J. OPERATOR THEORY
10(1983), 3--8

THE FUBINI THEOREM FOR EXACT C^*-ALGEBRAS

EBERHARD KIRCHBERG

1. INTRODUCTION

Let A and B be C^*-algebras. By $A \otimes B$ we denote the minimal (or spatial) C^*-tensor product of A and B, [7]. Further, algebra, subalgebra, tensor product, and ideal mean C^*-algebra, C^*-subalgebra, C^*-tensor product and, closed ideal, respectively. $A \otimes B$ is a subalgebra of the spatial W^*-tensor product $A^{**} \overline{\otimes} B^{**}$, see e.g. [6]. For every linear functional $\varphi \in A^*$ there is a unique ultraweakly continuous linear map L_φ from $A^{**} \overline{\otimes} B^{**}$ into B^{**} with $L_\varphi(a \otimes b) = \varphi(a)b$ for $a \in A$, $b \in B$, see e.g. [6], [8]. Hence $L_\varphi(A \otimes B) = B$ if $\varphi \neq 0$ and $\varphi \in A^* \to L_\varphi \in (A \otimes B, B)$ is an isometric linear map.

Now let D be a subalgebra of B. Then D^{**} is a W^*-subalgebra of B^{**} and $D^{**} \cap B = D$. The *Fubini algebra* $\mathrm{F}(A, D, A \otimes B)$ of A and D in $A \otimes B$ is the set of all $t \in A \otimes B$ with $L_\varphi(t) \in D$ for every state of A. From $D^{**} \cap B = D$, $L_\varphi(A \otimes B) \subseteq B$ and from the commutation theorem for W^*-algebras we get

$$\mathrm{F}(A, D, A \otimes B) = (A \otimes B) \cap F(A^{**}, D^{**}, A^{**} \overline{\otimes} B^{**})$$

and

$$\mathrm{F}(A^{**}, D^{**}, A^{**} \overline{\otimes} B^{**}) = A^{**} \overline{\otimes} D^{**},$$

i.e. $\mathrm{F}(A, D, A \otimes B)$ is a subalgebra of $A \otimes B$ containing $A \otimes D$.

For fixed A the minimal tensor product is a functor $B \curvearrowright A \otimes B$ in the category of C^*-algebras. We call A *exact* if this functor is an exact functor, [3]. According to [4], A is exact if and only if $\mathrm{F}(A, J, A \otimes B) = A \otimes J$ for every algebra B and every ideal J of B. By [5] quotiens of subalgebras of nuclear algebras are exact. Now let H be a Hilbert space of countable dimension and let $\mathscr{L}(H)$ and $\mathscr{C}\mathscr{L}(H)$ denote the algebras of bounded and compact linear operators on H, respectively. The kernel of the canonical epimorphism $A \otimes \mathscr{L}(H) \to A \otimes (\mathscr{L}(H)/\mathscr{C}\mathscr{L}(H))$ is $\mathrm{F}(A, \mathscr{C}\mathscr{L}(H), A \otimes \mathscr{L}(H))$. Hence

$$0 \to A \otimes \mathscr{C}\mathscr{L}(H) \to A \otimes \mathscr{L}(H) \to A \otimes (\mathscr{L}(H)/\mathscr{C}\mathscr{L}(H)) \to 0$$

is an exact sequence if and only if

$$\mathrm{F}(A, \mathscr{C}\mathscr{L}(H), A \otimes \mathscr{L}(H)) = A \otimes \mathscr{C}\mathscr{L}(H).$$

In Section 2 we prove the following Fubini type theorem.

THEOREM 1.1. *Let A be a C^*-algebra. The following properties of A are equivalent:*

(i) $\mathrm{F}(A, D, A \otimes B) = A \otimes D$ *for every C^*-algebra B and every hereditary C^*-subalgebra D of B* (i.e. $DBD \subseteq D$).

(ii) *A is exact.*

(iii) *The canonical sequence*

$$0 \to A \otimes \mathscr{C}\mathscr{L}(H) \to A \otimes \mathscr{L}(H) \to A \otimes (\mathscr{L}(H)/\mathscr{C}\mathscr{L}(H)) \to 0$$

is exact.

According to [4], subalgebras of exact algebras are exact. $\mathbf{C} \oplus M_2 \oplus M_3 \oplus \ldots$ $\ldots (\subseteq \mathscr{L}(H))$ is not exact and $C^*(G)$ is not exact for every discrete maximally almost periodic nonamenable group G (e.g. $G =$ free group of two generators).

COROLLARY 1.2. (Wassermann [9], [10]). *Let M be a W^*-algebra.*

$$M \otimes \mathscr{C}\mathscr{L}(H) \to M \otimes \mathscr{L}(H) \to M \otimes (\mathscr{L}(H)/\mathscr{C}\mathscr{L}(H))$$

is exact if and only if M is a subalgebra of $C(K, M_n) \cong M_n \otimes C(K)$ for some compact space K and some positive integer n. In particular,

$$\mathscr{L}(H) \otimes \mathscr{C}\mathscr{L}(H) \to \mathscr{L}(H) \otimes \mathscr{L}(H) \to \mathscr{L}(H) \otimes (\mathscr{L}(H)/\mathscr{C}\mathscr{L}(H))$$

is not exact.

COROLLARY 1.3.

$$C^*(G) \otimes \mathscr{C}\mathscr{L}(H) \to C^*(G) \otimes \mathscr{L}(H) \to C^*(G) \otimes (\mathscr{L}(H)/\mathscr{C}\mathscr{L}(H))$$

is not exact for every maximally almost periodic nonamenable discrete group G.

REMARK. A satisfies the *Fubini property* if $\mathrm{F}(A, D, A \otimes B) = A \otimes D$ for every B and every subalgebra D of B. It is easy to check that we can restrict to unital separable B and unital subalgebras. By the strong completely positive approximation property, nuclear algebras have the Fubini property. In general A has the Fubini property if

$$\mathrm{F}(A, C^*(F(S)) + J, A \otimes C^*(F(S \cup T))) = A \otimes (C^*(F(S)) + J)$$

for every ideal J of $C^*(F(S \cup T))$. Here $F(S)$ denotes the free group generated by S. S and T are countable sets. This follows from

$$D \cong (C^*(F(S)) + J)/J \subseteq C^*(F(S \cup T))/J \cong B$$

for some ideal J if B and D are unital and separable and from

$$\mathrm{F}(A, \pi^{-1}(D), A \otimes C) = (\mathrm{id} \otimes \pi)^{-1}(\mathrm{F}(A, D, A \otimes B))$$

if $\pi: C \to B$ is an epimorphism. Compare [1], [2], and [5] for further information.

2. PROOF OF THEOREM 1.1

Ideals J of B are hereditary subalgebras of B: $JBJ \subseteq B$. Thus the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. To show the implication (iii) $\Rightarrow$ (i) we need three lemmas.

LEMMA 2.1. *Let C be a separable C^*-subalgebra of B. There exists $h \geqslant 0$ in C such that $C \subseteq \overline{hBh}$.*

Proof. Put $h = \sum_{n=1}^{\infty} 2^{-n} h_n$, where $h_1, h_2, \ldots$ is a sequence dense in $\{k \in C \mid 0 \leqslant k, \|k\| \leqslant 1\}$. $\overline{hBh}$ is a hereditary subalgebra of B containing h^3, hence $h \in \overline{hBh}$ and $\{h_1, h_2, \ldots\} \subseteq \overline{hBh}$, i.e. $C \subseteq \overline{hBh}$. Q.E.D.

Now let A and B be C^*-algebras and $0 \leqslant h \in B$, $\|h\| \leqslant 1$. We denote by $m(B) = B \oplus B \oplus \ldots$ the algebra of bounded sequences in B and by $c_0(B)$ the zero sequences in B. Then $c_0(B)$ is an ideal of $m(B)$.

Put $j := (h, h^{1/2}, \ldots, h^{1/n}, \ldots)$ and $\varepsilon(b) = (b, b, \ldots, b, \ldots)$ if $b \in B$. Then $j \in M(B)$ and ε is a monomorphism from B into $m(B)$. So $\delta(b) = \varepsilon(b) - j\varepsilon(b)j$ defines a linear map from B into $m(B)$ and $\mathrm{id} \odot \delta : A \odot B \to A \odot m(B)$ (algebraic tensor product!) extends uniquely to a bounded linear map $\mathrm{id} \otimes \delta$ from $A \otimes B$ into $A \otimes m(B)$ with $\mathrm{id} \otimes \delta(t) = \mathrm{id} \otimes \varepsilon(t) - (1 \otimes j)\, \mathrm{id} \otimes \varepsilon(t)\, (1 \otimes j)$ and $L_\varphi(\mathrm{id} \otimes \delta(t)) = \delta(L_\varphi(t))$ for $t \in A \otimes B$, $\varphi \in A^*$.

Let p_n denote the canonical epimorphisms $p_n : (b_1, b_2, \ldots, b_n, \ldots) \in m(B) \to b_n \in B$. Then $\varkappa(t) := (\mathrm{id} \otimes p_1(t), \mathrm{id} \otimes p_2(t), \ldots)$ defines a canonical monomorphism $\varkappa$ from $A \otimes m(B)$ into $m(A \otimes B)$ such that $\varkappa(a \otimes (b_1, b_2, \ldots)) = (a \otimes b_1, a \otimes b_2, \ldots)$. $\varkappa$ extends to the canonical isomorphism between the W^*-algebras $A^{**} \mathbin{\overline{\otimes}} m(B^{**}) \supseteq A \otimes m(B)$ and $m(A^{**} \mathbin{\overline{\otimes}} B^{**}) \supseteq m(A \otimes B)$.

$\varkappa(a \otimes (b_1, b_2, \ldots)) \in c_0(A \otimes B)$ if $(b_1, b_2, \ldots) \in c_0(B)$. So $\varkappa(A \otimes c_0(B)) \subseteq c_0(A \otimes B)$. The span of elements $\varkappa(a \otimes (0, \ldots, 0, b, 0, \ldots)) = (0, \ldots, 0, a \otimes b, 0, \ldots)$ is dense in $c_0(A \otimes B)$, i.e. $\varkappa(A \otimes c_0(B)) = c_0(A \otimes B)$.

LEMMA 2.2. *Let $t \in A \otimes B$. Then*

(i) *$t \in A \otimes \overline{hBh}$ if and only if* $\mathrm{id} \otimes \delta(t) \in A \otimes c_0(B)$.

(ii) *$t \in \mathrm{F}(A, \overline{hBh}, A \otimes B)$ if and only if* $\mathrm{id} \otimes \delta(t) \in \mathrm{F}(A, c_0(B), A \otimes m(B))$.

Proof. $\overline{hBh} =: C$ is a hereditary subalgebra of B containing h^3. Hence

$$h^{1/n} \in C, \quad h^{1/n} B h^{1/n} \subseteq C,$$

$$(1 \otimes h^{1/n})\, A \odot B (1 \otimes h^{1/n}) = A \odot h^{1/n} B h^{1/n} \subseteq A \otimes C$$

and

$$(1 \otimes h^{1/n})\, A \otimes B (1 \otimes h^{1/n}) \subseteq A \otimes C$$

by continuity of $t \to (1 \otimes h^{1/n}) t (1 \otimes h^{1/n})$.

$\mathrm{id} \otimes \delta(t) \in A \otimes c_0(B)$ means $\varkappa(\mathrm{id} \otimes \delta(t)) \in c_0(A \otimes B)$, i.e.

$$\|\mathrm{id} \otimes p_n(\mathrm{id} \otimes \delta(t))\| = \|t - (1 \otimes h^{1/n})\, t(1 \otimes h^{1/n})\| \to 0 \quad \text{if } n \to \infty.$$

Thus $t \in A \otimes C$ if $\mathrm{id} \otimes \delta(t) \in A \otimes c_0(B)$. $h^{1/n}$ tends to h if $n \to \infty$, hence $\delta(hBh) \subseteq$ $\subseteq c_0(B)$, $\delta(C) \subseteq c_0(B)$, $\mathrm{id} \otimes \delta(A \odot C) \subseteq A \otimes c_0(B)$ and $\mathrm{id} \otimes \delta(A \otimes C) \subseteq A \otimes c_0(B)$ by continuity. (ii) follows from $L_\varphi(\mathrm{id} \otimes \delta(t)) = \delta(L_\varphi(t))$ if we put in (i) $A = \mathbf{C} :=$ $:=$ complex numbers. Q.E.D

Now let S be the set of completely positive maps V from $m(B)$ into $\mathscr{L}(H)$ with $V(c_0(B)) \subseteq \mathscr{C}\mathscr{L}(H)$. From the decomposition $V(b) = T^*D(b)T$ with $D: m(B) \to$ $\to \mathscr{L}(K)$ a $*$-representation and $T \in \mathscr{L}(H, K)$ we obtain that $\mathrm{id} \odot V: A \odot m(B) \to$ $\to A \odot \mathscr{L}(H)$ extends uniquely to a continuous linear map $\mathrm{id} \otimes V$ from $A \otimes m(B)$ into $A \otimes \mathscr{L}(H)$ with $\mathrm{id} \otimes V(A \otimes c_0(B)) \subseteq A \otimes \mathscr{C}\mathscr{L}(H)$ and $\|\mathrm{id} \otimes V\| = \|V\|$.

LEMMA 2.3. *Let $t \in A \otimes m(b)$. Then*

(i) *$t \in A \otimes c_0(B)$ if and only if $\mathrm{id} \otimes V(t) \in A \otimes \mathscr{C}\mathscr{L}(H)$ for every $V \in S$;*

(ii) *$t \in \mathrm{F}(A, c_0(B), A \otimes m(B))$ if and only if $\mathrm{id} \otimes V(t) \in \mathrm{F}(A, \mathscr{C}\mathscr{L}(H), A \otimes \mathscr{L}(H))$ for every $V \in S$.*

Proof. Let be $s \in A \otimes B$, then $\|s\| = \sup\{\|\mathrm{id} \otimes V(s)\| : V: B \to M_n$ completely positive, $\|V\| \leqslant 1, n = 1, 2, \dots\}$. In fact for fixed faithful representations $D_1: A \to \mathscr{L}(H_1)$ and $D_2: B \to \mathscr{L}(H_2)$ with $r = D_1 \otimes D_2(s) \in \mathscr{L}(H_1) \otimes \mathscr{L}(H_2) \subseteq$ $\subseteq \mathscr{L}(H_1 \otimes H_2)$ we have

$$\|s\| = \|r\| = \sup\{\langle rx, y\rangle \mid x, y \in H_1 \odot H_2,\ \|x\| = \|y\| = 1\}$$

and

$$|\langle rx, y\rangle| = \|(1 \otimes p)r(1 \otimes p)\| \leqslant \|\mathrm{id} \otimes V(s)\|$$

if $V(B) = pD_2(b)p \mid pH_2$, $p = p^*p$, $(\mathrm{id} \otimes p)x = x$, $(\mathrm{id} \otimes p)y = y$, $\dim pH_2 =$ $= n < \infty$, $\|x\| = \|y\| = 1$. Then $\mathscr{L}(pH_2) \cong M_n$, $\|V\| = 1$ and V is completely positive. On the other hand, $\|\mathrm{id} \otimes V(s)\| \leqslant \|V\| \cdot \|s\|$ if V is completely positive.

Now let $t \in A \otimes m(B) \setminus A \otimes c_0(B)$. Then $\varkappa(t) \in m(A \otimes B) \setminus c_0(A \otimes B)$. So there exist $\varepsilon > 0$ and a strictly increasing sequence $\nu(1), \nu(2), \dots$ of positive integers such that $\|\mathrm{id} \otimes P_{\nu(n)}(t)\| \geqslant 2\varepsilon$. Hence for every n there exist a positive integer $\tau(n)$ and a completely positive map $V_n: B \to M_{\tau(n)}$ with $\|V_n\| \leqslant 1$ and $\|\mathrm{id} \otimes V_n(\mathrm{id} \otimes p_{\nu(n)}(t))\| \geqslant \varepsilon$. Put $H = \mathbf{C}^{\nu(1)} \oplus \mathbf{C}^{\nu(2)} \oplus \dots$ (Hilbert space sum). Then $M = M_{\nu(1)} \oplus M_{\nu(2)} \oplus \dots \subseteq \mathscr{L}(H)$ and $V: (b_1, b_2, \dots) \in m(B) \to (V_1(b_{\nu(1)}),$ $V_2(b_{\nu(2)}), \dots) \in M \subseteq \mathscr{L}(H)$ defines a completely positive linear map from $m(B)$ into $\mathscr{L}(H)$ with $\|V\| \leqslant 1$ and $V(0, \dots, 0, b, 0, \dots) \in \mathscr{C}\mathscr{L}(H)$. So $V(c_0(B)) \subseteq \mathscr{C}\mathscr{L}(H)$, i.e. $V \in S$. Let $P_n: H \to 0 \oplus 0 \oplus \dots \oplus 0 \oplus \mathbf{C}^{\nu(n)} \oplus 0 \oplus \dots \subseteq H$ be the canonical orthogonal projection if $n \geqslant 1$ and $P_0 = 0$. Put $Q_n = 1 - \sum_{k=0}^{n-1} P_k$. Then $P_n \leqslant Q_n$

and

$$\|(1 \otimes Q_n)(\mathrm{id} \otimes V(t))(1 \otimes Q_n)\| \geqslant \|(1 \otimes P_n)(\mathrm{id} \otimes V(t))(1 \otimes P_n)\| =$$

$$= \|\mathrm{id} \otimes V_n(\mathrm{id} \otimes P_{\nu(n)}(t))\| \geqslant \varepsilon$$

because

$$\mathrm{id} \otimes V_n(\mathrm{id} \otimes P_{\nu(n)}(t)) = (1 \otimes P_n)(\mathrm{id} \otimes V(t))(1 \otimes P_n) \mid P_n H.$$

$\|(1 \otimes Q_n)s(1 \otimes Q_n)\| \to 0$ if $n \to \infty$ for every $s \in A \odot \mathscr{C}\mathscr{L}(H)$ and hence for every $s \in A \otimes \mathscr{C}\mathscr{L}(H)$ because $Q_n \to 0$ strongly and so $\|Q_n d Q_n\| \to 0$ for every $d \in \mathscr{C}\mathscr{L}(H)$. Hence id $\otimes V(t)$ is not in $A \otimes \mathscr{C}\mathscr{L}(H)$. (ii) follows from $L_\varphi(\mathrm{id} \otimes V(t)) = V(L_\varphi(t))$ by (i). Q.E.D.

Now assume A satisfies (iii), i.e. $\mathrm{F}(A, \mathscr{C}\mathscr{L}(H), A \otimes \mathscr{L}(H)) = A \otimes \mathscr{C}\mathscr{L}(H)$. By Lemma 2.2 and Lemma 2.3, A satisfies $\mathrm{F}(A, \overline{hBh}, A \otimes B) = A \otimes \overline{hBh}$ for every B and every $h \geqslant 0$ in B. Now let D be a hereditary subalgebra of B and $t \in \mathrm{F}(A, D, A \otimes B)$. The linear map $S_t : \varphi \in A \to L_\varphi(t) \in D$ is compact because S_t is in $\mathscr{L}(A^*, B)$ the norm limit of linear maps S_r, $r \in A \odot B$, and the S_r's are of finite rank. Hence $S_t(A^*)$ is contained in a subalgebra C of D generated by a compact subset of C, i.e. C is separable. By Lemma 2.1 there exists $h \geqslant 0$ in $C \subseteq D$ such that $C \subseteq \overline{hBh} \subseteq \overline{DBD} \subseteq D$. This means

$$t \in \mathrm{F}(A, C, A \otimes B) \subseteq \mathrm{F}(A, \overline{hBh}, A \otimes B) = A \otimes \overline{hBh} \subseteq A \otimes D.$$

Q.E.D.

REFERENCES

1. Batty, C. J. K., Fubini and martingale theorems in C^*-algebras, *Math. Proc. Cambridge Philos. Soc.*, **86**(1979), 57–69.
2. Huruya, T., An intersection result for tensor products of C^*-algebras, *Proc. Amer. Math. Soc.*, **75**(1979), 186–187.
3. Kirchberg, E., Positive maps and C^*-nuclear algebras, *Proceedings of the International Conference on operator algebras, ideals and their applications in theoretical physics* (Leipzig, 1977), pp. 327–328, Teubner, Leipzig, 1978.
4. Kirchberg, E., On exact C^*-algebras, ZIMM-Preprint.
5. Kirchberg, E., On subalgebras of nuclear C^*-algebras, ZIMM-Preprint.
6. Sakai, S., *C^*-algebras and W^*-algebras*, Springer-Verlag, 1971.
7. Takesaki, M., On the cross-norm of the direct product of C^*-algebras, *Tôhoku Math. J.*, **16**(1964), 111–122.
8. Tomiyama, J., Applications of Fubini type theorem to the tensor product of C^*-algebras, *Tôhoku Math. J.*, **19**(1967), 213–226.
9. Wassermann, S., On tensor products of certain group C^*-algebras, *J. Functional Analysis*, **23**(1976), 239–254.

10. WASSERMANN, S., A pathology in the ideal space of $L(H)\otimes L(H)$, *Indiana Univ. Math. J.*, 27(1978), 1012–1020.
11. WASSERMANN, S., The slice map problem for C^*-algebras, *Proc. London Math. Soc.*, (3), 32(1976), 537–559.

E. KIRCHBERG
Akademie der Wiessenschaften der DDR,
Zentralinstitut für Mathematik und Mechanik,
1080 Berlin, Mohrenstrasse 39,
G.D.R.

Received September 25, 1980.