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LIFTING PROJECTIONS FROM QUOTIENT C^*-ALGEBRAS

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1. INTRODUCTION

This paper deals with various aspects of the following lifting problem: let $\mathscr{I}$ be a closed two-sided ideal of a C^*-algebra $\mathfrak{A}$, and let $\Pi\colon \mathfrak{A} \to \mathfrak{A}/\mathscr{I}$ be the quotient map; for each projection $P \in \mathfrak{A}/\mathscr{I}$, must there exist a projection $Q \in \mathfrak{A}$ satisfying $\Pi(Q) = P$? The "concrete" results formulated here may serve, to a certain extent, as the C^*-algebraic realization of some K-theory implication. However, no working knowledge of K-theory is needed for the reading of this paper.

Notably, there is an appropriate K-theory approach to an abstract counterpart of the lifting problem. Utilizing such fact, Larry Brown has been successful to offer an affirmative answer for the lifting problem, under the assumption that the ideal $\mathscr{I}$ is an AF C^*-algebra. Moreover, Brown's result, in conjunction with George Elliott's observation [3, Corollary 3.3, p. 5] leads further to a remarkable structure theorem: The extensions of AF C^*-algebras by AF ideals are AF. (Readers are referred to the expository lecture notes by Edward Effros [2, Sections 8–9] for all details.)

While we may greatly admire the fascinating achievements attributed to C^*-algebraic K-theory techniques in recent research, we may also become fully aware of the appearance of certain mysterious phenomena or indirect constructions associated with the sophisticated machinery. For the sake of better understanting it is often helpful to have explicit constructions and simple examples to show the reality of K-theory miracles. This paper, motivated by K-theory methods, is written for the exposition of certain plain facts and concrete solutions towards the lifting problem. Along with other outcomes, a complete direct proof yielding, in particular, Brown's result is presented.

This paper is organized as follows. § 2 is intended for preliminaries. § 3 consists of the main results, establishing an affirmative answer for the lifting problem provided that the ideal $\mathscr{I}$ has a certain approximation property. Finally, § 4 is devoted to remarks and examples pertinent to K-theory considerations.

2. PRELIMINARIES

Throughout this paper, let $\mathscr{I}$ be a closed two-sided ideal of a C^*-algebra $\mathfrak{A}$. Without loss of generality, assume further that $\mathfrak{A}$ has the unit I. $\mathscr{B}(\mathscr{H})$ denotes the algebra of all bounded operators on a Hilbert space $\mathscr{H}$. For each operator $T \in \mathscr{B}(\mathscr{H})$, $C^*(T)$ is the unital C^*-algebra generated by T. M_n denotes the algebra of all $n \times n$ complex matrices. $M_n(\mathfrak{A})$ is the algebra of $n \times n$ matrices over $\mathfrak{A}$. Note that the quotient map $\Pi: \mathfrak{A} \to \mathfrak{A}/\mathscr{I}$ extends naturally to quotient map: $M_n(\mathfrak{A}) \to$ $\to M_n(\mathfrak{A})/M_n(\mathscr{I}) \simeq M_n(\mathfrak{A}/\mathscr{I})$.

In this paper, we employ elementary functional calculus frequently. Specifically, if T is a normal operator in a unital C^*-algebra $\mathfrak{B}$ and f: $\operatorname{sp}(T) \to \mathbf{C}$ is a continuous function defined on the spectrum of T, then $f(T)$ is a well-defined normal operator in $\mathfrak{B}$ with $\operatorname{sp}(f(T)) = \{f(\lambda): \lambda \in \operatorname{sp}(T)\}$. In particular, if T is a hermitian operator in a unital C^*-algebra $\mathfrak{B}$, and χ_X is the characteristic function for a set $X \subseteq \mathbf{R}$, and if $X \cap \operatorname{sp}(T)$ is both closed and open relative to $\operatorname{sp}(T)$, then $\chi_X(T)$ is a projection in $\mathfrak{B}$.

Next, we present several simple lemmas which are probably known to many readers. Since no handy references are available, we provide short proofs for completeness.

LEMMA 1. *Let $\Pi: \mathfrak{A} \to \mathfrak{A}/\mathscr{I}$ be the quotient map. If $T \in \mathfrak{A}/\mathscr{I}$ is a hermitian operator satisfying $0 \leqslant T \leqslant I$, then there exists a hermitian operator $A \in \mathfrak{A}$ satisfying $\Pi(A) = T$ and $0 \leqslant A \leqslant I$.*

Proof. Let $S \in \mathfrak{A}$ be an arbitrary operator satisfying $\Pi(S) = T$. Let

$$f(x) = \begin{cases} 1 & \text{if } x \geqslant 1 \\ x & \text{if } x \in [0,1], \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

and $A = f\left(\frac{S + S^*}{2}\right)$. Then A is a hermitian operator satisfying $0 \leqslant A \leqslant I$ and

$$\Pi(A) = \Pi\left(f\left(\frac{S + S^*}{2}\right)\right) = f\left(\Pi\left(\frac{S + S^*}{2}\right)\right) = f(T) = T,$$

as desired. ∎

LEMMA 2. *Let f: $S^1 \to \mathbf{C}$ be a continuous function defined on the unit circle S^1. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f(U) - f(V)\| < \varepsilon$ whenever U and V are unitary operators in $\mathscr{B}(\mathscr{H})$ satisfying $\|U - V\| < \delta$.*

Proof. For each $\varepsilon > 0$, there exists a positive integer n_0 and complex numbers α_n $(-n_0 \leqslant n \leqslant n_0)$ such that

$$|f(z) - \sum_{-n_0 \leqslant n \leqslant n_0} \alpha_n z^n| < \varepsilon/3$$

for all $z \in S^1$. Suppose $\|U - V\| < \delta$ for some δ to be selected. Then for each positive integer k,

$$\|U^k - V^k\| = \|U^{k-1}(U - V) + U^{k-2}(U - V)\, V + \ldots + (U - V)V^{k-1}\| < k\delta$$

and

$$\|U^{-k} - V^{-k}\| = \|(U^k - V^k)^*\| = \|U^k - V^k\| < k\delta.$$

Thus

$$\|f(U) - f(V)\| \leqslant \|f(U) - \sum \alpha_n U^n\| + \|f(V) - \sum \alpha_n V^n\| + \|\sum \alpha_n (U^n - V^n)\| <$$

$$< \varepsilon/3 + \varepsilon/3 + \sum_{-n_0 \leqslant n \leqslant n_0} |\alpha_n n|\, \delta$$

which is smaller than ε when δ is sufficiently small. ▨

LEMMA 3. *Let U and T be operators in $\mathscr{B}(\mathscr{H})$. If U is unitary, then there exists a unitary operator $V \in C^*(T)$ satisfying $\|U - V\| \leqslant 2\|U - T\|$.* (Cf. [2, Appendix to Section 8, Lemma A 8.3].)

Proof. If $\|U - T\| \geqslant 1$, then letting $V = I$, we are done.
Now suppose $\|U - T\| < 1$; then

$$\|I - U^*T\| = \|U^*(U - T)\| < 1,$$

so U^*T is invertible, thus T is invertible. By polar decomposition, $T = V(T^*T)^{1/2}$ where V is a unitary operator in $C^*(T)$. Hence, it follows by direct computation,

$$(U - V)\,[I + (T^*T)^{1/2}] = (U - T) - U(U - T)^*V;$$

i.e., $U - V = [(U - T) - U(U - T)^*V]\,[I + (T^*T)^{1/2}]^{-1}$. Therefore $\|U - V\| \leqslant 2\|U - T\|$, since $\|[I + (T^*T)^{1/2}]^{-1}\| \leqslant 1$. ▨

LEMMA 4. *If N_0 and N_1 are two normal operators in $\mathscr{B}(\mathscr{H})$, and if N_1 has finite spectrum, then there exists a normal operator $N_2 \in C^*(N_1)$ satisfying $\mathrm{sp}(N_2) \subseteq \mathrm{sp}(N_0)$ and $\|N_0 - N_2\| \leqslant 2\|N_0 - N_1\|$.*

Proof. Each $\alpha \in \mathrm{sp}(N_1)$ is an eigenvalue; i.e., there exists a nonzero vector $x \in \mathscr{H}$ such that $N_1 x = \alpha x$. Thus,

$$\|N_0 - N_1\| \geqslant \|(N_0 - N_1)x\|/\|x\| = \|(N_0 - \alpha I)x\|/\|x\| = \|\,|N_0 - \alpha I|x\|/\|x\| \geqslant$$

$$\geqslant \inf\{\|\,|N_0 - \alpha I|y\|/\|y\| : y \in \mathscr{H}\} =$$

$$= \inf \mathrm{sp}(|N_0 - \alpha I|) = \inf\{|\lambda - \alpha| : \lambda \in \mathrm{sp}(N_0)\}.$$

Define a function $g\colon \mathrm{sp}(N_1) \to \mathrm{sp}(N_0)$ such that if $\alpha \in \mathrm{sp}(N_1)$, then $g(\alpha)$ is any point in $\mathrm{sp}(N_0)$ satisfying

$$|g(\alpha) - \alpha| = \inf\{|\lambda - \alpha| : \lambda \in \mathrm{sp}(N_0)\}.$$

Thus $|g(\alpha) - \alpha| \leqslant \|N_0 - N_1\|$. Let $N_2 = g(N_1)$. Then N_2 is a normal operator in $C^*(N_1)$ satisfying $\mathrm{sp}(N_2) \subseteq \mathrm{sp}(N_0)$ and

$$\|N_0 - N_2\| \leqslant \|N_0 - N_1\| + \|G(N_1) - N_1\| \leqslant 2\|N_0 - N_1\|. \qquad \blacksquare$$

3. MAIN RESULTS

We call attention to the situation when the ideal $\mathscr{I}$ satisfies one of the following conditions.

Condition (A). The unitary group of $\mathbf{C}I + \mathscr{I}$ is the norm closure of the union of its finite abelian subgroups; i.e., for each $\delta > 0$ and each unitary operator U in $\mathbf{C}I + \mathscr{I}$, there exists a unitary operator V in $\mathbf{C}I + \mathscr{I}$ of finite spectrum with $\|U - V\| < \delta$. (Cf. the condition (FS) in [6].)

Condition (B). The group $\{I + T : T \in \mathscr{I},\ I + T \text{ is unitary}\}$ is the norm closure of the set $\{e^{2\pi i B} : B \in \mathscr{I}, 0 \leqslant B \leqslant I\}$; i.e., for each $\delta > 0$ and each unitary operator U with $U - I \in \mathscr{I}$, there exists a hermitian operator $B \in \mathscr{I}$ such that $0 \leqslant B \leqslant I$ and $\|U - e^{2\pi i B}\| < \delta$.

It is easy to see that Condition (A) $\Rightarrow$ Condition (B). Because of Lemmas 3—4, Condition (A) is actually equivalent to several other statements with various stronger or weaker assertions. Moreover, it follows from basic definitions that if $\mathscr{I}$ is an AW* algebra [4] or an AF C^*-algebra [1] (see also Lemma 3), then $\mathscr{I}$ satisfies the Conditions (A) and (B).

The following theorem is general enough to cover several cases of interest, including Brown's result. Note that the proof is fairly elementary since only rudiments of functional calculus are involved.

THEOREM 1. *Let $\mathscr{I}$ be a closed two-sided ideal of a unital C^*-algebra $\mathfrak{A}$ and let $\Pi\colon \mathfrak{A} \to \mathfrak{A}/\mathscr{I}$ be the quotient map. Suppose $\mathscr{I}$ satisfies the Condition* (A). *Then for each projection $P \in \mathfrak{A}/\mathscr{I}$, there exists a projection $Q \in \mathfrak{A}$ such that $\Pi(Q) = P$.*

Proof. (i) By Lemma 1, P can be lifted to a hermitian operator $A \in \mathfrak{A}$ with $0 \leqslant A \leqslant I$. Hence $\Pi(e^{2\pi i A} - I) = e^{2\pi i P} - I = 0$, so $e^{2\pi i A}$ is a unitary operator in $\mathbf{C}I + \mathscr{I}$.

(ii) By the Condition (A), there exists a unitary operator $V \in \mathbf{C}I + \mathscr{I}$ of finite spectrum such that $\|e^{2\pi i A} - V\| <$ any prescribed positive number $\delta/2$. Write $V = \alpha(I + T)$ with $\alpha \in \mathbf{C}$, $T \in \mathscr{I}$. Then

$$|\alpha|^2 = \|\Pi(V^*)\Pi(V)\| = \|\Pi(V^*V)\| = \|\Pi(I)\| = 1,$$

and $\alpha^{-1}V = I + T$ is a unitary operator of finite spectrum, and thus $\alpha^{-1}V$ can be written (uniquely) as $e^{2\pi i B}$ with $B \in C^*(V)$ and $0 \leqslant B < I$. It follows that

$$I = \Pi(I + T) = \Pi(e^{2\pi i B}) = e^{2\pi i \Pi(B)},$$

so $\Pi(B)$ is a projection. But, $\|\Pi(B)\| \leqslant \|B\| < 1$; thus $\Pi(B) = 0$ and so $B \in \mathscr{I}$. Furthermore,

$$\|e^{2\pi i A} - e^{2\pi i B}\| = \|(e^{2\pi i A} - V) - \alpha^{-1}(1 - \alpha)V\| \leqslant$$

$$\leqslant \|e^{2\pi i A} - V\| + |1 - \alpha| \leqslant \|e^{2\pi i A} - V\| + \|\Pi(e^{2\pi i A} - V)\| \leqslant$$

$$\leqslant 2\|e^{2\pi i A} - V\| < \delta.$$

(iii) Since B is of finite spectrum, it follows that $E = \chi_{(1/3,\,2/3)}(B)$ is a well-defined projection in $\mathscr{I}$. Let $C = (I - E)A(I - E)$. We *claim* that $1/2 \notin \mathrm{sp}(C)$ if δ is sufficiently small. Henceforth, $Q = \chi_{(1/2,\,\infty)}(C)$ is a projection in $\mathfrak{A}$ satisfying

$$\Pi(Q) = \Pi(\chi_{(1/2,\,\infty)}(C)) = \chi_{(1/2,\,\infty)}(\Pi(C)) =$$

$$= \chi_{(1/2,\,\infty)}(\Pi((I - E)A(I - E))) = \chi_{(1/2,\,\infty)}(\Pi(A)) = \chi_{(1/2,\,\infty)}(P) = P$$

as desired.

(iv) It remains to establish the "claim" of (iii). Apply Lemma 2 to functions f_1 and f_2, where

$$f_1(e^{2\pi i t}) = t - t^2, \qquad f_2(e^{2\pi i t}) = t^2 - t^3,$$

with $t \in [0,1]$. It follows that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|e^{2\pi i A} - e^{2\pi i B}\| < \delta$ implies

$$\|(A - A^2) - (B - B^2)\| < \varepsilon \quad \text{and} \quad \|(A^2 - A^3) - (B^2 - B^3)\| < \varepsilon.$$

Now, using the fact $E = \chi_{(1/3,\,2/3)}(B)$ and

$$\inf\{b - b^2 : b \in (1/3, 2/3)\} = \sup\{b - b^2 : b \in [0, 1/3] \cup [2/3, 1]\} = 2/9,$$

we derive that

$$0 \leqslant \frac{2}{9} E \leqslant B - B^2 \quad \text{and} \quad 0 \leqslant (B - B^2)(I - E) \leqslant \frac{2}{9} I.$$

Thus

$$(I - E)AEA(I - E) = (I - E)(A - B)E(A - B)(I - E) \leqslant$$

$$\leqslant \frac{9}{2}(I - E)(A - B)(B - B^2)(A - B)(I - E)$$

has a norm

$$\leqslant \frac{9}{2}\|(A-B)(B-B^2)\| \leqslant$$

$$\leqslant \frac{9}{2}\|A[(B-B^2)-(A-A^2)]\| + \frac{9}{2}\|(A^2-A^3)-(B^2-B^3)\| <$$

$$< \frac{9}{2}\cdot\varepsilon + \frac{9}{2}\cdot\varepsilon = 9\varepsilon .$$

Therefore

$$C - C^2 = (I-E)[A - A^2 + AEA](I-E) \leqslant$$

$$\leqslant (I-E)\left[\frac{2}{9}I - (B-B^2) + (A-A^2) + AEA\right](I-E)$$

has a norm

$$\leqslant \left\|\frac{2}{9}I\right\| + \|(A-A^2)-(B-B^2)\| + \|(I-E)AEA(I-E)\| <$$

$$< \frac{2}{9} + \varepsilon + 9\varepsilon < \frac{1}{4}$$

if ε is sufficiently small. Using the implications

$$\frac{1}{2}\in \mathrm{sp}(C) \Rightarrow \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}\in \mathrm{sp}(C-C^2) \Rightarrow \|C-C^2\| \geqslant \frac{1}{4},$$

we deduce $1/2 \notin \mathrm{sp}(C)$ as claimed. ∎

Roughly speaking, the proof above is an account of "non-commutative" $\varepsilon - \delta$ manipulations. The same technique can also be employed to tackle the following

THEOREM 2. *Let $\mathscr{I}$ be a closed two-sided ideal of a unital C^*-algebra $\mathfrak{A}$, and let $\Pi: \mathfrak{A} \to \mathfrak{A}/\mathscr{I}$ be the quotient map. Suppose $\mathscr{I}$ satisfies the Condition* (B). *Then for each projection $P \in \mathfrak{A}/\mathscr{I}$, there exists a projection $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix} \in M_2(\mathfrak{A})$ such that $\Pi(A_{11}) = P$ and $\Pi(A_{12}) = \Pi(A_{21}) = \Pi(A_{22}) = 0$.*

Proof. (i) Again by Lemma 1, lift P to a hermitian $A \in \mathfrak{A}$ with $0 \leqslant A \leqslant I$.

(ii) Let $A' = I - A$. Since

$$\Pi(e^{2\pi i A'} - I) = e^{2\pi i \Pi(A')} - I = e^{2\pi i (I-P)} - I = 0,$$

it follows that $e^{2\pi i A'} - I \in \mathscr{J}$. By the Condition (B), there exists a hermitian operator $B \in \mathscr{J}$ such that $0 \leqslant B \leqslant I$ and $\|e^{2\pi i A'} - e^{2\pi i B}\| <$ any prescribed positive number δ.

(iii) Let $Y = \begin{bmatrix} A & (A - A^2)^{1/2} \\ (A - A^2)^{1/2} & B \end{bmatrix}$. We *claim* that $1/2 \notin \mathrm{sp}(Y)$ if δ is sufficiently small. Let $\tilde{\Pi}$ denote the quotient map: $M_2(\mathfrak{A}) \to M_2(\mathfrak{A})/M_2(\mathscr{J}) \simeq \simeq M_2(\mathfrak{A}/\mathscr{J})$ induced by Π. Henceforth, $\chi_{(1/2,\infty)}(Y)$ is a projection in $M_2(\mathfrak{A})$ satisfying

$$\tilde{\Pi}(\chi_{(1/2,\infty)}(Y)) = \chi_{(1/2,\infty)}(\tilde{\Pi}(Y)) =$$

$$= \chi_{(1/2,\infty)}\left(\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$$

as desired.

(iv) It remains to prove the "claim". Apply Lemma 2 to functions f_j where $f_1(e^{2\pi i t}) = t - t^2$, $f_2(e^{2\pi i t}) = (t - t^2)^{1/2}$, and $f_3(e^{2\pi i t}) = (t - t^2)^{1/2} t$ with $t \in [0, 1]$. It follows that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|e^{2\pi i A'} - e^{2\pi i B}\| < \delta$ implies

$$\|(A' - A'^2) - (B - B^2)\| < \varepsilon,$$

$$\|(A' - A'^2)^{1/2} - (B - B^2)^{1/2}\| < \varepsilon,$$

and

$$\|(A' - A'^2)^{1/2} A' - (B - B^2)^{1/2} B\| < \varepsilon.$$

Thus by direct computation, we have

$$Y - Y^2 = \begin{bmatrix} 0 & X \\ X^* & W \end{bmatrix}$$

where

$$X = (A - A^2)^{1/2}(I - A - B) = (A' - A'^2)^{1/2}(A' - B) =$$

$$= [(A' - A'^2)^{1/2} A' - (B - B^2)^{1/2} B] + [(B - B^2)^{1/2} - (A' - A'^2)^{1/2}] B$$

is of norm $< 2\varepsilon$ and

$$W = (A - A^2) - (B - B^2) = (A' - A'^2) - (B - B^2)$$

is of norm $< \varepsilon$. Therefore, $\|Y - Y^2\| < 1/4$ if ε is sufficiently small. Again using the implications

$$\frac{1}{2} \in \mathrm{sp}(Y) \Rightarrow \frac{1}{4} \in \mathrm{sp}(Y - Y^2) \Rightarrow \|Y - Y^2\| \geqslant \frac{1}{4}$$

we deduce $1/2 \notin \mathrm{sp}(Y)$ as claimed. ∎

4. REMARKS AND EXAMPLES

There is a fascinating K-theory approach to the lifting problem. Specifically, the quotient map: $\mathfrak{A} \to \mathfrak{A}/\mathscr{I}$, which extends naturally to quotient map $M_n(\mathfrak{A}) \to M_n(\mathfrak{A})/M_n(\mathscr{I}) \simeq M_n(\mathfrak{A}/\mathscr{I})$, also induces a natural homomorphism from $K_0(\mathfrak{A})$ *onto* $K_0(\mathfrak{A})/\mathscr{I})$ provided that $K_1(\mathscr{I})$ is trivial [7, Theorem, p. 169; 2, Theorem 9.1]. In a more precise language, "a special case" of such an abstract result can be restated as follows:

If the unitary group of $M_n(CI + \mathscr{I})$ is connected for every positive integer n, then for each projection $P \in \mathfrak{A}/\mathscr{I}$, there exists non-negative integers m and k such that

$$P \oplus \underbrace{I \oplus \ldots \oplus I}_{m\text{ times}} \oplus \underbrace{0 \oplus \ldots \oplus 0}_{k\text{ times}} \in \underbrace{\mathfrak{A}/\mathscr{I} \oplus \ldots \oplus \mathfrak{A}/\mathscr{I}}_{1+m+k\text{ times}} \subseteq M_{1+m+k}(\mathfrak{A}/\mathscr{I})$$

is liftable to a projection in $M_{1+m+k}(\mathfrak{A})$. (See, e.g., [2, Lemma 9.6].)

There arise natural questions: what is m? what is k? how to eliminate the unwanted I's and 0's? Unfortunately, all these explicit questions extend beyond the boundary of abstract K-theory manipulations. Indeed, all known proofs of the statement above in italics depend upon heavy machinery, such as the Bott Periodicity Theorem, which does not lend itself to a really practical construction. In contrast, the elementary proofs of Theorems 1—2 may provide more satisfactory results towards a better exposition of K-theory miracles. Along these lines, we present two examples below to illustrate some phenomena pertinent to K-theory considerations.

EXAMPLE 1. A projection $P \in \mathfrak{A}/\mathscr{I}$ that is not liftable to any projection in $\mathfrak{A}$ but $\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathfrak{A}/\mathscr{I})$ is liftable to a projection in $M_2(\mathfrak{A})$.

Let $C([0,1], M_2)$ be the C^*-algebra of all continuous functions from [0,1] into M_2 with the sup norm. Let

$$\mathfrak{A} = \{F \in C([0,1], M_2) : F(0) \text{ is a diagonal matrix and } F(1) \text{ is a scalar matrix}\},$$

$$\mathscr{I} = \left\{F \in C([0,1], M_2) : F(0) = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \text{ and } F(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ with } \alpha \in \mathbf{C}\right\}.$$

It is easy to see that $\mathfrak{A}$ has no non-trivial projection. In fact, if $G \in \mathfrak{A}$ is a projection, then $G(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and by continuity, $G(t)_{t\in[0,1]}$ must be all $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or all $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now, let $H \in \mathfrak{A}$ with $H(t) = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}$ for $t \in [0,1]$; then $P = \Pi(H)$ is a projection in $\mathfrak{A}/\mathscr{I}$ that cannot be liftable to any projection in $\mathfrak{A}$. On the other hand, let $Q = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in M_2(\mathfrak{A})$ with

$$F_{11}(t) = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}, \quad F_{12}(t) = \begin{bmatrix} 0 & (t-t^2)^{1/2} \\ 0 & 0 \end{bmatrix},$$

$$F_{21}(t) = \begin{bmatrix} 0 & 0 \\ (t-t^2)^{1/2} & 0 \end{bmatrix}, \quad F_{22}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1-t \end{bmatrix};$$

then obviously, Q is a projection serving as a lifting for $\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$.

EXAMPLE 2. A projection $P \in \mathfrak{A}/\mathscr{I}$ such that for each n, $\begin{bmatrix} P & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} \in M_n(\mathfrak{A}/\mathscr{I})$ is not liftable to any projection in $M_n(\mathfrak{A})$, although the unitary group of $\mathbf{C}I + \mathscr{I}$ is connected.

Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space. Let

$$\mathfrak{A} = C([0,1], \mathscr{B}(\mathscr{H})) = \{\text{all norm-continuous functions} : [0,1] \to \mathscr{B}(\mathscr{H})\}$$

and $\mathscr{I} = \{F \in C([0,1], \mathscr{B}(\mathscr{H})) : F(0) = F(1) = 0\}$.

Since the unitary group of $\mathscr{B}(\mathscr{H})$ is simply connected (see, e.g., [5]), it follows immediately that

$$\mathbf{C}I + \mathscr{I} = \{F \in C([0,1], \mathscr{B}(\mathscr{H})) : F(0) = F(1) \in \mathbf{C}I\}$$

has a connected unitary group. (In fact, by a routine verification, it also follows that $K_1(\mathscr{I})$ is trivial.) On the other hand, define $G \in C([0,1], \mathscr{B}(\mathscr{H}))$ by $G(t) = tI$; then $P = \Pi(G) \in \mathfrak{A}/\mathscr{I}$ is a projection satisfying the required property. Indeed, if $\begin{bmatrix} P & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} \in M_n(\mathfrak{A}/\mathscr{I})$ were liftable to a projection $[F_{ij}]_{1 \leqslant i,j \leqslant n} \in M_n(\mathfrak{A})$, then

$[F_{ij}(0)]_{i,j}$ would be the zero matrix-operator in $M_n(\mathscr{B}(\mathscr{H}))$, and by continuity, $[F_{ij}(t)]_{i,j}$ would be the zero matrix-operator for all t, thus leading to contradiction. This justifies the construction of P.

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Added in proofs: Larry Brown's result has already appeared in his article [Extensions of AF algebras, the projection lifting problem, *Proc. Symp. Pure Math.*, **38**(1982), Part 1, 175–176].