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SPECTRAL ASYMPTOTICS FOR THE "SOFT" SELFADJOINT EXTENSION OF A SYMMETRIC ELLIPTIC DIFFERENTIAL OPERATOR

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1. INTRODUCTION

The purpose of this paper is to give an affirmative answer to a question raised by A. Alonso and B. Simon in [2], concerning the asymptotic behavior of the nonzero eigenvalues of the so-called soft extension A_M (or Kreĭn extension or von Neumann extension) of the minimal operator $A_{\min}$ associated with a strongly elliptic formally selfadjoint elliptic operator A of order $2m$ on a bounded smooth domain $\Omega \subset \mathbf{R}^n$. We show that the number $N(t; A_M)$ of eigenvalues in $]0, t[$ satisfies

$$(1.1) \qquad N(t; A_M) - c_A t^{n/2m} = O(t^{(n-\theta)/2m}) \quad \text{for } t \to \infty,$$

where c_A is the same constant as for the Dirichlet problem, and $\theta = \max\{1/2 - \varepsilon, 2m/(2m + n - 1)\}$ for any $\varepsilon > 0$. (Here $\theta = 2m/(2m + n - 1)$ when $2m \geqslant n - 1$.)

The method consists of reducing the eigenvalue problem for A_M to the eigenvalue problem for a certain compact operator S, whose deviation from the inverse A_γ^{-1} of the Dirichlet realization A_γ of A,

$$(1.2) \qquad G = S - A_\gamma^{-1},$$

is an operator with the spectral behavior

$$(1.3) \qquad \mu_j(|G|) = O(j^{-2m/(n-1)}) \quad \text{for } j \to \infty.$$

The eigenvalues of S are then estimated by application of a perturbation argument to a well known estimate for the eigenvalues of A_γ.

There are several ways to prove (1.3). A method is to use the very particular structure of G to derive (1.3) from a result essentially due to M. S. Birman [5], by use of operator-theoretical arguments (involving a theorem of Birman, Koplienko and Solomiak [6]). Another method is to view G as a kind of "singular Green operator", as introduced by L. Boutet de Monvel [8], for which the spectral asymptotics were studied systematically in G. Grubb [15], [17], [18]. Since the second method involves more technical machinery than the first one, we present the first method, noting however that the second point of view should be of value in the treatment of more general problems.

2. BACKGROUND MATERIAL

It was shown by M. G. Kreĭn in [20] that the family of selfadjoint nonnegative extensions $\tilde{A}$ of a positive, symmetric, closed, densely defined operator $A_{\min}$ in a Hilbert space H, can be described as the family of selfadjoint operators "lying between" (in a certain sense defined via sesquilinear forms) two special selfadjoint extensions A_γ and A_M, that he called, respectively, the "hard" extension and the "soft" extension. The hard extension simply equals the Friedrichs extension of $A_{\min}$, whereas the soft extension is the restriction of $A_{\max} = A_{\min}^*$ with domain

$$\mathrm{D}(A_M) = \mathrm{D}(A_{\min}) \dotplus \mathrm{Z}(A_{\max}). \tag{2.1}$$

(We denote by $\mathrm{D}(T)$, $\mathrm{R}(T)$ and $\mathrm{Z}(T)$, respectively, the domain, range and nullspace of a linear operator T in H; and we define its lower bound $m(T)$ by

$$m(T) = \inf\{\operatorname{Re}(Tu, u) \mid u \in \mathrm{D}(T), \|u\| = 1\} \geqslant -\infty.) \tag{2.2}$$

The operator A_M was originally introduced by J. von Neumann [22] as a solution to the problem of finding nonnegative selfadjoint extensions of $A_{\min}$.

The systematic study of selfadjoint and more general extensions $\tilde{A}$ (operators lying between $A_{\min}$ and $A_{\max}$) was carried further on in the works of M. I. Vishik [23] and M. S. Birman [4]. Vishik studied the solvability properties (characterizing, among other things, the extensions with the Fredholm property), and he introduced a concept of generalized boundary value problem applicable to the extensions $\tilde{A}$ with closed range, associated with a (second order) elliptic operator in a bounded domain $\Omega \subset \mathbf{R}^n$. Birman studied in particular the lower semibounded extensions, and placed the emphasis on the operator representing the difference between the inverses of a general (positive) extension and the hard extension A_γ.

The present author took up the subject again in her Stanford dissertation (1966), with a more concrete application to (nonlocal) elliptic boundary problems in bounded domains in $\mathbf{R}^n$, made possible by the regularity theory for elliptic boundary problems that had been developed in the meanwhile; in particular the theory of Lions and Magenes [21]. (The explicit application by Kreĭn [20, II] is concerned with *ordinary* differential equations, where $Z(A_{max})$ has *finite* dimension, in contrast with the case $n \geqslant 2$.) The material from the dissertation, containing both an abstract part extending the theories of [20], [23], [4], and its concrete interpretations is included in [10] and [11] (the latter especially concerned with sesquilinear forms). Let us also mention that non-selfadjoint cases were included in [12]—[14], where the structure of the realizations (and extensions) is thoroughly analyzed; among other things it is shown how lower boundedness and other estimates for realizations are related (by the use of pseudo-differential operators) to formulas and estimates for operators over the boundary.

In a recent volume of the Journal of Operator Theory (1980), A. Alonso and B. Simon [2] presented a version of the theory of Kreĭn, Vishik and Birman, unaware of the above mentioned development (cf. the addenda [3]), since most of their results are contained in the abstract part of [10], [11]. It may be remarked that the hypothesis, appearing in several places in [2] (e.g. in Theorem 2.3), that either $\dim Z(A_{max}) < \infty$ or a certain operator B is $\geqslant 0$ (in order to assure semiboundedness of the corresponding extension $\widetilde{A}$), has been gradually removed in the present author's papers, where no assumption on $\dim Z(A_{max})$ is made. According to [4] and [10], the hypothesis $B \geqslant 0$ can be replaced by a condition on the lower bound of B; in [11] it is replaced by a Gårding type inequality for B; and in [14] the hypothesis is completely removed, in the case where A_γ^{-1} is compact (as it is in our application).

Alonso and Simon end their paper with some observations on the soft extension A_M, showing (as Kreĭn remarked) that when A_γ has a discrete spectrum, then the spectrum of A_M outside of 0 is discrete. Moreover,

$$\lambda_j(A_M) \geqslant \lambda_j(A_\gamma), \quad j = 1, 2, \ldots, \tag{2.3}$$

where $\lambda_j(A_M)$, resp. $\lambda_j(A_\gamma)$, denotes the j'th positive eigenvalue of A_M, resp. A_γ, arranged nondecreasingly and repeated according to multiplicity. They raise the question as to whether $\lambda_j(A_M)$ has the same asymptotic behavior as $\lambda_j(A_\gamma)$, for $j \to \infty$. We show below that it is indeed so, with the accuracy indicated in (1.1) above.

3. SPECTRAL PROPERTIES OF THE SOFT REALIZATION OF AN ELLIPTIC OPERATOR

Let Ω be a bounded open subset of $\mathbb{R}^n$ ($n > 1$) with smooth boundary, and let, for s integer $\geqslant 0$, $H^s(\Omega)$ denote the Sobolev space of complex square integrable function on Ω with square integrable derivatives up to order s (so $H^0(\Omega) = L^2(\Omega)$). With $C_0^\infty(\Omega)$ denoting the space of C^∞ functions on Ω with compact support in Ω, we let $H_0^s(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$; it can also be described as the set of functions $u \in H^s(\Omega)$ with boundary values

$$\gamma_0 u = \gamma_1 u = \ldots = \gamma_{s-1} u = 0,$$

where γ_j stands for (a suitable generalization of) the trace operator

$$\gamma_j : u \mapsto (D_n^j u) \mid \partial\Omega ,$$

D_n a normal derivative. The spaces $H^s(\Omega)$ can be generalized to $s \in \mathbf{R}$, and to manifolds (e.g. Ω replaced by $\partial\Omega$), cf. Lions-Magenes [21].

Let A be a differential operator of order $2m$ (m integer > 0) on $\bar{\Omega}$ (with the notation $D^\alpha = (-i\partial/\partial x_1)^{\alpha_1} \ldots (-i\partial/\partial x_n)^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$)

$$(3.1) \qquad Au = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u, \quad a_\alpha(x) \in C^\infty(\bar{\Omega}),$$

which is *formally selfadjoint*, i.e.

$$(3.2) \qquad (Au, v) = (u, Av) \quad \text{for } u, v \in C_0^\infty(\Omega)$$

(with respect to the $L^2(\Omega)$ scalar product), and *strongly elliptic*, i.e. for some $c > 0$,

$$(3.3) \qquad a^0(x, \xi) \equiv \sum_{|\alpha| = 2m} a_\alpha(x) \xi^\alpha \geqslant c |\xi|^{2m} \quad \text{for } x \in \bar{\Omega}, \xi \in \mathbf{R}^n.$$

(The results extend straightforwardly to square matrix formed systems.) The *realizations* of A in $L^2(\Omega)$ are operators acting like A in the distribution sense. Let $A_{\min}$ be the realization with domain

$$(3.4) \qquad \mathrm{D}(A_{\min}) = H_0^{2m}(\Omega) = \{u \in H^{2m}(\Omega) \mid \rho u = 0\},$$

where $\rho u = \{\gamma_0 u, \ldots, \gamma_{2m-1} u\}$ denotes the Cauchy data of u (of order $2m$). Then $A_{\min}$ is a closed, densely defined operator in $L^2(\Omega)$. Because of (3.2), $A_{\min}$ is symmetric, and because of (3.3), it is lower semibounded (by the Gårding inequality); so we may assume, by adding a constant to A if necessary, that $A_{\min}$ is positive (i.e. $m(A_{\min}) > 0$).

The adjoint of $A_{\min}$, as an operator in $L^2(\Omega)$, identifies with $A_{\max}$, the realization of A with domain

$$\mathrm{D}(A_{\max}) = \{u \in L^2(\Omega) \mid Au \in L^2(\Omega) \text{ in the distribution sense}\}.$$

The "hard" extension of $A_{\min}$, alias the Friedrichs extension, is the same as the Dirichlet realization A_γ with domain

$$(3.5) \qquad \mathrm{D}(A_\gamma) = \{u \in H^{2m}(\Omega) \mid \gamma u = 0\},$$

where $\gamma u = \{\gamma_0 u, \ldots, \gamma_{m-1} u\}$ denotes the Dirichlet data of u; here $m(A_\gamma) = m(A_{\min}) > 0$, so A_γ is invertible. The "soft" extension of $A_{\min}$, i.e. the realization A_M with domain

(3.6)
$$\begin{aligned} D(A_M) &= D(A_{\min}) \dot{+} Z(A_{\max}) = \\ &= H_0^{2m}(\Omega) \dot{+} Z(A_{\max}), \end{aligned}$$

equals the realization defined by a certain non-elliptic, pseudo-differential boundary condition; namely the condition

$$Mu \equiv \nu u - P_{\gamma,\nu} \gamma u = 0,$$

where $\nu u = \{\gamma_m u, \ldots, \gamma_{2m-1} u\}$ (the Neumann data), and $P_{\gamma,\nu}$ is, briefly speaking, the pseudo-differential operator on $\partial\Omega$ sending γz into νz when $z \in Z(A_{\max})$. (More details in [10]—[12], [14], where the operators $A_{\min} \subset \tilde{A} \subset A_{\max}$ are studied systematically. A_M is described in [10, Theorem III.1.2] and [11, Remarque 1.3].) When $u \in D(A_M)$ has the decomposition $u = v + z$ according to (3.6), then v is determined by u via the formula

(3.7)
$$v = A_\gamma^{-1} A u.$$

The space $Z(A_{\max})$ is infinite dimensional; in fact γ defines a very useful *isomorphism* of $Z(A_{\max})$ onto the product of Sobolev spaces $\prod_{j=0}^{m-1} H^{-j-1/2}(\partial\Omega)$, that was a basic tool in [10]—[14]. So 0 is an eigenvalue for $D(A_M)$ of infinite multiplicity. For the eigenvalues distinct from 0, we observe

PROPOSITION 1. *Let $\lambda \neq 0$. Then there exists $u \in D(A_M) \setminus \{0\}$ with*

(3.8)
$$Au - \lambda u = 0,$$

if and only if there exists $v \in C^\infty(\bar{\Omega}) \setminus \{0\}$ so that

(3.9)
$$\begin{aligned} A^2 v - \lambda A v &= 0, \\ \rho v &= 0; \end{aligned}$$

here the solutions u of (3.8) *are in a* 1-1 *correspondence with the solutions v of* (3.9) *by the formulas*

(3.10)
$$\begin{aligned} v &= A_\gamma^{-1} A u, \\ u &= \frac{1}{\lambda} A v. \end{aligned}$$

Proof. Let $u \in D(A_M)$, uniquely decomposed into $u = v + z$ with $v \in H_0^{2m}(\Omega)$ and $z \in Z(A_{\max})$. Then $Au = \lambda u$ if and only if

$$0 = (A - \lambda)(v + z) = (A - \lambda) v - \lambda z,$$

and in that case, $z = \frac{1}{\lambda}(A - \lambda)v$ and $u = -\frac{1}{\lambda}Av$, so $u \neq 0$ if and only if $v \neq 0$. Furthermore, we then have

$$A(A - \lambda)v = \lambda Az = 0,$$

so that, since $v \in H_0^{2m}(\Omega)$ satisfying (3.4), v is a solution of (3.9). Since ρ is the Dirichlet boundary operator for the strongly elliptic operator $A^2 - \lambda A$, it follows from the ellipticity of this Dirichlet problem that $v \in C^\infty(\bar{\Omega})$.

Conversely, if v is a solution of (3.9), then $u = v + \frac{1}{\lambda}(A - \lambda)v$ satisfies (3.8). Here $u = \frac{1}{\lambda}Av$. ▨

The study of the eigenvalues outside of 0 is thus reduced to the study of the "eigenvalues" of (3.9).

In [7], M. S. Birman and M. Z. Solomiak gave an account of the development of the theory of eigenvalue problems of the form $A_1u = \lambda A_2u$, initiated by Å. Pleijel. In particular, a result of Birman and Solomiak (from 1971) implies, when combined with Proposition 1,

$$N(t; A_M) = c_A t^{n/2m} + o(t^{n/2m}) \quad \text{for } t \to \infty,$$

where c_A is the usual constant

$$c_A = (2\pi)^{-n} \int_\Omega dx \int_{a^0(x,\xi) < 1} d\xi. \tag{3.11}$$

In a recent announcement [19], V. A. Kozlov presents a result implying the following amelioration:

$$N(t; A_M) = c_A t^{n/2m} + O(t^{(n-\theta)/2m}) \quad \text{for } t \to \infty, \tag{3.12}$$

for any $\theta < 1/2$.

We shall show below, by an independent method, that (3.12) in fact holds with

$$\theta = 2m/(2m + n - 1), \tag{3.13}$$

which improves the estimate following from Kozlov's result, when $2m \geq n - 1$ (e.g. for the Laplace operator in $\mathbf{R}^3$).

For the inverse of the Dirichlet realization $(A^2)_\rho$ of A^2, we introduce the notation

$$((A^2)_\rho)^{-1} = R_\rho,$$

and note the following properties: R_ρ is a compact nonnegative selfadjoint operator in $L^2(\Omega)$, mapping $L^2(\Omega)$ bijectively onto

$$\mathrm{D}((A^2)_\rho) = H^{4m}(\Omega) \cap H_0^{2m}(\Omega).$$

Its square root $R_\rho^{1/2}$ is a well defined compact selfadjoint operator in $L^2(\Omega)$; and it follows from the definition of $(A^2)_\rho$ as the Friedrichs extension of $(A^2)_{\min}$, that $R_\rho^{1/2}$ is an isomorphism

$$R_\rho^{1/2} : L^2(\Omega) \xrightarrow{\sim} H_0^{2m}(\Omega), \tag{3.14}$$

and extends to an isomorphism

$$R_\rho^{1/2} : H^{-2m}(\Omega) \xrightarrow{\sim} L^2(\Omega), \tag{3.15}$$

where $H^{-2m}(\Omega)$ is the dual space of $H_0^{2m}(\Omega)$ with respect to an extension of the scalar product on $L^2(\Omega)$. (These are standard facts from the variational theory of elliptic boundary problems, see e.g. Lions-Magenes [21] and its references.)

$R_\rho^{1/2}$ enters in our study of the eigenvalue problem for A_M as follows:

PROPOSITION 2. *The spectrum of A_M consists of the eigenvalue 0 (with infinite multiplicity) and a sequence of positive eigenvalues $\lambda_j(A_M)$ (counted with multiplicities) going to $+\infty$; the latter are exactly the inverses of the eigenvalues $\mu_j(S)$ of the compact selfadjoint operator*

$$S = R_\rho^{1/2} A R_\rho^{1/2}. \tag{3.16}$$

Proof. The eigenvalue 0 was accounted for above. Let $\lambda \neq 0$. In order for u to solve the problem, for $f \in L^2(\Omega)$,

$$\begin{aligned} &Au - \lambda u = f \\ &u = v + z \quad \text{with } v \in H_0^{2m}(\Omega),\ z \in \mathrm{Z}(A_{\max}), \end{aligned} \tag{3.17}$$

one must have

$$A(A - \lambda) v = Af, \tag{3.18}$$

where $Af \in H^{-2m}(\Omega)$. In view of (3.14) and (3.15), we can set $v = R_\rho^{1/2} w$ with $w \in L^2(\Omega)$, and apply $R_\rho^{1/2}$ to both sides of (3.18), obtaining

$$R_\rho^{1/2} A^2 R_\rho^{1/2} w - \lambda R_\rho^{1/2} A R_\rho^{1/2} w = R_\rho^{1/2} Af \in L^2(\Omega).$$

Here $R_\rho^{1/2} A^2 R_\rho^{1/2} w = R_\rho^{1/2} R_\rho^{-1} R_\rho^{1/2} w = w$, so setting $\lambda^{-1} = \mu$ and defining S by (3.16), we reduce (3.18) to

$$\mu w - Sw = \mu R_\rho^{1/2} Af. \tag{3.19}$$

On one hand, when $f = 0$, (3.19) has a nontrivial solution w if and only if μ is an eigenvalue of S; in that case $v = R_\rho^{1/2} w$ solves (3.18) with $\lambda = \mu^{-1}$, and $u = \mu A v$ solves (3.17) (as seen already in Proposition 1); this gives the identification of the nonzero eigenvalues of A_M with the inverses of the eigenvalues of S. Here S is a bounded operator from $L^2(\Omega)$ to $H_0^{2m}(\Omega)$, and nonnegative, selfadjoint and injective in $L^2(\Omega)$, since $R_\rho^{1/2}$ maps $L^2(\Omega)$ into $H_0^{2m}(\Omega)$, so that $S = R_\rho^{1/2} A_\gamma^{-1} R_\rho^{1/2}$. It follows that S is a compact operator in $L^2(\Omega)$ with positive eigenvalues.

Now on the other hand, if $\mu \neq 0$ is not an eigenvalue of S, then $S - \mu$ is surjective from $L^2(\Omega)$ onto $L^2(\Omega)$, so (3.18) with $\lambda = \mu^{-1}$ has a (unique) solution $v \in H_0^{2m}(\Omega)$ for any $f \in L^2(\Omega)$. Then it is seen that (3.17) has the solution

$$u = v + \lambda^{-1}[(A - \lambda)v - f],$$

so $A_M - \lambda$ is surjective, and λ is not in the spectrum of A_M. ▨

When T is a compact selfadjoint nonnegative operator, its eigenvalue sequence will be denoted $\mu_j(T)$ $(j = 1,2,\ldots)$, arranged in a nonincreasing sequence and repeated according to multiplicities. More generally, when T is a compact operator, its characteristic values $s_j(T)$ are the eigenvalues $\mu_j(|T|)$ of $|T| = (T^*T)^{1/2}$.

The eigenvalues of S will be estimated by use of some well known eigenvalue estimates. Recall first the estimate for the Dirichlet realization: For any $\varepsilon > 0$,

$$\mu_j(A_\gamma^{-1}) = c_A^{2m/n} j^{-2m/n} + O(j^{-(2m+1-\varepsilon)/n}) \quad \text{for } j \to \infty; \tag{3.20}$$

this follows from Brüning [9], and we omit various known improvements that will not be of use here. (The estimate (3.20) is equivalent with the estimate

$$N(t; A_\gamma) = c_A t^{n/2m} + O(t^{(n-1+\varepsilon)/2m}) \quad \text{for } t \to \infty,$$

cf. e.g. [16, Lemma 6.2].)

Next, we recall the useful information that when A_B is a selfadjoint positive elliptic realization of A, then the difference between the inverses

$$G_1 = A_B^{-1} - A_\gamma^{-1} \tag{3.21}$$

is a nonnegative selfadjoint compact operator in $L^2(\Omega)$, whose nonzero part is *isometric* with a positive selfadjoint elliptic pseudo-differential operator on $\partial\Omega$ of order $-2m$ (Grubb [14]). In particular, the eigenvalues satisfy

$$\mu_j(G_1) = c_1 j^{-2m/(n-1)} + o(j^{-2m/(n-1)}) \quad \text{for } j \to \infty, \tag{3.22}$$

for a constant c_1 depending on A_B; here the remainder $o(j^{-2m/(n-1)})$ can be sharpened to $O(j^{-2m/(n-2)})$ in view of results of L. Hörmander and V. Ja. Ivrii, but we shall actually just need a weak version of (3.22) with the whole right hand side replaced by

$O(j^{-2m/(n-1)})$, shown in the second order case by M. S. Birman in [5]. It will be used in the case where A is replaced by A^2:

LEMMA 3. *Let $(A^2)_B$ be a positive selfadjoint elliptic realization of A^2. Then $G_2 = (A^2)_B^{-1} - R_\rho$ is a compact nonnegative operator, whose nonzero eigenvalues satisfy*

$$\mu_j(G_2) \leqslant C j^{-4m/(n-1)} \quad \textit{for } j = 1,2, \ldots,$$

for some constant $C > 0$.

The following general results will be used:

LEMMA 4. (Birman, Koplienko and Solomiak [6, Theorem 3]). *Let L and M be compact nonnegative operators on a Hilbert space H, let $\gamma > 0$ and $c > 0$, and denote $L - M = G$ and $L^\sigma - M^\sigma = G^{(\sigma)}$ for $\sigma \in]0,1[$. Then the estimate*

$$\mu_j(|G|) \leqslant cj^{-\gamma} \quad \textit{for } j = 1, 2, \ldots$$

implies, for any $\sigma \in]0, 1[$,

$$\mu_j(|G|^{(\sigma)}) \leqslant c_\sigma j^{-\gamma\sigma} \quad \textit{for } j = 1, 2, \ldots$$

with positive constants c_σ.

LEMMA 5. (Grubb [16, Proposition 6.1]). *Let L and M be compact operators on a Hilbert space H, let a, b and c be positive constants, and let $\beta > \alpha > 0$, and $\gamma > \alpha$. Denote $L - M = G$. The estimates*

$$|\mu_j(|M|) - aj^{-\alpha}| \leqslant bj^{-\beta} \quad \textit{for all } j,$$

$$\mu_j(|G|) \leqslant cj^{-\gamma} \quad \textit{for all } j,$$

imply

$$|\mu_j(|L|) - aj^{-\alpha}| \leqslant b'j^{-\beta'} \quad \textit{for all } j,$$

where $b' > 0$ and

$$\beta' = \min\{\beta, \gamma(1 + \alpha) / (1 + \gamma)\}.$$

Now we can show:

THEOREM. *The eigenvalue sequence of S satisfies*

$$\mu_j(S) - c_A^{2m/n} j^{-2m/n} = O(j^{-(2m+\theta)/n}) \quad \textit{for } j \to \infty, \tag{3.23}$$

with $\theta = 2m/(2m + n - 1)$, and c_A defined by (3.11). Consequently, the sequence of nonzero eigenvalues of A_M satisfies

$$N(t; A_M) - c_A t^{n/2m} = O(t^{(n-\theta)/2m}) \quad \textit{for } t \to \infty, \tag{3.24}$$

with

$$\theta = \max\{1/2 - \varepsilon, \ 2m/(2m + n - 1)\} \tag{3.25}$$

for any $\varepsilon > 0$.

Proof. Consider the operator

$$G_3 = S - A_\gamma^{-1}. \tag{3.26}$$

For this operator we have, since $A_\gamma^{-1}Au = u$ for $u \in D(A_{\min}) = H_0^{2m}(\Omega) = R(R_\rho^{1/2})$,

$$G_3 = S - A_\gamma^{-1} = R_\rho^{1/2}AR_\rho^{1/2} - A_\gamma^{-1} =$$

$$= (R_\rho^{1/2} - A_\gamma^{-1})AR_\rho^{1/2} + R_\rho^{1/2} - A_\gamma^{-1} =$$

$$= (R_\rho^{1/2} - A_\gamma^{-1})(I + AR_\rho^{1/2}),$$

where $I + AR_\rho^{1/2}$ is a bounded operator on $L^2(\Omega)$. It is then a simple consequence of the maximum-minimum principle that

$$\mu_j(|G_3|) \leqslant \mu_j(|R_\rho^{1/2} - A_\gamma^{-1}|)\|I + AR_\rho^{1/2}\|. \tag{3.27}$$

Concerning the operator

$$G_4 = R_\rho^{1/2} - A_\gamma^{-1},$$

we observe that, in the notation of Lemma 4,

$$G_4 = G_5^{(1/2)}, \quad \text{where } G_5 = R_\rho - A_\gamma^{-2};$$

and here A_γ^{-2} is the inverse of the realization $(A^2)_B$ of A^2 with boundary condition

$$\gamma u = 0 \quad \text{and} \quad \gamma Au = 0;$$

this realization is obviously elliptic and positive. By Lemma 3, we have that

$$\mu_j(|G_5|) \leqslant Cj^{-4m/(n-1)},$$

and hence by Lemma 4

$$\mu_j(|G_4|) \leqslant C'j^{-2m/(n-1)},$$

so that we may conclude from (3.27) that

$$\mu_j(|G_3|) \leqslant C''j^{-2m/(n-1)} \quad \text{for all } j. \tag{3.28}$$

Now we can combine (3.20) and (3.28) in an application of Lemma 5 with $L = S$, $M = A_\gamma^{-1}$, $\alpha = 2m/n$, $\beta = (2m + 1 - \varepsilon)/n$ and $\gamma = 2m/(n-1)$, obtaining

$$|\mu_j(S) - c_A^{2m/n} j^{-2m/n}| \leqslant C'''j^{-\beta'},$$

with

$$\beta' = \min\left\{\frac{2m+1-\varepsilon}{n}, \frac{2m}{n-1}\,\frac{1+2m/n}{1+2m/(n-1)}\right\} =$$

$$= \left(2m + \frac{2m}{2m+n-1}\right)\frac{1}{n},$$

since ε can be taken arbitrarily small. This shows (3.23) with $\theta = 2m/(2m + n - 1)$. Now the statement (3.23) for a $\theta > 0$ is equivalent with (3.24) with the same θ (cf. e. g. [16, Lemma 6.2]), so in view of Kozlov's result (3.12), we have (3.24) with (3.25). ▨

REMARK 1. The various operators $G_1, \ldots, G_5$ are all related to the concept of *singular Green operators* (s.g.o.s) as introduced by L. Boutet de Monvel in [8]. In fact, G_1, G_2 and G_5 are genuine s.g.o.s, whereas G_3 and G_4 are a kind of generalized s.g.o.s arizing from taking fractional powers of elliptic realizations; such operators were studied in [18]. These generalized s.g.o.s share many properties with the usual s.g.o.s; the property that is of importance here is that the spectral behavior of these operators is governed by the *dimension of the boundary* $(n - 1)$ rather than the dimension of the interior (n) of the considered manifold $\bar{\Omega}$. More details can be found in Grubb [15], [17] and [18].

The proof of the main theorem of the present paper was originally based on the general study of s.g.o.s, instead of the above, more operator-theoretical version. The author is indebted to professor M. S. Birman for pointing out the applicability of the result of Birman, Koplienko and Solomiak [6].

REMARK 2. It should be expected that the range of θ in (3.23)—(3.24) can be further improved, e.g. to all values $\theta < 1$. This seems to require either some sharper perturbation arguments, or a better knowledge of the iterative properties of the involved generalized singular Green operators (in order to use that $\theta \to 1$ for $2m \to \infty$ in (3.25)), or possibly another point of view for the whole problem.

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