{n-1}, \alpha_n, \beta_{n+1}, \alpha_{n+2}, \ldots$.

The corresponding points $a_n = \tilde{g}^{-1}(\alpha_n)$ and $b_n = \tilde{g}^{-1}(\beta_n)$ on the open arc $C \setminus \{1\}$ must then be situated in the same order, and by virtue of property c) they must converge in both directions to 1, that is,

$$1 \leftarrow \ldots, b_{n+2}, a_{n+1}, b_n, a_{n-1}, \ldots, a_1, b_1, \ldots, b_{n-1}, a_n, b_{n+1}, a_{n+2}, \ldots \to 1$$

as $n \to \infty$. The segments l_n themselves are mapped by g^{-1} onto disjoint open Jordan arcs $j_n = g^{-1}(l_n)$ lying in $\mathbf{D}$ and having their endpoints a_n, b_n on C. Each of the closed arcs j_n^- disconnects $\mathbf{D}^-$ and, again by property c), the convergence $l_n^- \to e$

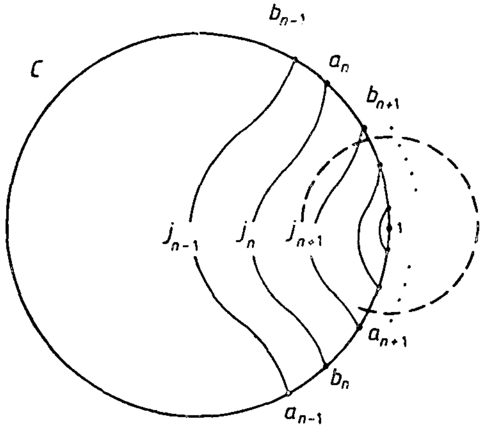

Fig. 2

implies the convergence $j_n^- \to 1$ (in the sense that every open disc centered at 1 contains j_n^- for n sufficiently large). See Figure 2.

4. PROOF OF THEOREM 2.1

Let $E = E_\varepsilon$, $0 < \varepsilon < 2\pi$, be a fixed subarc of C with 1 as its midpoint, as described in the Remark after Theorem 2.1, and let G_ε and $g = g_{E_\varepsilon}$ be, respectively, the Carathéodory domain contained in $\mathbf{D}$ and the Riemann mapping function of $\mathbf{D}$ onto G_ε described in Section 3. Let also A be any given operator in $(\mathscr{P})$.

We note first that since $G_\varepsilon \subset \mathbf{D}$, it follows immediately from what was said in Section 3 that $g(A)$ and $h(A)$ are completely nonunitary contractions and that $\mathscr{A}(A) = \mathscr{A}(g(A)) = \mathscr{A}(h(A))$. This proves (4).

We will show below, using only the fact that $\sigma(A)$ is connected and contains the point 1, that

$$E_\varepsilon \subset \sigma_e(g(A)), \tag{11}$$

and this will certainly imply that $1 \in \sigma(g(A))$. Furthermore, once we have shown that $\sigma(g(A))$ is connected and hence that $g(A) \in (\mathscr{P})$, then a repetition of the argument (with $g(A)$ in place of A) shows that $h(A) \in (\mathscr{P})$. Thus we now show, using (11), that $\sigma(g(A))$ is connected. Suppose, to the contrary, that $\sigma(g(A)) = F_1 \cup F_2$, where F_1 and F_2 are disjoint nonempty compact sets, and let S be the spectral idempotent associated with $g(A)$ corresponding to this decomposition, so arranged that the invariant subspaces [illegible] range S of $g(A)$ satisfy $\sigma(g(A)|\mathscr{H}_1) = F_1$

and $\sigma(g(A) \mid \mathscr{R}) = F_2$. Since $\mathscr{A}(A) = \mathscr{A}(g(A))$, we know that $\mathscr{K}$ and $\mathscr{R}$ are also invariant subspaces for A, and we write $A_1 = A \mid \mathscr{K}$, $A_2 = A \mid \mathscr{R}$. Since $\mathscr{K}$ and $\mathscr{R}$ are complements of one another, it follows easily that $\sigma(A) = \sigma(A_1) \cup \sigma(A_2)$, and since $\sigma(A)$ is connected we must have $\sigma(A_1) \cap \sigma(A_2) \neq \emptyset$. Furthermore it is clear that A_1 and A_2 are completely nonunitary, since A is, and that

$$g(A_1) = g(A) \mid \mathscr{K}, \quad g(A_2) = g(A) \mid \mathscr{R}.$$

If there were a point λ_0 in $\mathbf{D}$ belonging to $\sigma(A_1) \cap \sigma(A_2)$, then by [9, Corollary 3.1], $g(\lambda_0)$ would belong to $F_1 \cap F_2$, contrary to hypothesis. Furthermore, if there were a point ζ_0 in $(\mathbf{C} \setminus \{1\}) \cap \sigma(A_1) \cap \sigma(A_2)$, then by virtue of property a) of g and Proposition (FM) of [10] (proved but not explicitly stated in [9]), once again we would have $F_1 \cap F_2 \neq \emptyset$.

Thus, the only remaining possibility to be dealt with is the case $\sigma(A_1) \cap \cap\, \sigma(A_2) = \{1\}$. In this situation it is obvious that the connectedness of $\sigma(A)$ implies that of $\sigma(A_1)$ and $\sigma(A_2)$. But then A_1 and A_2 belong to $(\mathscr{P})$, and hence by (II), $E_\varepsilon \subset \sigma(g(A_1)) \cap \sigma(g(A_2)) = F_1 \cap F_2$. Since this also contradicts the hypothesis that $F_1 \cap F_2 = \emptyset$, we have proved that $\sigma(g(A))$ is connected, and consequently that $g(A)$ and $h(A)$ belong to $(\mathscr{P})$.

We show next, assuming only that A belongs to $(\mathscr{P})$, that $\sigma_e(g(A)) \cap C = E_\varepsilon$. The first step is to show that $E_\varepsilon \subset \sigma_e(g(A))$.

Suppose, to the contrary, that there is a point e in E_ε which is *not* in $\sigma_e(g(A))$. Then we must also have $e \notin \sigma(g(A))$. (For, if $e \in \sigma(g(A)) \setminus \sigma_e(g(A))$, then since e lies on the unit circle and $g(A)$ is a contraction, it follows easily that e is an eigenvalue of $g(A)$ and that the corresponding eigenspace is reducing for $g(A)$, contrary to the fact that $g(A)$ is completely nonunitary.) The remaider of the proof is very similar to the proof of the Theorem in [10], but for clarity certain changes have been made, and we give the remainder of the proof in full detail.

Since $\sigma(g(A))$ is compact, there is a neighborhood N of e such that $\sigma(g(A)) \cap \cap\, N = \emptyset$, and we may move e slightly on E_ε, if necessary, so that it remains in N and is different from the endpoints of E_ε. It follows from what was said in Section 3 that for n sufficiently large, say $n \geqslant n_0$, the endpoints α_n and β_n of l_n appear in the order indicated in (I). Furthermore, since $l_n^- \to e$, we may suppose that n_0 has been chosen large enough that $l_n^- \subset N$ for $n \geqslant n_0$, and hence that $\sigma(g(A)) \cap l_n^- = \emptyset$ for such n.

By virtue of [9, Corollary 3.1], we have $u(\sigma(A) \cap \mathbf{D}) \subset \sigma(u(A))$ for every u in H^∞, so we infer that

$$g(\sigma(A) \cap \mathbf{D}) \cap l_n = \emptyset, \quad n \geqslant n_0,$$

and consequently, because $g(j_n) = l_n$, it follows that

$$\sigma(A) \cap j_n = (\sigma(A) \cap \mathbf{D}) \cap j_n = \emptyset, \quad n \geqslant n_0 .$$

Moreover, since $a_n, b_n \in C\setminus\{1\}$ for all n, it follows from property a) of g above that $\tilde{g}$ is continuous at a_n and b_n, and since $\tilde{g}(a_n) = \alpha_n$, $\tilde{g}(b_n) = \beta_n$, we know from [10, Proposition (FM)] and the fact that $\alpha_n, \beta_n \in N$ for $n \geqslant n_0$ that neither a_n nor b_n can belong to $\sigma(A)$ for such n. Thus

$$\sigma(A) \cap j_n^- = \emptyset, \quad n \geqslant n_0 .$$

Since $\sigma(A)$ is connected, $j_n^- \to 1$, and each j_n^- disconnects $\mathbf{D}^-$, we conclude that $\sigma(A)$ must consist of the singleton $\{1\}$.

But this implies, by [20, Chapter VI], that the characteristic function $\theta_A(\lambda)$ of A is a contractive, operator valued, analytic function on $\mathbf{D}^-\setminus\{1\}$ that is unitary valued on $C\setminus\{1\}$. Moreover, $\theta_A(\lambda)^{-1}$ exists for every $\lambda \in \mathbf{D}^-\setminus\{1\}$ and this function is an analytic function on $\mathbf{D}$ that is continuous on $\mathbf{D}^-\setminus\{1\}$. From these facts it follows that $\|\theta_A(\lambda)^{-1}\|$ is subharmonic on $\mathbf{D}$ and equal to one on $C\setminus\{1\}$ Hence, if for $n \geqslant n_0$ we denote by $\mathbf{D}_n^-$ the connected subset of $\mathbf{D}^-$ whose boundary is the union of j_n^- and the arc $\widehat{a_n b_n}$ on C which does not contain the point 1, we have

$$\text{(II)} \qquad \mathbf{D}_{n_0}^- \subset \mathbf{D}_{n_0+1}^- \subset \ldots \quad \text{and} \quad \bigcup_{n=n_0}^{\infty} \mathbf{D}_n^- = \mathbf{D}^-\setminus\{1\}.$$

Using the fact that $\theta_A(\lambda)$ is a contraction for λ in $\mathbf{D}$ (so that $\|\theta_A(\lambda)^{-1}\| \geqslant 1$ on $\mathbf{D}$) and the maximum principle for subharmonic functions, we deduce that for each $n \geqslant n_0$ there exists at least one point λ_n of j_n at which the maximum of $\|\theta_A(\lambda)^{-1}\|$ on $\mathbf{D}_n^-$ is attained.

Suppose now that the (obviously increasing) sequence $\{\|\theta_A(\lambda_n)^{-1}\|\}$ is bounded. Then, by virtue of (III), it follows that $\|\theta_A(\lambda)^{-1}\|$ is bounded on the open unit disc $\mathbf{D}$, and that implies, in turn, by [20, Theorem IX.1.2], that A is similar to some unitary operator U. Thus $\sigma(U) = \sigma(A) = \{1\}$, so U must be the identity operator, which implies the same for A. But this contradicts the fact that A is completely nonunitary, so we conclude that

$$\text{(IV)} \qquad \lim_n \|\theta_A(\lambda_n)^{-1}\| = +\infty .$$

Since $\lambda_n \in j_n$, we have $g(\lambda_n) \in l_n$, and hence $g(\lambda_n) \to e$ as $n \to \infty$. Furthermore, by virtue of (IV) and the inequality

$$(1 - |\lambda|)\|(A - \lambda)^{-1}\| \leqslant \|\theta_A(\lambda)^{-1}\| \leqslant 1 + 2(1 - |\lambda|)\, \|(A - \lambda)^{-1}\| ,$$

valid for every λ in $\mathbf{D}$ (cf. [20, Proposition VI. 4.2]), there exists a sequence $\{\eta_n\}$ of positive numbers converging to zero such that

$$\text{(V)} \qquad \|(A - \lambda_n)^{-1}\|^{-1} < \eta_n(1 - |\lambda_n|), \quad n \geqslant n_0.$$

Since the left hand member of (V) is just the lower bound of the operator $A - \lambda_n$, it follows that there exists a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that

$$\text{(VI)} \qquad \|(A - \lambda_n)x_n\| < \eta_n(1 - |\lambda_n|), \quad n \geq n_0 .$$

Moreover, we may write $g(\lambda) - g(\lambda_n) = (\lambda - \lambda_n)k_n(\lambda)$ for each such integer $n \geq n_0$, and it is clear that $k_n \in H^\infty$ and satisfies $\|k_n\|_\infty \leq 2/(1 - |\lambda_n|)$. Thus, employing (VI), we have

$$\begin{aligned}\|(g(A) - e)x_n\| &\leq \|(g(A) - g(\lambda_n))x_n\| + |g(\lambda_n) - e| \leq \\ &\leq \|k_n(A)\| \, \|(A - \lambda_n)x_n\| + |g(\lambda_n) - e| \leq \\ &\leq \{2/(1 - |\lambda_n|)\}\eta_n(1 - |\lambda_n|) + |g(\lambda_n) - e| \leq \\ &\leq 2\eta_n + |g(\lambda_n) - e| \to 0,\end{aligned}$$

from which it follows that $e \in \sigma(g(A))$, a manifest contradiction.

We conclude that $E_\varepsilon \subset \sigma(g(A))$, and, moreover, by what was shown earlier, $E_\varepsilon \subset \sigma_e(g(A))$. Since $E_\varepsilon \subset C$ and $g(A)$ is a contraction, we must have $E_\varepsilon \subset \partial\sigma_e(g(A))$. Furthermore, if $a \in C \setminus E_\varepsilon$, then the function $(g(\lambda) - a)^{-1}$ clearly belongs to H^∞ and is the inverse there of $g(\lambda) - a$. Hence, since the functional calculus is a homomorphism, $g(A) - a$ is invertible, and consequently $a \notin \sigma(g(A))$. Thus $\partial\sigma_e(g_\varepsilon(A)) \cap C = E_\varepsilon$, and (2) is proved.

To complete the proof of Theorem 2.1, it remains only to prove (3) and $0 \notin \sigma(h(A))$. To this end, let e_1 and e_2 denote the endpoints on C of the arc E_ε. Then, using property a) of g, we know that there exist unique points λ_1 and λ_2 belonging to the path J_ε such that $\tilde{g}(e_i) = \lambda_i$, $i = 1, 2$. If P_ε denotes the open subpath of J_ε that joins λ_1 to λ_2 and is bounded away from C, then it follows from properties a) and c) of g that $\tilde{g}$ maps the set $E_\varepsilon \setminus \{1\}$ onto $J_\varepsilon \setminus P_\varepsilon$, which is itself a union of two subpaths of J_ε, and, more importantly, a dominating set for the arc E_ε. (In fact, every point e^{it} of $E_\varepsilon \setminus \{e_1, e_2\}$ is the limit of a sequence of points of $J_\varepsilon \setminus P_\varepsilon$ that lie on the radius $\{re^{it}: 0 < r < 1\}$.) Furthermore, we can apply [10, Proposition (FM)] (with $T = g(A)$, $u = g$, and ζ_0 any point on $E_\varepsilon \setminus \{1\}$) to conclude that $J_\varepsilon \setminus P_\varepsilon \subset \sigma(h(A))$. Finally, an argument like one given above shows that no point of $\mathbf{D} \setminus (G_\varepsilon \cup J_\varepsilon)$ can belong to $\sigma(h(A))$, so, in particular, $0 \notin \sigma(h(A))$ and $J_\varepsilon \setminus P_\varepsilon \subset \partial\sigma(h(A))$. Since every point of $J_\varepsilon \setminus P_\varepsilon$ is an accumulation point of $J_\varepsilon \setminus P_\varepsilon$, it follows (cf. [16, Corollary 1.26]) that $J_\varepsilon \setminus P_\varepsilon \subset \partial\sigma_e(h(A))$, and the proof of Theorem 2.1 is complete.

5. INVERSES OF (BCP)-OPERATORS

In this section we first make good our promise to show that the invariant subspace problem for the class $(\mathscr{F})$ can be solved by solving the "inverse" problem for invertible (BCP) operators. The main tool is the following theorem.

THEOREM 5.1. *There exists a function k in H^∞ that maps $\mathbf{D}$ conformally onto a Carathéodory domain $K \subset \mathbf{C} \setminus \mathbf{D}^-$ and is such that if $g = g_{E_\varepsilon}$, $0 < \varepsilon < 2\pi$, is any of the family of conformal mappings from Theorem 2.1, and A is any operator in $(\mathscr{P})$, then*

(1) *$(k \circ g)(A)$ is invertible and $[(k \circ g)(A)]^{-1} = (1/k)(g(A))$ is a completely nonunitary contraction,*

(2) *$\partial\sigma_e([(k \circ g)(A)]^{-1}) \cap \mathbf{D}$ is dominating for C, and*

(3) *$\mathscr{A}(A) = \mathscr{A}(g(A)) = \mathscr{A}((k \circ g)(A))$.*

The desired corollary follows easily from this result.

COROLLARY 5.2. *If the inverse of every invertible operator in* (BCP) *has a nontrivial invariant subspace, then every operator in $(\mathscr{F})$ has a nontrivial invariant subspace.*

Proof. To prove the corollary, according to Proposition 1.1 it suffices to show that an arbitrary operator A in $(\mathscr{P})$ has a nontrivial invariant subspace. Let $g = g_{E_\varepsilon}$, $0 < \varepsilon < 2\pi$, be any one of the family of conformal mappings from Theorem 2.1 and let k be the conformal mapping from Theorem 5.1. Then, according to (1) and (2) of the latter theorem, $B = [(k \circ g)(A)]^{-1}$ is an invertible (BCP)-operator. It then follows from the hypothesis of the corollary that $\mathrm{Lat}(B^{-1}) \neq \{(0), \mathscr{H}\}$, and since $\mathrm{Lat}(B^{-1}) = \mathrm{Lat}(g(A)) = \mathrm{Lat}(A)$ from conclusion (3) of the theorem, the result follows.

Proof of Theorem 5.1. Let K be the snakelike domain, spiraling down on the unit circle, determined by the line segment $L = [11/6, 2]$ and the curves

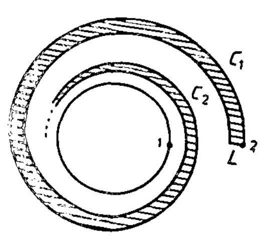

Fig. 3

$$C_1 = \left\{ \left(\frac{2+t}{1+t} \right) e^{it} : 0 \leq t < +\infty \right\}$$

and

$$C_2 = \left\{ \left(\frac{11/5 - t}{6/5 + t} \right) e^{it} : 0 \leq t < +\infty \right\}.$$

(See Figure 3 for a sketch of K.)

It is easy to see that K is simply connected, that ∂K is the union of the open Jordan arc $J = C_1 \cup L \cup C_2$ and the unit circle C, and that all of ∂K is outer boundary. Thus K is a Carathéodory domain. It is also clear that K possesses exactly one prime end $\hat{E}$ whose impression E contains more than one point, and that $E = C$. Moreover, no two distinct prime ends of K have overlapping impressions. Thus if k is a conformal map of $\mathbf{D}$ onto K, so normalized that the point 1 of $\mathbf{D}^-$ corresponds to the prime end $\hat{E}$, then by the theorem of Carathéodory, k can be extended to a homeomorphism $\tilde{k}$ of $\mathbf{D}^-$ onto the quotient space obtained from K^- by identifying all the points of C (with a single point). In particular, $\tilde{k}$ maps $\mathbf{D}^-\setminus\{1\}$ onto $K^-\setminus C$.

Suppose now that A is any operator in $(\mathscr{P})$ and that $g = g_{E_\varepsilon}$, $0 < \varepsilon < 2\pi$, is any (fixed) member of the family of conformal maps constructed in Theorem 2.1. Then, according to that theorem, we know that $g(A) \in (\mathscr{P})$ and thus, in particular, by [20, Theorem III.2.1], we have that $(k \circ g)(A) = k(g(A))$. Moreover, since k is obviously invertible in H^∞ and $\|1/k\|_\infty = 1$, it follows easily from [20, Theorem III.2.1] that

$$\text{(VII)} \qquad [k(g(A))]^{-1} = \left(\frac{1}{k}\right)(g(A))$$

and that this operator is a completely nonunitary contraction. Thus (1) is proved.

To prove (3), we simply observe that $\mathscr{A}(A) = \mathscr{A}(g(A))$ from Theorem 2.1, and since K is Carathéodory, that $\mathscr{A}(g(A)) = \mathscr{A}(k(g(A)))$ follows as in Section 3.

To establish (2), we note first that since $\sigma(k(g(A))) \subset K^-$, it suffices to show that there exists a $\delta > 0$ such that all points on $C_1 \cup C_2$ whose distance from C is less than δ belong to $\sigma(k(g(A)))$. (For, such points will then be accumulation points of $\partial\sigma(k(g(A)))$, and thus by [17, Corollary 1.26] will belong to $\partial\sigma_e(k(g(A)))$. Furthermore, by the spectral mapping theorem, $\partial\sigma_e([k(g(A))]^{-1}) = [\partial\sigma_e(k(g(A)))]^{-1}$.) To see that such a δ exists, we observe that it follows from conclusion (2) of Theorem 2.1 that the arc E_ε of C corresponding to the function $g = g_{E_\varepsilon}$ is contained in $\sigma(g(A))$. Hence, since $\tilde{k}$ is continuous on $C\setminus\{1\}$ and maps this set onto $J = C_1 \cup L \cup C_2$, we know from [10, Proposition (FM)] that $\sigma(k(g(A)))$ contains the image $\tilde{k}(E_\varepsilon\setminus\{1\})$. Since $\tilde{k}(C\setminus E_\varepsilon)$ must be a subarc of J that is bounded away from C, it follows that there exists a $\delta > 0$ such that $\tilde{k}(E_\varepsilon\setminus\{1\})$ contains all points ζ on $C_1 \cup C_2$ such that $\operatorname{dist}(\zeta, C) < \delta$, and the argument is complete.

Theorem 5.1 also enables us to prove the following rather interesting analog of Corollary 2.5 that was mentioned in the introduction.

COROLLARY 5.3. *There exists a function h in H^∞ such that for every operator A in $(\mathscr{P})$,*

(1) *$h(A)$ is an invertible operator in* (BCP),

(2) $\operatorname{Lat}(h(A)) \cap \operatorname{Lat}(h(A)^{-1}) = \operatorname{Lat}(A)$, *and*

(3) $\operatorname{Lat}(h(A))$ *contains a lattice L that is disjoint from* $\operatorname{Lat}(A)$ *and is isomorphic to the lattice of all subspaces of $\mathscr{H}$.*

Proof. Let h be the composition $h = (1/k) \circ g$, where k and g are as in Theorem 5.1. Then we know from Theorem 5.1 and (VII) that $h(A)$ is an invertible operator in (BCP) and that $\operatorname{Lat}(h(A)^{-1}) = \operatorname{Lat}(A)$. Since obviously $\operatorname{Lat}(h(A)) \supset \operatorname{Lat}(A)$,

(2) follows. To prove (3), we recall from [3] that since $h(A) \in$ (BCP), there is a subspace $\mathscr{M} \in \mathrm{Lat}(h(A))$ such that $\mathscr{M} \ominus h(A)\mathscr{M}$ has dimension $\aleph_0$. Thus the required lattice L in (3) can be taken to be

$$L = \{h(A)\mathscr{M} \oplus \mathscr{N} : \mathscr{N} \text{ is a subspace of } \mathscr{M} \ominus h(A)\mathscr{M}\},$$

since none of the subspaces in L is invariant under $h(A)^{-1}$.

6. AN EXAMPLE

Despite the fact, mentioned earlier, that we now have considerable information about the structure of (BCP)-operators, we still do not know whether the inverse of every invertible (BCP)-operator has a nontrivial invariant subspace, nor whether every square root in ($\mathscr{P}$) of an invertible (BCP) operator has a nontrivial invariant subspace. (We do know, however, from [3], that if A is any invertible (BCP)-operator, then there exist nontrivial invariant subspaces $\mathscr{M}$ and $\mathscr{N}$ for A such that $A(\mathscr{M} \cap \mathscr{N}) = \mathscr{M} \cap \mathscr{N}$, so that either A^{-1} has a nontrivial invariant subspace or $\mathscr{M} \cap \mathscr{N} = (0)$.)

In this section we give a specific example that illustrates some of the difficulties inherent in trying to solve the "square root" and "inverse" problems for (BCP) operators.

EXAMPLE 6.1. Let H^2 be, as usual, the Hilbert space consisting of all functions $u(\zeta)$, holomorphic on $\mathbf{D}$, such that the norm

$$\|u\|_2 = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |u(re^{it})|^2 \, dt \right)^{1/2}$$

is finite, and let U_+ denote the (unilateral shift) operator M_ζ of multiplication by the position function on H^2. Furthermore, let $m(\zeta)$ be the singular inner function $m(\zeta) = e^{\frac{\zeta+1}{\zeta-1}}$ in H^∞. One knows that mH^2 is an invariant subspace of U_+, so that $\mathscr{H}(m) = H^2 \ominus mH^2$ is an invariant subspace for U_+^*. Let $A^* = U_+^* | \mathscr{H}(m)$, so that A is the operator in $\mathscr{L}(\mathscr{H}(m))$ defined by

$$\text{(VIII)} \qquad Av(\zeta) = P_{\mathscr{H}(m)}\zeta v(\zeta), \quad v(\zeta) \in \mathscr{H}(m).$$

It is well known (cf. [20, p. 124]) that A is a completely nonunitary contraction of class C_0 (in the terminology of [20]) whose minimal function is m. Furthermore one knows, because of the connection between the spectrum and the minimal function [20, Theorem III.5.1], that $\sigma(A) = \{1\}$, so A is an invertible operator in ($\mathscr{P}$). Finally, because of the relation between the divisors of m and the invariant subspaces of A [20, pp. 129—139], one knows that the invariant subspaces of A are exactly the spaces $\{H^2 \ominus m_t H^2\}_{0 \leqslant t \leqslant 1}$, where m_t is the singular inner function $m_t(\zeta) = e^{t\left(\frac{\zeta+1}{\zeta-1}\right)}$, $0 \leqslant t \leqslant 1$. Thus Lat($A$) is isomorphic to the closed interval [0,1], and all of the invariant subspaces of A are hyperinvariant for A [20, Proposition III.7.6].

With A as in (VIII), let $g = g_{E_\varepsilon}$ and h be the conformal maps of Theorem 2.1 corresponding to any (fixed) one of the arcs E_ε, $\pi \leqslant \varepsilon < 2\pi$. Then, according to Theorem 2.1 and Corollary 2.5, one has that Lat($h(A)$) is isomorphic to [0,1], while $h(A)^2 \in$ (BCP) and its invariant subspace lattice Lat($h(A)^2$) is so large that it contains a lattice L disjoint from Lat($h(A)$) that is isomorphic to the lattice of all subspaces of $\mathscr{H}$. Thus, if one is to solve the square root problem for (BCP)-operators, one must somehow "find" an element of Lat($h(A)$) among the much larger set Lat($h(A)^2$). Note also that since Lat($h(A)$) is linearly ordered, $h(A)$ is not a reflexive operator, but $[h(A)]^2$ is reflexive since it is a (BCP)-operator [1].

A similar situation occurs if we take A to be as in (VIII) and h to be the function of Corollary 5.3. In this case, one has that $h(A) \in$ (BCP) and has a huge lattice containing Lat(A), while Lat($h(A)^{-1}$) = Lat(A) $\simeq$ [0,1]. So to solve the inverse problem for (BCP)-operators, one must somehow locate one of the elements of this tiny lattice Lat(A) among the huge lattice Lat($h(A)$). Note also that in this case $h(A)$ is reflexive, while $h(A)^{-1}$ is not reflexive. To our knowledge, this is the first example exhibited of a reflexive invertible operator on a Hilbert space whose inverse is not reflexive. (This phenomenon cannot occur on a finite dimensional space.)

REMARKS. 1) Enhancement of the technique used to prove Theorem 2.1 actually yields (cf. [11]) this theorem: *If* $h \in H^\infty$, $\iint_{\mathbf{D}} |h'| < +\infty$, *and h has a continuous extension to almost every point of C, then for every A in* ($\mathscr{P}$), *the nontangential cluster set of h at* 1 *belong to* $\sigma(h(A))$.

2) An example of a completely nonunitary contraction A and an H^∞-function h such that $\sigma(A) = \{1\}$ but $\sigma(h(A))$ is not a singleton was given in [12]. (The first such example had been given earlier by C. Foiaş.) The specific operator A with this property exhibited in [12] was the operator in (VIII) above, and the domain corresponding to the conformal mapping h was similar to that in Figure 1. The authors of [12] say that their idea was partly due to R. G. Douglas.

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C. FOIAŞ
Department of Mathematics,
Indiana University,
Bloomington, IN 47401,
U.S.A.

C. M. PEARCY
Department of Mathematics,
University of Michigan,
Ann Arbor, Michigan 48109,
U.S.A.

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